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INTERSECTIONS OF SETS OF DISTANCE

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Abstract

We isolate conditions on the relative asymptotic size of sets of natural numbers A, B that guarantee a nonempty intersection of the corresponding sets of distances. Such conditions apply to a large class of zero density sets. We also show that a variant of Khintchine's Recurrence Theorem holds for all infinite sets $A = \{a_1 < a_2 < \ldots\}$ where $a_n \lll n^{3/2}$.

1. Introduction

It is a well-known phenomenon that if a set of natural numbers A has positive upper asymptotic density $\overline{d}(A) > 0$, then its *set of distances* (or *Delta-set*)

$$\Delta(A) \ = \ \{a - a' \mid a, a' \in A,\ a > a'\}$$

has a very rich combinatorial structure. An old problem attributed to Paul Erdős was whether the distance sets of two sets of positive upper density must necessarily meet.

- *Does* $\Delta(A) \cap \Delta(B) \neq \emptyset$ *whenever* $\overline{d}(A), \overline{d}(B) > 0$ *?*

The answer was shortly shown to be positive, and in fact the following much stronger intersection property holds:

- *If the upper density* $\overline{d}(A) = \alpha > 0$ *is positive, then* $\Delta(A) \cap \Delta(B) \neq \emptyset$ *whenever the set* B *contains more than* $1/\alpha$*-many elements.*

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The proof consists of a straightforward application of the *pigeonhole principle*. The key observation is that if one takes distinct elements $b_1, \ldots, b_N$ with $N > 1/\alpha$, then the shifted sets $A+b_i$ cannot be pairwise disjoint, as otherwise $\overline{d}(\bigcup_{i=1}^N A+b_i) = \sum_{i=1}^N \overline{d}(A+b_i) = N \cdot \overline{d}(A) > 1$. The argument is then completed by noticing that $(A+b) \cap (A+b') \neq \emptyset$ for some $b \neq b'$ in B if and only if $\Delta(A) \cap \Delta(B) \neq \emptyset$.

In the last forty years, the research on the combinatorial properties of distance sets and difference sets[2] has produced many interesting results (see, e.g., [10, 5, 11, 13, 14, 15, 16, 1, 9, 3, 7, 12, 8, 4]) which are almost always grounded on the hypothesis of positive density. In this paper, we look for general properties that include the zero density case, and investigate the size of intersections $\Delta(A) \cap \Delta(B)$ depending on the relative density of A with respect to B. More generally, for $k \in \mathbb{N}$, we will consider intersections $R_k(A) \cap \Delta(B)$ where

$$R_k(A) \;=\; \{x \in \mathbb{N} : |A \cap (A+x)| \geq k\}$$

is the *k-recursion set* of A. Elements of $R_k(A)$ are those natural numbers that are the distance of at least k-many different pairs of elements in A; in particular, $R_1(A) = \Delta(A)$.

The main results presented here (see Corollaries 3.5 and 4.3) can be summarized as follows.

Main Theorem. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and let $\vartheta : \mathbb{N} \to \mathbb{R}^+$ be such that $\limsup_{n\to\infty} \frac{a_n}{n \cdot \vartheta(n)} < \infty$.*

1. *If $\lim_{n\to\infty} \frac{\vartheta(b_n)}{n} = 0$, then $R_k(A) \cap \Delta(B)$ is infinite for all k.*
2. *If $\lim_{n\to\infty} \frac{\vartheta(n \cdot b_n)}{n} = 0$, then there exists a sequence $\langle x_n \mid n \in \mathbb{N}\rangle$ of elements of $\Delta(B)$ such that*

$$\limsup_{n\to\infty} \left(\frac{\frac{|A \cap (A+x_n) \cap [1,n]|}{n}}{\left(\frac{|A \cap [1,n]|}{n}\right)^2} \right) \;\geq\; 1\,.$$

We remark that the above results apply to a large class of zero density sets; e.g., when $B = \mathbb{N}$, (1) applies whenever $a_n \lll n^2$, and (2) applies whenever $a_n \lll n^{3/2}$. By way of examples, we list below three consequences (see Examples 3.7, 3.9, and 4.5).

Example 1. If $\sum_{n=1}^\infty \frac{1}{a_n} = \infty$ and $B = \{b_n\}$ is such that $\log b_n \lll n^{1-\varepsilon}$ for some $\varepsilon > 0$, then the intersections $R_k(A) \cap \Delta(B)$ are infinite for all k.

[2] By a *difference set* is meant a set of the form $A - B = \{a - b \mid a \in A, b \in B\}$. So, the set of distances $\Delta(A)$ is the positive part of $A - A$.

Example 2. Let $A = \{\lfloor K \cdot n\sqrt{n} \rfloor\}$ and $B = \{M \cdot n^3\}$. If $K^2 \cdot M < \frac{4}{27}$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k.

Example 3. Let $A = \{a_n\}$ have the same asymptotic size as the set of prime numbers, i.e., $\lim_{n\to\infty} \frac{a_n}{n \cdot \log n} = 1$, and assume that $B = \{b_n\}$ is sub-exponential, i.e., $\log b_n \lll n$. Then for every $\varepsilon > 0$ there exist infinitely many n and elements $x_n \in \Delta(B)$ such that

$$|A \cap (A + x_n) \cap [1, a_n]| \;\geq\; \frac{n}{\log n} \cdot (1 - \varepsilon).$$

Of course, the asymptotic conditions considered in our theorems about sequences $A = \{a_n\}$ may be reformulated by using the corresponding counting functions $A(u) = |\{a \in A \mid a \leq u\}|$. For instance, the role of a_n/n is played by $u/A(u)$, and so forth.

Notation. The natural numbers $\mathbb{N}$ are the set of *positive* integers. Letters $n, m, h, k, s, t, \nu, \mu, N$ will be used for natural numbers, and upper-case letters A, B, C, will be used for sets of natural numbers. For infinite sets $A \subseteq \mathbb{N}$ we use the brace notation $A = \{a_n\}$ to mean that elements a_n are arranged in increasing order:

$$A \;=\; \{a_n\} \;=\; \{a_1 < a_2 < \ldots < a_n < \ldots\}.$$

We write $A + x = \{a + x \mid a \in A\}$ to denote the *shift* of A by x. For functions $f : \mathbb{N} \to \mathbb{R}^+$ taking positive real values, we write $a_n \lll f(n)$ to mean that $\lim_{n\to\infty} a_n/f(n) = 0$.[3] By $\lfloor x \rfloor = \max\{k \in \mathbb{Z} \mid k \leq x\}$ is denoted the *integer part* of a real number x. Finally, recall the notion of *upper asymptotic density* $\overline{d}(A)$ for sets $A \subseteq \mathbb{N}$:

$$\overline{d}(A) \;=\; \limsup_{n\to\infty} \frac{|A \cap [1, n]|}{n}.$$

2. Preliminary Results

Let us start with a straightforward consequence of the *pigeonhole principle*.

Proposition 2.1. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers. If there exist N, ν such that $a_N + b_\nu \leq N \cdot \nu$ then $\Delta(A) \cap \Delta(B) \neq \emptyset$. In particular, if $\liminf_{n\to\infty} \frac{a_n + b_n}{n^2} < 1$ then $\Delta(A) \cap \Delta(B) \neq \emptyset$.*

Proof. Fix N, ν as in the hypothesis. The sumset

$$\{a_i + b_j \mid 1 \leq i \leq N;\; 1 \leq j \leq \nu\} \subseteq [2, a_N + b_\nu] \subseteq [2, N \cdot \nu]$$

[3] This is equivalent to Landau notation $a_n = o(f(n))$.

contains at most $N \cdot \nu - 1$ elements. So, by the *pigeonhole principle*, there exist $(i,j) \neq (i',j')$ such that $a_i + b_j = a_{i'} + b_{j'}$. Clearly $i \neq i'$, say $i > i'$. Then $a_i - a_{i'} = b_{j'} - b_j \in \Delta(A) \cap \Delta(B) \neq \emptyset$. If $\liminf_n \frac{a_n+b_n}{n^2} < 1$, pick N such that $\frac{a_N+b_N}{N^2} < 1$, and apply the above argument with $N = \nu$. □

Remark 2.2. The above result is best possible because there exist infinite sets $A = \{a_n\}$ and $B = \{b_n\}$ such that $\liminf_n \frac{a_n+b_n}{n^2} = 1$ but $\Delta(A) \cap \Delta(B) = \emptyset$. The following example is due to P. Erdős and R. Freud [6].

- Let A be the set of all natural numbers that are sums of even powers of 2, including $1 = 2^0$.
- Let B be the set of all natural numbers that are sums of odd powers of 2.

It only takes a little computation to verify that:

- $b_n = 2 \cdot a_n$ for all n;
- $\liminf_{n\to\infty} a_n/n^2 = 1/3$ is attained on the subsequence $n_k = 2^k - 1$;
- $\liminf_{n\to\infty} \frac{a_n+b_n}{n^2} = 1$.

Besides, since every natural number is uniquely written as a sum of powers of 2, an equality $a_i - a_j = b_s - b_t \Leftrightarrow a_i + b_t = a_j + b_s$ holds if and only if $i = j$ and $s = t$. It follows that $\Delta(A) \cap \Delta(B) = \emptyset$.

In order to improve on the previous result, we will use the following elementary inequality.

Lemma 2.3. *Let $A = \{a_1 < \ldots < a_N\}$ and $B = \{b_1 < \ldots < b_\nu\}$ be finite sets of natural numbers. For every $h \leq \nu/2$ there exists $x \in \Delta(B)$ such that $x \geq h$ and*

$$|\, A \cap (A+x) \,| \;\geq\; \frac{N^2}{a_N + b_\nu} - \frac{N \cdot (2h-1)}{\nu} \;=\; \frac{N^2}{a_N} \cdot \frac{1 - \frac{(a_N+b_\nu)(2h-1)}{N \cdot \nu}}{1 + \frac{b_\nu}{a_N}}\,.$$

The above inequality is strict except when $h = 1$ and $N \cdot \nu = a_N + b_\nu$.

Proof. Let us first consider the case $h = 1$. Let I be the interval $[1, a_N + b_\nu]$, and for every $i = 1, \ldots, \nu$, let $\chi_i : I \to \{0,1\}$ be the characteristic function of the shifted sets $A + b_i \subseteq I$. Notice that

$$\sum_{x \in I} \left(\sum_{i=1}^{\nu} \chi_i(x) \right) \;=\; \sum_{i=1}^{\nu} \left(\sum_{x \in I} \chi_i(x) \right) \;=\; \sum_{i=1}^{\nu} |A + b_i| \;=\; N \cdot \nu.$$

By the Cauchy-Schwartz inequality, we obtain:

$$\begin{aligned} N^2 \cdot \nu^2 &= \left(\sum_{x \in I} 1 \cdot \left(\sum_{i=1}^{\nu} \chi_i(x) \right) \right)^2 \leq \left(\sum_{x \in I} 1^2 \right) \cdot \sum_{x \in I} \left(\sum_{i=1}^{\nu} \chi_i(x) \right)^2 \\ &= |I| \cdot \sum_{x \in I} \left(\sum_{i,j=1}^{\nu} \chi_i(x) \cdot \chi_j(x) \right) = |I| \cdot \sum_{i,j=1}^{\nu} \left(\sum_{x \in I} \chi_i(x) \cdot \chi_j(x) \right) \\ &= (a_N + b_\nu) \cdot \sum_{i,j=1}^{\nu} |(A + b_i) \cap (A + b_j)|. \end{aligned}$$

If $M = \max\{|(A + b_i) \cap (A + b_j)| : 1 \leq i < j \leq \nu\}$ then

$$\begin{aligned} \sum_{i,j=1}^{\nu} |(A + b_i) \cap (A + b_j)| &= \sum_{i=1}^{\nu} |A + b_i| + 2 \cdot \sum_{1 \leq i < j \leq \nu} |(A + b_i) \cap (A + b_j)| \\ &\leq N \cdot \nu + 2 \cdot \binom{\nu}{2} \cdot M = \nu \cdot (N + (\nu - 1) \cdot M). \end{aligned}$$

(The above expressions are well-defined because we are assuming $h \leq \nu/2$, and hence $\nu \geq 2$.) By combining with the previous inequalities, we get that

$$N^2 \cdot \nu \leq (a_N + b_\nu) \cdot (N + (\nu - 1) \cdot M),$$

and hence

$$M \geq \frac{\nu}{\nu - 1} \cdot \frac{N^2}{a_N + b_\nu} - \frac{N}{\nu - 1} = \frac{\nu}{\nu - 1} \cdot \left(\frac{N^2}{a_N + b_\nu} - \frac{N}{\nu} \right) \geq \frac{N^2}{a_N + b_\nu} - \frac{N}{\nu}.$$

Notice that the last inequality is strict provided that $\frac{N^2}{a_N + b_\nu} - \frac{N}{\nu} > 0$ or, equivalently, when $N \cdot \nu > a_N + b_\nu$. Notice also that, since $M \geq 0$, the strict inequality trivially holds also when $N \cdot \nu < a_N + b_\nu$. Observe that if $M = |(A + b_s) \cap (A + b_t)|$ then $M = |A \cap (A + x)|$ where $x = b_s - b_t \in \Delta(B)$, and this completes the proof of the case $h = 1$.

Now let $h \geq 2$. Let μ be such that $\mu h \leq \nu < (\mu + 1)h$, and consider the set $B' = \{b'_1 < \ldots < b'_\mu\} \subset B$ where $b'_i = b_{ih}$. Notice that $\mu \geq 2$, because we are assuming $h \leq \nu/2$, and so we can apply the property proved above to prove the existence of an element $x \in \Delta(B')$ such that

$$|A \cap (A + x)| \geq \frac{N^2}{a_N + b'_\mu} - \frac{N}{\mu} \geq \frac{N^2}{a_N + b_\nu} - \frac{N}{\mu}.$$

For suitable indexes $1 \leq s < t \leq \mu$, one has that $x = b'_t - b'_s = b_{th} - b_{sh} \geq th - ts \geq h$. Finally, notice that $\frac{N}{\mu} = \frac{N}{\nu} \cdot \frac{\nu}{\mu} < \frac{N}{\nu} \cdot \frac{(\mu+1)h}{\mu}$, and since $\frac{(\mu+1)h}{\mu} \leq 2h - 1$, the proof is completed. Indeed, $\mu h + h \leq 2\mu h - \mu \Leftrightarrow \mu h \geq \mu + h$, and the last inequality holds because $\mu, h \geq 2$. □

Theorem 2.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that*

$$\liminf_{n\to\infty} \frac{a_n + b_n}{n^2} \; = \; 0\,.$$

Then $R_k(A) \cap \Delta(B)$ is infinite for all k.[4]

Proof. Fix an arbitrary $h \in \mathbb{N}$. For every $n \geq 2h$, apply Lemma 2.3 to the finite sets $A_n = \{a_1 < \ldots < a_n\}$ and $B_n = \{b_1 < \ldots < b_n\}$, and get the existence of an element $x_n \in \Delta(B_n)$ such that $x_n \geq h$ and

$$|A \cap (A + x_n) \cap [1, a_n + b_n]| \; \geq \; |A_n \cap (A_n + x_n)| \; > \; \frac{n^2}{a_n + b_n} - (2h - 1)\,.$$

By the hypothesis, the sequence on the right side is unbounded as n goes to infinity and so, for every k, there exists $x_n \in \Delta(B_n) \subseteq \Delta(B)$ with $x_n \geq h$ and $|A \cap (A + x_n)| \geq k$. As h was arbitrary, this proves that the intersections $R_k(A) \cap \Delta(B)$ are infinite. □

Next, we prove that when A has positive asymptotic density, the set of all possible shifts x that yield "large" intersections $A \cap (A + x)$ is "combinatorially large," in the sense that it meets all sufficiently large Delta-sets.

Theorem 2.5. *Let A be a set of natural numbers with $\overline{d}(A) = \alpha > 0$. Then for every $\varepsilon > 0$ and for every set B with $|B| \geq \alpha/\varepsilon$, one has*

$$\{x \mid \overline{d}(A \cap (A + x)) \geq \alpha^2 - \varepsilon\} \cap \Delta(B) \neq \emptyset\,.$$

Proof. Notice first that the limit superior for the upper asymptotic density is attained along intervals of the form $[1, a_n]$; so, by passing to a subsequence if necessary, we can directly assume that $\lim_{n\to\infty} n/a_n = \alpha$. Without loss of generality, let us assume that $B = \{b_1 < \ldots < b_\nu\}$ is finite with $\nu \geq \alpha/\varepsilon$. For every n, apply Lemma 2.3 to the finite sets $A_n = \{a_1 < \ldots < a_n\}$ and B (with $h = 1$) and obtain the existence of an element $x_n \in \Delta(B)$ such that

$$|A \cap (A + x_n) \cap [1, a_n]| \; \geq \; |A \cap (A + x_n) \cap [1, a_n + b_\nu]| - b_\nu \; \geq$$

$$|A_n \cap (A_n + x_n)| - b_\nu \; \geq \; \frac{n^2}{a_n} \cdot \frac{1 - \frac{a_n + b_\nu}{n \cdot \nu}}{1 + \frac{b_\nu}{a_n}} - b_\nu\,.$$

Since ν is fixed, by passing to the limit as n goes to infinity, we get

$$\lim_{n\to\infty} \frac{|A \cap (A + x_n) \cap [1, a_n]|}{a_n} \; \geq$$

[4] Recall that $R_k(A) = \{x \in \mathbb{N} : |A \cap (A + x)| \geq k\}$.

$$\lim_{n\to\infty} \frac{n^2}{a_n^2} \cdot \frac{1 - \frac{a_n}{n\nu} - \frac{b_\nu}{n\nu}}{1 + \frac{b_\nu}{a_n}} - \frac{b_\nu}{a_n} \;=\; \alpha^2 \cdot \left(1 - \frac{1}{\alpha \cdot \nu}\right) \;\geq\; \alpha^2 - \varepsilon\,.$$

Now notice that the sequence $\langle x_n \mid n \in \mathbb{N}\rangle$ takes values in the finite set $\Delta(B)$, and so there exists an element $x \in \Delta(B)$ such that the limit superior is attained along a subsequence $\{n_k\}$ where $x_{n_k} = x$ for all k. Such an element x yields the theorem because

$$\overline{d}(A \cap (A + x)) \;\geq\; \limsup_{k\to\infty} \frac{|A \cap (A + x) \cap [1, a_{n_k}]|}{a_{n_k}} \;=$$

$$=\; \limsup_{k\to\infty} \frac{|A \cap (A + x_{n_k}) \cap [1, a_{n_k}]|}{a_{n_k}} \;\geq\; \alpha^2 - \varepsilon\,.$$

□

As a straight corollary, we obtain the well-known density version of Khintchine's Recurrence Theorem for sets of integers (see, e.g., Section 5 of [2]).

Corollary 2.6. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers. If $\overline{d}(A) > 0$ then for every $\varepsilon > 0$ the following intersection is infinite:*

$$\{x \mid \overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon\} \cap \Delta(B).$$

In consequence:

1. *All intersections $R_k(A) \cap \Delta(B)$ are infinite;*
2. $\limsup_{x\in\Delta(B)} \frac{\overline{d}(A\cap(A+x))}{\overline{d}(A)^2} \geq 1\,.$

Proof. For every h, by applying the previous theorem to A and $B^h = \{b_{hn}\}$, one gets the existence of an element $x_h = b_{hs} - b_{ht} \in \Delta(B^h) \subseteq \Delta(B)$ with $\overline{d}(A \cap (A + x_h)) \geq \overline{d}(A)^2 - \varepsilon$. Notice that $x_h \geq hs - ht \geq h$. This proves that there are arbitrarily large elements in the intersection $\{x \mid \overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon\} \cap \Delta(B)$, as desired.

(1). Every set of positive upper density is infinite, and so, for every k, we have $\{x \mid \overline{d}(A \cap (A + x)) > \overline{d}(A)^2 - \varepsilon\} \subseteq R_k(A)$ whenever $0 < \varepsilon < \overline{d}(A)^2$.

(2). By what is proved above, for every $\varepsilon > 0$ there are infinitely many elements $x \in \Delta(B)$ such that $\overline{d}(A + (A + x)) \geq \overline{d}(A)^2 - \varepsilon$; but then $\limsup_{x\in\Delta(B)} \overline{d}(A \cap (A + x)) \geq \overline{d}(A)^2 - \varepsilon$. Since $\varepsilon > 0$ can be taken arbitrarily small, the result follows. □

Further on in this paper, we will show that a result similar to (2) can be proved for a large class of zero density sets (see Corollary 4.3).

3. Intersection Properties

We saw in Theorem 2.4 that $\Delta(A) \cap \Delta(B) \neq \emptyset$ whenever both A and B are asymptotically larger than the set of squares. We now sharpen that result, and prove a general intersection property that also applies when b_n/n^2 goes to infinity.

Theorem 3.1. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers where $a_n \lll n^2$. Let $f(n) = a_n/n$ and $g(n) = b_n/n$.*

1. *If there exists a constant $c > 1$ such that*
$$\liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;<\; 1 - \frac{1}{c}$$
then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k.

2. *If for arbitrarily large constants c one has*
$$\liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;=\; 0$$
then $R_k(A) \cap \Delta(B)$ is infinite for all k.

3. *If there exists a constant $\varepsilon > 0$ such that*
$$\liminf_{n\to\infty} \frac{f(\lfloor \varepsilon \cdot b_n \rfloor)}{n} \;<\; 1$$
then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k.

4. *If there exists a constant $\varepsilon > 0$ such that*
$$\liminf_{n\to\infty} \frac{f(\lfloor \varepsilon \cdot b_n \rfloor)}{n} \;=\; 0$$
then $R_k(A) \cap \Delta(B)$ is infinite for all k.

Proof. In the following, without loss of generality, we will always assume that $n \lll a_n$. Indeed, $n \lll a_n$ fails if and only if the upper asymptotic density $\overline{d}(A)$ is positive, and in this case the four properties above are all proved by Corollary 2.6.

(1). Let
$$\liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;=\; l \;<\; 1 - \frac{1}{c}.$$
For every n, let $\tau(n) = \lfloor c \cdot f(n) \rfloor$, and apply Lemma 2.3 with $h = 1$ to the sets $A_n = \{a_1 < \ldots < a_n\}$ and $B_{\tau(n)} = \{b_1 < \ldots < b_{\tau(n)}\}$. We obtain the existence of an element $x_n \in \Delta(B_{\tau(n)}) \subseteq \Delta(B)$ such that:
$$|A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| \;\geq\; |A_n \cap (A_n + x_n)| \;\geq\; \frac{n^2}{a_n} \cdot \frac{1 - \frac{a_n + b_{\tau(n)}}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}}\,.$$

Since we are assuming $n \lll a_n$, we have that $\lim_{n\to\infty} f(n) = \infty$, and so

$$\lim_{n\to\infty} \frac{a_n}{n \cdot \tau(n)} \;=\; \lim_{n\to\infty} \frac{f(n)}{\lfloor c \cdot f(n) \rfloor} \;=\; \frac{1}{c}\,.$$

Besides,

$$\liminf_{n\to\infty} \frac{b_{\tau(n)}}{n \cdot \tau(n)} \;=\; \liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;=\; l\,,$$

and

$$\liminf_{n\to\infty} \frac{b_{\tau(n)}}{a_n} \;=\; \liminf_{n\to\infty} \frac{\lfloor c \cdot f(n) \rfloor}{f(n)} \cdot \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;=\; c \cdot l\,.$$

Notice that the two limit inferiors above are attained along the same subsequence, and so

$$\limsup_{n\to\infty} \frac{1 - \frac{a_n + b_{\tau(n)}}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}} \;=\; \frac{1 - \left(\frac{1}{c} + l\right)}{1 + c \cdot l} \;>\; 0\,.$$

By using the hypothesis $a_n \lll n^2$, i.e., $\lim_{n\to\infty} n^2/a_n = \infty$, we can then conclude that

$$\limsup_{n\to\infty} |A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| \;=\; \infty\,.$$

This shows that for every k one finds elements $x_n \in \Delta(B)$ such that $|A \cap (A + x_n)| \geq k$, and hence $R_k(A) \cap \Delta(B) \neq \emptyset$.[5]

(2). Fix $h > 1$. For every n, let $\tau(n) = \lfloor 2h \cdot f(n) \rfloor$, and apply Lemma 2.3 to the sets A_n and $B_{\tau(n)}$ so as to get the existence of an element $x_n \in \Delta(B_{\tau(n)}) \subseteq \Delta(B)$ such that $x_n \geq h$ and

$$|A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}| \;\geq\; \frac{n^2}{a_n} \cdot \frac{1 - \frac{(a_n + b_{\tau(n)})(2h-1)}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}}.$$

Now use the same arguments as in the proof of the previous property (1). Since in our case $c = 2h$ and $l = 0$, we obtain that

$$\limsup_{n\to\infty} \frac{1 - \frac{(a_n + b_{\tau(n)})(2h-1)}{n \cdot \tau(n)}}{1 + \frac{b_{\tau(n)}}{a_n}} \;=\; \frac{1 - \left(\frac{1}{c} + l\right)(2h-1)}{1 + c \cdot l} \;=\; 1 - \frac{2h-1}{2h} \;>\; 0.$$

By the hypothesis $a_n \lll n^2$, we conclude that

$$\limsup_{n\to\infty} |A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| \;=\; \infty.$$

So, for every k, there exist elements $x_n \in \Delta(B_n) \subseteq \Delta(B)$ such that $x_n \geq h$ and $|A \cap (A + x_n)| \geq k$. Since h is arbitrary, this shows that the intersection $R_k(A) \cap \Delta(B)$ is infinite, as desired.

[5] We remark that the map $n \mapsto x_n$ may not be 1-1, and so the above argument *does not* prove that $R_k(A) \cap \Delta(B)$ contains infinitely many elements.

(3). The proof is entirely similar to the proof of (1), by applying Lemma 2.3 to the sets $A_{\sigma(n)}$ and B_n where $\sigma(n) = \lfloor \varepsilon \cdot b_n \rfloor$. Indeed, notice that

$$\liminf_{n\to\infty} \frac{a_{\sigma(n)}}{\sigma(n)\cdot n} \;=\; \liminf_{n\to\infty} \frac{f(\lfloor \varepsilon \cdot b_n \rfloor)}{n} \;=\; l \;<\; 1.$$

Besides,

$$\lim_{n\to\infty} \frac{b_n}{\sigma(n)} \;=\; \lim_{n\to\infty} \frac{b_n}{\lfloor \varepsilon \cdot b_n \rfloor} \;=\; \frac{1}{\varepsilon} \;<\; \infty\,,$$

and so

$$\lim_{n\to\infty} \frac{b_n}{\sigma(n)\cdot n} \;=\; 0 \quad \text{and} \quad \lim_{n\to\infty} \frac{b_n}{a_{\sigma(n)}} \;=\; \lim_{n\to\infty} \frac{b_n}{\sigma(n)} \cdot \frac{\sigma(n)}{a_{\sigma(n)}} \;=\; 0.$$

Thus we have the existence of elements $x_n \in \Delta(B)$ such that

$$\limsup_{n\to\infty} |A \cap (A + x_n) \cap [1, a_{\sigma(n)} + b_n]| \;\geq\; \limsup_{n\to\infty} \frac{\sigma(n)^2}{a_{\sigma(n)}} \cdot \frac{1 - \frac{a_{\sigma(n)}+b_n}{\sigma(n)\cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} \;=\; \infty\,,$$

and the result is proved.

(4). For fixed $h > 1$, we proceed as in (3) and obtain the existence of elements $x_n \in \Delta(B_n) \subseteq \Delta(B)$ with $x_n \geq h$ and such that

$$\limsup_{n\to\infty} |A \cap (A + x_n) \cap [1, a_{\sigma(n)} + b_n]| \;\geq\; \limsup_{n\to\infty} \frac{\sigma(n)^2}{a_{\sigma(n)}} \cdot \frac{1 - \frac{(a_{\sigma(n)}+b_n)(2h-1)}{\sigma(n)\cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}}\,.$$

As we are assuming $l = 0$, the above limit superior is infinite. Finally, since h can be taken arbitrarily large, the property is proved. □

Remark 3.2. Under the (mild) hypothesis that $g(n)$ be non-decreasing, one can prove (3) and (4) as consequences of (1) and (2), which are therefore basically stronger properties. Indeed, given $\varepsilon > 0$, let us assume that $\tau(n) = f(\lfloor \varepsilon \cdot b_n \rfloor)/n$ satisfies the condition $\liminf_{n\to\infty} \tau(n) = l < 1$. Then for every constant c such that $c \cdot l < 1$, we have that $c \cdot \tau(n) \cdot n < n$ for infinitely many n, and so

$$\begin{aligned}\liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;&\leq\; \liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(\lfloor \varepsilon \cdot b_n \rfloor) \rfloor)}{\lfloor \varepsilon \cdot b_n \rfloor} \\ =\; \liminf_{n\to\infty} \frac{g(\lfloor c \cdot \tau(n) \cdot n \rfloor)}{\lfloor \varepsilon \cdot n \cdot g(n) \rfloor} \;&=\; \frac{1}{\varepsilon} \cdot \liminf_{n\to\infty} \frac{g(\lfloor c \cdot \tau(n) \cdot n \rfloor)}{n \cdot g(n)} \\ &\leq\; \frac{1}{\varepsilon} \cdot \liminf_{n\to\infty} \frac{g(n)}{n \cdot g(n)} \;=\; 0\,.\end{aligned}$$

Notice that, since $l < 1$, we can pick constants $c > 1$ such that $c \cdot l < 1$, and this completes the proof of (1) ⇒ (3). Besides, if $l = 0$, every constant $c > 1$ trivially satisfies $c \cdot l < 1$, and also (2) ⇒ (4) follows.

As a consequence of the previous theorem, one can isolate a large class of sets B such that $R_k(A) \cap \Delta(B) \neq \emptyset$, in terms of their density relative to A.

Corollary 3.3. *Let $A = \{a_n = n \cdot f(n)\}$ be an infinite set of natural numbers where $f : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing unbounded function, and assume that the infinite set of natural numbers $B = \{b_n\}$ is such that*

$$\lim_{n\to\infty} \frac{b_n/n}{f^{-1}(\varepsilon \cdot n)} \;=\; 0 \quad \textit{for all } \varepsilon > 0\,.$$

Then the intersections $R_k(A) \cap \Delta(B)$ are infinite for all k.

Proof. Fix $c > 1$, and let $\tau(n) = \lfloor c \cdot f(n) \rfloor$ and $\varepsilon = 1/c$. Then $f^{-1}(\varepsilon \cdot \tau(n)) \leq n$ and

$$0 \;\leq\; \lim_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \;\leq\; \lim_{n\to\infty} \frac{g(\tau(n))}{f^{-1}(\varepsilon \cdot \tau(n))} \;=\; \lim_{n\to\infty} \frac{b_{\tau(n)}/\tau(n)}{f^{-1}(\varepsilon \cdot \tau(n))} \;=\; 0\,.$$

Thus (2) of the previous Theorem applies, and the corollary is proved. ☐

When $\varepsilon = 1$, items (3) and (4) in Theorem 3.1 have the advantage that they can be reformulated in the following simpler form:

Corollary 3.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers where $a_n \lll n^2$, and let*

$$\liminf_{n\to\infty} \frac{a_{b_n}}{n \cdot b_n} \;=\; l\,.$$

If $l < 1$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k; and if $l = 0$ then $R_k(A) \cap \Delta(B)$ is infinite for all k.

A consequence that is easily applied in several examples is the following:

Corollary 3.5. *Given a function $\vartheta : \mathbb{N} \to \mathbb{R}^+$ and infinite sets of natural numbers $A = \{a_n\}$ and $B = \{b_n\}$, denote by:*

$$\liminf_{n\to\infty} \frac{a_n}{n \cdot \vartheta(n)} = \underline{\ell}\,; \quad \limsup_{n\to\infty} \frac{a_n}{n \cdot \vartheta(n)} = \overline{\ell}\,;$$

$$\liminf_{n\to\infty} \frac{\vartheta(b_n)}{n} = \underline{\ell}'\,; \quad \limsup_{n\to\infty} \frac{\vartheta(b_n)}{n} = \overline{\ell}'\,.$$

If $\underline{\ell} \cdot \overline{\ell}' < 1$ or $\overline{\ell} \cdot \underline{\ell}' < 1$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k; and if $\underline{\ell} \cdot \overline{\ell}' = 0$ or $\underline{\ell} \cdot \overline{\ell}' = 0$ then $R_k(A) \cap \Delta(B)$ is infinite for all k.[6]

[6] By writing $\underline{\ell} \cdot \overline{\ell}' < 1$ or $\underline{\ell} \cdot \overline{\ell}' = 0$, it is implicitly assumed that both $\underline{\ell}$ and $\overline{\ell}'$ are finite; and similarly in the other cases.

Proof. It is a direct application of Corollary 3.4. Indeed, if $\underline{\ell}$ and $\overline{\ell}\,'$ are finite, then

$$\liminf_{n\to\infty} \frac{a_{b_n}}{n \cdot b_n} \;\leq\; \liminf_{n\to\infty} \frac{a_{b_n}}{b_n \cdot \vartheta(b_n)} \cdot \limsup_{n\to\infty} \frac{\vartheta(b_n)}{n} \;\leq\; \underline{\ell} \cdot \overline{\ell}\,' ;$$

and if $\overline{\ell}$ and $\underline{\ell}\,'$ are finite, then

$$\liminf_{n\to\infty} \frac{a_{b_n}}{n \cdot b_n} \;\leq\; \limsup_{n\to\infty} \frac{a_{b_n}}{b_n \cdot \vartheta(b_n)} \cdot \liminf_{n\to\infty} \frac{\vartheta(b_n)}{n} \;\leq\; \overline{\ell} \cdot \underline{\ell}\,' .$$

□

As witnessed by the results proved above, if A has zero density but still it is "large" enough, then its set of distances intersect sets of distances of very "sparse" sets B. We give below two examples to illustrate this phenomenon.

Example 3.6. Let $P = \{p_n\}$ be the set of prime numbers, and let $B = \{2^n\}$ be the set of powers of 2. By the Prime Number Theorem,

$$\lim_{n\to\infty} \frac{p_n}{n \cdot \log n} \;=\; 1.$$

Since $(\log 2^n)/n = \log 2 < 1$, by the previous corollary we can conclude that for every k, there exist numbers of the form $2^m - 2^n$ which are the distance of at least k-many pairs of primes. Actually, there exist infinitely many such numbers, since the function $(n, m) \mapsto 2^m - 2^n$ is 1-1; indeed, first pick $2^{n_1} - 2^{m_1} \in R_k(P) \cap \Delta(B)$, then consider $B^{(1)} = B \setminus \{2^{n_1}, 2^{n_2}\}$ and pick $2^{n_2} - 2^{m_2} \in R_k(P) \cap \Delta(B^{(1)})$, and so forth.

Example 3.7. Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \;=\; \infty \quad \text{and} \quad \log b_n \;\lll\; n^{1-\varepsilon} \quad \text{for some } \varepsilon > 0 .$$

Then $R_k(A) \cap \Delta(B)$ is infinite for all k.

Proof. If we let $\vartheta(n) = (\log n)^{\frac{1}{1-\varepsilon}}$, the hypotheses imply that

$$\liminf_{n\to\infty} \frac{a_n}{n \cdot \vartheta(n)} = 0 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\vartheta(b_n)}{n} = \left(\limsup_{n\to\infty} \frac{\log b_n}{n^{1-\varepsilon}} \right)^{\frac{1}{1-\varepsilon}} = 0 ,$$

and the desired intersection property follows from Corollary 3.5. □

E.g., if $A = \{a_n\}$ is such that $\sum_{n=1}^{\infty} \frac{1}{a_n} = \infty$, then for every exponent $\alpha < 1$ and for every k, there exist infinitely many numbers of the form $\lfloor 10^{n^\alpha} \rfloor - \lfloor 10^{m^\alpha} \rfloor$, everyone of which is the distance of at least k-many different pairs of elements of A.

Let us now focus on powers of n.

Theorem 3.8. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that, for all sufficiently large n,*

$$a_n \leq K \cdot n^{1+\alpha} \quad \text{and} \quad b_n \leq M \cdot n^{1+\beta}.$$

1. *If $\alpha < 1$ and $\beta < 1/\alpha$ then $R_k(A) \cap \Delta(B)$ is infinite for all k.*
2. *If $\alpha < 1$ and $\beta = 1/\alpha$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k whenever $K^\beta M < \frac{\alpha}{(1+\alpha)^{\beta+1}}$.*
3. *If $\alpha = \beta = 1$ then $\Delta(A) \cap \Delta(B) \neq \emptyset$ whenever $KM < \frac{1}{4}$.*

Proof. Notice first that, without loss of generality, we can assume $n \lll a_n$, and hence $\alpha > 0$. Indeed, otherwise $\overline{d}(A) > 0$, and the thesis is proved by Corollary 2.6.

(1). This property follows from (2) of Theorem 3.1 since $a_n \lll n^2$ and for every constant $c > 1$ one has that

$$\liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \leq \lim_{n\to\infty} \frac{M \cdot (c \cdot K \cdot n^\alpha)^\beta}{n} = \lim_{n\to\infty} M \cdot c^\beta \cdot K^\beta \cdot \frac{n^{\alpha\beta}}{n} = 0.$$

(2). We use (1) of Theorem 3.1. Given a constant $c > 1$, under our hypotheses one has that

$$\liminf_{n\to\infty} \frac{g(\lfloor c \cdot f(n) \rfloor)}{n} \leq M \cdot c^\beta \cdot K^\beta.$$

Now,

$$M \cdot c^\beta \cdot K^\beta < 1 - \frac{1}{c} \iff M \cdot K^\beta < \frac{c-1}{c^{\beta+1}},$$

and the greatest possible value of the last expression is attained when $c = 1 + \alpha$, namely $\frac{\alpha}{(1+\alpha)^{\beta+1}}$, as one can directly verify.

(3). Fix a constant $c > 0$. For every given n, let $N = n$ and $\nu = \tau(n) = \lfloor c \cdot \sqrt{K/M} \cdot n \rfloor$. By Lemma 2.3, there exists an element $x_n \in \Delta(B)$ such that

$$|A \cap (A + x_n) \cap [1, a_n + b_{\tau(n)}]| \geq \frac{n^2}{a_n + b_{\tau(n)}} - \frac{n}{\tau(n)} \geq$$

$$\geq \frac{n^2}{Kn^2 + Mc^2 \cdot \frac{K}{M} \cdot n^2} - \frac{n}{\lfloor c \cdot \sqrt{\frac{K}{M}} \cdot n \rfloor} = \frac{1}{K} \cdot \left(\frac{1}{1+c^2} - \frac{\sqrt{KM}}{c} \cdot \psi(n) \right)$$

where $\psi(n) = \frac{c\cdot\sqrt{K/M}\cdot n}{\lfloor c\cdot\sqrt{K/M}\cdot n\rfloor} \longrightarrow 1$ as $n \to \infty$. So, the last quantity above is positive for all sufficiently large n if and only if $\sqrt{KM} < \frac{c}{1+c^2}$. Now, it is easily checked that the greatest possible value of the latter expression is $1/2$, which is attained when $c = 1$. This means that if $KM < 1/4$ then there exist elements $x_n \in \Delta(A) \cap \Delta(B)$, and the proof is completed. □

Example 3.9. Let $A = \{\lfloor K \cdot n\sqrt{n}\rfloor\}$ and $B = \{n^3\}$. If $K^2 \cdot M < 4/27$ then $R_k(A) \cap \Delta(B) \neq \emptyset$ for all k. Indeed, we can apply (2) of the theorem above, where $1/\alpha = \beta = 2$.

4. A Variant of Khintchine's Theorem

In this final section we exploit further consequences of Lemma 2.3 and prove a result for a class of zero density sets that resembles Khintchine's Recurrence Theorem.

Let us first introduce some notation. For sets $A \subseteq \mathbb{N}$, we write $\mathfrak{d}(A)_n$ to denote the relative density of A on the interval $[1, n]$, *i.e.*,

$$\mathfrak{d}(A)_n \;=\; \frac{|A \cap [1, n]|}{n}\,.$$

As already pointed out, the limit superior given by the upper asymptotic density is attained along intervals of the form $[1, a_n]$; so one has

$$\overline{d}(A) \;=\; \limsup_{n\to\infty} \mathfrak{d}(A)_{a_n} \;=\; \limsup_{n\to\infty} \frac{n}{a_n}.$$

Theorem 4.1. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers, and assume that*

$$\liminf_{n\to\infty} \frac{a_{n\cdot b_n}}{n^2 \cdot b_n} \;=\; l \;<\; \frac{1}{2}\,.$$

Then there exists a sequence $\langle x_n \mid n \in \mathbb{N}\rangle$ of elements of $\Delta(B)$ such that

$$\limsup_{n\to\infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2}\right) \;\geq\; 1 - 2l \;>\; 0\,.$$

Proof. For every n, let $\sigma(n) = n \cdot b_n$, and apply Lemma 2.3 with $h = 1$ to the sets $A_{\sigma(n)} = \{a_1 < \ldots < a_{\sigma(n)}\}$ and $B_n = \{b_1 < \ldots < b_n\}$. We obtain the existence of an element $x_n \in \Delta(B_n) \subseteq \Delta(B)$ such that:

$$|A \cap (A + x_n) \cap [1, a_{\sigma(n)}]| \;\geq\; |A \cap (A + x_n) \cap [1, a_{\sigma(n)} + b_n]| - b_n \;\geq$$

$$|A_{\sigma(n)} \cap (A_{\sigma(n)} + x_n)| - b_n \;\geq\; \frac{\sigma(n)^2}{a_{\sigma(n)}} \cdot \frac{1 - \frac{a_{\sigma(n)}+b_n}{\sigma(n)\cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} - b_n\,.$$

By combining, one gets

$$\frac{\mathfrak{d}(A \cap (A+x_n))_{a_{\sigma(n)}}}{(\mathfrak{d}(A)_{a_{\sigma(n)}})^2} \;=\; \frac{|A \cap (A+x_n) \cap [1, a_{\sigma(n)}]|}{\frac{\sigma(n)^2}{a_{\sigma(n)}}} \;\geq\; \frac{1 - \frac{a_{\sigma(n)}+b_n}{\sigma(n)\cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} - \frac{a_{\sigma(n)} \cdot b_n}{\sigma(n)^2}\,.$$

Now notice that:

- $\liminf_{n\to\infty} \frac{a_{\sigma(n)}}{\sigma(n)\cdot n} = \liminf_{n\to\infty} \frac{a_{n\cdot b_n}}{n^2 \cdot b_n} = l\,;$
- $\lim_{n\to\infty} \frac{b_n}{\sigma(n)\cdot n} = \lim_{n\to\infty} \frac{1}{n^2} = 0\,;$
- $\lim_{n\to\infty} \frac{b_n}{a_{\sigma(n)}} = \lim_{n\to\infty} \frac{n\cdot b_n}{a_{n\cdot b_n}} \cdot \frac{1}{n} \leq \lim_{n\to\infty} \frac{1}{n} = 0\,;$
- $\liminf_{n\to\infty} \frac{a_{\sigma(n)}\cdot b_n}{\sigma(n)^2} = \liminf \frac{a_{n\cdot b_n}}{n^2\cdot b_n} = l\,.$

By considering the inequalities proved above, and by passing to the limit superiors as n goes to infinity, we finally get:

$$\limsup_{n\to\infty} \left(\frac{\mathfrak{d}(A \cap (A+x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) \;\geq\; \limsup_{n\to\infty} \left(\frac{\mathfrak{d}(A \cap (A+x_n))_{a_{\sigma(n)}}}{(\mathfrak{d}(A)_{a_{\sigma(n)}})^2} \right) \;\geq$$

$$\limsup_{n\to\infty} \left(\frac{1 - \frac{a_{\sigma(n)}}{\sigma(n)\cdot n} - \frac{b_n}{\sigma(n)\cdot n}}{1 + \frac{b_n}{a_{\sigma(n)}}} - \frac{a_{\sigma(n)} \cdot b_n}{\sigma(n)^2} \right) \;=\; 1 - 2l \;>\; 0\,.$$

□

Corollary 4.2. *Let $A = \{a_n\}$ be an infinite set of natural numbers. If $a_n \lll n^{3/2}$ then there exists a sequence of shifts $\langle x_n \mid n \in \mathbb{N}\rangle$ such that*

$$\limsup_{n\to\infty} \left(\frac{\frac{|A\cap(A+x_n)\cap[1,n]|}{n}}{\left(\frac{|A\cap[1,n]|}{n}\right)^2} \right) \;\geq\; 1\,.$$

Proof. Let $B = \mathbb{N}$. Then the previous theorem applies where $l = 0$, and the desired result easily follows. □

Similarly as Corollary 3.5 is derived from Theorem 3.4, one proves the following property as a straight consequence of Theorem 4.1.

Corollary 4.3. *Assume that, for a suitable $\vartheta : \mathbb{N} \to \mathbb{R}^+$, the infinite sets of natural numbers $A = \{a_n\}$ and $B = \{b_n\}$ satisfy*

$$\limsup_{n\to\infty} \frac{a_n}{n \cdot \vartheta(n)} \;=\; l_1 \;<\; \infty \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\vartheta(n \cdot b_n)}{n} \;=\; l_2 \;<\; \infty$$

where $l_1 \cdot l_2 < 1/2$. Then there exists a sequence $\langle x_n \mid n \in \mathbb{N}\rangle$ of elements of $\Delta(B)$ such that

$$\limsup_{n\to\infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) \;\geq\; 1 - 2\, l_1 l_2 \;>\; 0\,.$$

Proof. Theorem 4.1 applies, since

$$\liminf_{n\to\infty} \frac{a_{n\cdot b_n}}{n^2 \cdot b_n} \;\leq\; \limsup_{n\to\infty} \frac{a_{n\cdot b_n}}{n \cdot b_n \cdot \vartheta(n \cdot b_n)} \cdot \liminf_{n\to\infty} \frac{\vartheta(n \cdot b_n)}{n} \;\leq\; l_1\, l_2 \;<\; \frac{1}{2}\,.$$

□

To illustrate the use of the above corollary, let us see a property that holds for all sets $A = \{a_n\}$ having the same asymptotic size as the set of primes.

Proposition 4.4. *Let $A = \{a_n\}$ and $B = \{b_n\}$ be infinite sets of natural numbers such that*

$$\lim_{n\to\infty} \frac{a_n}{n \log n} = \; 1 \quad \textit{and} \quad \liminf_{n\to\infty} \frac{\log b_n}{n} = 0.$$

Then for every $\varepsilon > 0$ there exist infinitely many n and elements $x_n \in \Delta(B)$ such that

$$|A \cap (A + x_n) \cap [1, a_n]| \;\geq\; \frac{n}{\log n} \cdot (1 - \varepsilon)\,.$$

Proof. Let $\vartheta(n) = \log n$. By the hypotheses,

$$\lim_{n\to\infty} \frac{a_n}{n \cdot \vartheta(n)} \;=\; 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\vartheta(n \cdot b_n)}{n} \;=\; \lim_{n\to\infty} \frac{\log n + \log b_n}{n} \;=\; 0\,.$$

So, the previous corollary applies, and we get the existence of elements $x_n \in \Delta(B)$ such that

$$\limsup_{n\to\infty} \left(\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \right) \;\geq\; 1\,.$$

Now notice that

$$\frac{\mathfrak{d}(A \cap (A + x_n))_{a_n}}{(\mathfrak{d}(A)_{a_n})^2} \;=\; |A \cap (A + x_n) \cap [1, a_n]| \cdot \frac{a_n}{n^2}\,.$$

So, for every $\delta > 0$, there exist infinitely many n that satisfy

$$|A \cap (A + x_n) \cap [1, a_n]| \cdot \frac{a_n}{n^2} \;\geq\; 1 - \delta\,.$$

By our hypothesis on $\{a_n\}$, we know that $\frac{n \cdot \log n}{a_n} \geq 1 - \delta$ for all sufficiently large n, and so we can conclude that there exist infinitely many n and elements $x_n \in \Delta(B)$ such that:

$$|A \cap (A + x_n) \cap [1, a_n]| \ \geq \ \frac{n^2}{a_n} \cdot (1 - \delta) \ = \ \frac{n}{\log n} \cdot \frac{n \log n}{a_n} \cdot (1 - \delta) \ \geq \ \frac{n}{\log n} \cdot (1 - \delta)^2 .$$

The proof is completed by choosing δ in such a way that $(1 - \delta)^2 \geq 1 - \varepsilon$. □

Example 4.5. Let $P = \{p_n\}$ be the set of prime numbers. Then, for any given $\varepsilon > 0$, there exist arbitrarily large n such that one finds "nearly" $(n/\log n)$-many pairs of primes $p, p' \leq p_n$ which have a common distance $p - p' = d$. Moreover, such a distance d can be taken to belong to any prescribed set of distances $\Delta(B)$, provided $B = \{b_n\}$ is not too sparse in the precise sense that $\log b_n \lll n$ (*e.g.*, one can take $b_n = \lfloor 10^{\frac{n}{\log n}} \rfloor$).

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