PACKING POLYNOMIALS ON SECTORS OF $\mathbb{R}^2$

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Abstract

If S is a region in the plane and I its set of lattice points, we say that a polynomial $P(x, y)$ is a packing polynomial on S if when we restrict $P(x, y)$ to I, the resulting map is a bijection to $\mathbb{N}$. In this paper we give a necessary condition for the existence of quadratic packing polynomials on rational sectors, and determine all quadratic packing polynomials on integral sectors.

1. Introduction

Let S be any region in $\mathbb{R}^2$, and I the set of lattice points contained in S. We call a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$ a **packing polynomial** if its restriction to I gives a bijection to $\mathbb{N}$. Fueter and Pólya [1] showed the following result when S is the first quadrant of $\mathbb{R}^2$.

Theorem 1 (Fueter and Pólya). *Let $S = \{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0\}$. Then the only quadratic packing polynomials on S are:*

$$f(x,y) = \frac{(x+y)^2}{2} + \frac{x+3y}{2}$$
$$g(x,y) = \frac{(x+y)^2}{2} + \frac{3x+y}{2}.$$

Vsemirnov gave two elementary proofs of this result [4]. Though Lew and Rosenberg [2] showed that there are no cubic or quartic packing polynomials on S, it is still an open problem whether or not higher degree packing polynomials exist.

In this paper we will be concerned with the existence of quadratic packing polynomials on a certain family of regions in $\mathbb{R}^2$. In particular, for each $\alpha \in \mathbb{R}_{\geq 0}$ we define $S(\alpha)$ to be the convex hull of the rays spanned by $(1, 0)$ and $(1, \alpha)$, and $I(\alpha)$ the set of lattice points contained in $S(\alpha)$. Note that for S as defined in Fueter and Pólya's theorem, $S = S(\infty)$.

It is already known [3] that quadratic packing polynomials exist when $\alpha = n \in \mathbb{N}$. They can be constructed by defining $J_m = \{(m, y) \mid y \in \mathbb{N}, y \leq mn\}$, and then counting the lattice points for each successive J_m upwards from $(m, 0)$ to (m, mn). This ordering gives us the following quadratic polynomial:

$$f_n(x, y) = \sum_{m=0}^{x-1} |J_m| + y = \sum_{m=0}^{x-1} (mn + 1) + y = (n/2)x^2 + (1 - n/2)x + y.$$

The second quadratic packing polynomial is obtained by instead counting each J_m from top to bottom, which gives:

$$g_n(x, y) = \sum_{m=0}^{x-1} |J_m| + nx - y = \sum_{m=0}^{x-1} (mn + 1) + nx - y = (n/2)x^2 + (1 + n/2)x - y.$$

We call $S(\alpha)$ a **rational sector** if $\alpha \in \mathbb{Q}$. In this paper we will give a necessary condition for the existence of quadratic packing polynomials on rational sectors, and find all quadratic packing polynomials on $S(n)$ when $n \in \mathbb{N}$.

2. A Necessary Condition

Using a similar method as in Lew and Rosenberg's proof [2] of Fueter and Pólya's theorem, we can determine the necessary form of the homogeneous quadratic part of any quadratic packing polynomial on $S(\alpha)$, for $\alpha \in \mathbb{Q}$. It turns out that for some values of α, this homogeneous quadratic part, denoted $P_2(x, y)$, causes $P(x, y)$ to necessarily take non-integer values at some lattice points, thus implying that no packing polynomials exist for such α.

The following theorem gives the necessary homogeneous quadratic part of any packing polynomial $P(x, y)$ on $S(\alpha)$, and provides a necessary condition on α such that the $P_2(x, y)$ allows $P(x, y)$ to take only integer values on $I(\alpha)$.

Theorem 2. *Let $\frac{n}{m} \geq 1$, with $(n, m) = 1$. Then if $S(n/m)$ has a quadratic packing polynomial, $n \mid (m - 1)^2$. Furthermore, such a polynomial has homogeneous quadratic part:*

$$P_2(x, y) = \frac{n}{2}\left(x - \frac{(m-1)y}{n}\right)^2.$$

Note that we may always assume that $\frac{n}{m} \geq 1$, since if not we bijectively map $I(n/m)$ to $I\left(\frac{n}{m - n\lfloor m/n \rfloor}\right)$ by means of the map:

$$W = \begin{pmatrix} 1 & -\lfloor m/n \rfloor \\ 0 & 1 \end{pmatrix}.$$

Precomposing with W gives a bijection between quadratic packing polynomials on $S(n/m)$ and $S\left(\frac{n}{m-n\lfloor m/n\rfloor}\right)$.

Now we prove the theorem. We start by mapping $S(n/m)$ to $S(\infty)$ by means of the linear transformation:

$$U_{n/m} = \begin{pmatrix} 1 & -m/n \\ 0 & 1 \end{pmatrix}.$$

Note that this map does not necessarily send lattice points to lattice points.

Letting $\hat{I}(n/m)$ denote $U_{n/m}(I(n/m))$, we can see that quadratic packing polynomials on $\hat{I}(n/m)$ are in bijection with quadratic packing polynomials on $I(n/m)$. Clearly $\phi : P \mapsto P \circ U_{n/m}^{-1}$ maps packing polynomials on $I(n/m)$ to those on $\hat{I}(n/m)$, and the inverse map $\phi^{-1} : \hat{P} \mapsto \hat{P} \circ U_{n/m}$ maps packing polynomials on $\hat{I}(n/m)$ to those on $I(n/m)$. We need only check that these maps never send quadratic packing polynomials to linear packing polynomials.

By Lew and Rosenberg's Corollary 3.5 [2, pg. 204], there are no packing polynomials on $I(n/m)$ of degree less than 2, and therefore ϕ^{-1} always sends quadratic packing polynomials to quadratic packing polynomials. Thus we must only make sure that if P is a quadratic packing polynomial on $I(n/m)$, then $P \circ U_{n/m}^{-1}$ is quadratic. But this is clearly true, since if this polynomial were linear, then $P = P \circ U_{n/m}^{-1} \circ U_{n/m}$ would also be linear, which is a contradiction.

In light of this bijection, we may find quadratic packing polynomials on $I(n/m)$ by instead looking for quadratic packing polynomials on $\hat{I}(n/m)$.

Let $\hat{P}(x,y) = (a/2)x^2 + bxy + (c/2)y^2 + dx + ey + f$ be a quadratic packing polynomial on $\hat{I}(n/m)$ with homogeneous quadratic part denoted by $\hat{P}_2(x,y)$.

Proposition 1. *The coefficients of $\hat{P}(x,y)$ are rational.*

Proof. We see that a, c, d, and f are rational by simple calculation:

$$\begin{aligned} a &= \hat{P}(2,0) - 2\hat{P}(1,0) + \hat{P}(0,0) \in \mathbb{Z} \\ c &= (1/n^2)(\hat{P}(0,2n) - 2\hat{P}(0,n) + \hat{P}(0,0)) \in \frac{1}{n^2}\mathbb{Z} \\ d &= (-1/2)\hat{P}(2,0) + 2\hat{P}(1,0) - (3/2)\hat{P}(0,0) \in \frac{1}{2}\mathbb{Z} \\ f &= \hat{P}(0,0) \in \mathbb{Z}. \end{aligned}$$

Since $e = (1/n)(\hat{P}(0,n) - f - c/2)$, we also have that $e \in \mathbb{Q}$. Finally, we see that $b \in \mathbb{Q}$, since $\hat{P}(1,n) = a/2 + bn + (c/2)n^2 + d + en + f$. □

Knowing that these coefficients are rational, we may adapt some of the notation and methods that Lew and Rosenberg use in their proof of Fueter and Pólya's theorem [2], and prove the following analogous result.

Lemma 1. *Let $\hat{P}_2(x,y) = (a/2)x^2 + bxy + (c/2)y^2$ be the homogeneous quadratic part of a quadratic packing polynomial on $\hat{I}(n/m)$. Then the coefficients of $\hat{P}_2(x,y)$ satisfy $b = 1 = ac$.*

Proof of Lemma. Lew and Rosenberg's Lemma 4.1 [2, pg. 205] tells us that

$$\begin{aligned} a &> 0 \\ c &> 0 \\ b &> -\sqrt{ac} \end{aligned}$$

holds when P gives an injection from $I(\infty)$ to $\mathbb{N}$, a function known as a storing polynomial.

Using their result, we can show that these inequalities must also hold for any quadratic packing polynomial on $\hat{I}(n/m)$. Suppose $\hat{P}(x,y) = (a/2)x^2 + bxy + (c/2)y^2 + dx + ey + f$ were such a polynomial. Let W denote the linear transformation:

$$W = \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix}.$$

Clearly if $(x,y) \in \mathbb{N}^2$, $W(x,y) = (x + my - mny/n, ny) \in \hat{I}(n/m)$. Then $\hat{P} \circ W(x,y) = (a/2)x^2 + nbxy + (cn^2/2)y^2 + dx + ey + f$ gives a storing function on $I(\infty)$, and thus by Lew and Rosenberg's result, we have:

$$\begin{aligned} a &> 0 \\ n^2c &> 0 \\ bn &> -\sqrt{n^2ac}. \end{aligned}$$

Since $n > 0$, we obtain the desired result.

Following their method, we let $\gamma = \frac{b}{\sqrt{ac}}$, $D(\hat{P},k) = \{(x,y) \in S(\infty) \mid \hat{P}_2(x,y) \leq k\}$, and $A(\hat{P},k)$ denote the area of $D(\hat{P},k)$. We define the density of $\hat{P}$ to be:

$$\hat{I}(n/m) \div \hat{P} = \lim_{l\to\infty} (1/l)[\#(\hat{I}(n/m) \cap \hat{P}^{-1}([0,l]))]$$

In order for $\hat{P}$ to be a packing polynomial we must have that $\hat{I}(n/m) \div \hat{P} = 1$, in which case we say that $\hat{P}$ has unit density. By Lew and Rosenberg's argument [2, pg. 207], $\frac{A(\hat{P},k)}{k}$ is independent of choice of k. Furthermore this ratio equals $\hat{I}(n/m) \div \hat{P}$, a result that they derive by showing that the number of points in $\hat{I}(n/m) \cap \hat{P}([0,l])$ has asymptotic behavior similar to $A(\hat{P},l)$. Thus we have that $1 = \hat{I}(n/m) \div \hat{P} = A(\hat{P},1)$.

Because the coefficients of $\hat{P}_2(x,y)$ are rational, we have that $\gamma = 1$ (otherwise $A(\hat{P},1)$ is transcendental [2, pg. 208]), and therefore we have that $ac = b^2$.

We may thus write:

$$\hat{P}_2(x,y) = \frac{(\sqrt{a}x + \sqrt{c}y)^2}{2}.$$

The region $D(\hat{P}, 1)$ is the triangle with endpoints $(0,0)$, $(\sqrt{2/a}, 0)$, and $(0, \sqrt{2/c})$, which has area $1/\sqrt{ac} = 1/b$. So in order for $\hat{P}(x,y)$ to be a packing polynomial, we need $b = 1$. □

Our polynomial is therefore of the form:

$$\hat{P}(x,y) = \frac{(\sqrt{a}x + \sqrt{c}y)^2}{2} + dx + ey + f$$

for some $a, c, d, e, f \in \mathbb{Q}$, with $ac = 1$.

To pin down the precise values of a and c, the following proposition will be useful.

Proposition 2. *We have*

$$a \in \mathbb{Z} \tag{1}$$

$$c \in \frac{1}{n^2}\mathbb{Z} \tag{2}$$

$$am^2 - amn - 2mn + cn^2 \equiv 0 \pmod{n^2}. \tag{3}$$

Proof. We recall that (1) and (2) were shown in the proof of Proposition 1. To prove (3), define points:

$$A = U_{n/m}(2,2)$$
$$B = U_{n/m}(1,1)$$
$$C = U_{n/m}(2,1)$$
$$D = U_{n/m}(1,0).$$

Note that since we assumed $n/m \geq 1$, we know that all of these points are in $I(n/m)$. We see that $am^2 - amn - 2mn + cn^2 = n^2(P(A) - P(B) - P(C) + P(D) - 1) \in n^2\mathbb{Z}$, which completes the proof. □

We may now determine the values of a and c.

Proposition 3. *The homogeneous quadratic part of $\hat{P}(x,y)$ is $\hat{P}_2(x,y) = (n/2)(x + y/n)^2$.*

Proof. We need to show that $a = n$ and $c = 1/n$. Suppose we have prime p dividing n. Let α and β denote the highest powers of p dividing n and a, respectively. By Lemma 1, $c = 1/a$. Then by Proposition 2, $a|n^2$ and the following congruence holds:

$$am^2 - amn - 2mn + \frac{n^2}{a} \equiv 0 \pmod{p^{2\alpha}}.$$

Evaluating modulo p^α and multiplying through by a, we get:

$$a^2m^2 \equiv 0 \pmod{p^\alpha}$$

and thus, since n and m are relatively prime, we have that $p^\alpha | a^2$.

Multiplying the original congruence through by a, we obtain:

$$\begin{aligned} 0 &\equiv a^2m^2 - a^2mn - 2amn + n^2 \pmod{p^{2\alpha}} \\ &\equiv a^2m^2 - 2amn \pmod{p^{2\alpha}} \\ &\equiv am(am - 2n) \pmod{p^{2\alpha}}. \end{aligned}$$

Therefore we have that $p^{2\alpha-\beta}|(am-2n)$. Suppose $\beta < \alpha$. Then we have $p^\alpha|(am-2n)$, and thus $p^\alpha|am$, which is a contradiction. Therefore we have that $\beta \geq \alpha$ and thus $p^\alpha|a$.

Now write $a = p^\alpha l$, where $l|\frac{n^2}{p^\alpha}$. The necessary congruence modulo p^α is now:

$$\begin{aligned} 0 &\equiv p^\alpha lm^2 - p^\alpha lmn - 2mn + \frac{n^2}{p^\alpha l} \pmod{p^\alpha} \\ &\equiv \frac{n^2}{p^\alpha l} \pmod{p^\alpha} \end{aligned}$$

and thus $p \nmid l$. Therefore $\beta = \alpha$, and $(a, c) = (n, 1/n)$. □

We are now ready to complete the proof of Theorem 2. Precomposing with $U_{n/m}$, we get that the homogeneous quadratic part of $P(x, y)$ is:

$$P_2(x, y) = \frac{n}{2}\left(x - \frac{(m-1)y}{n}\right)^2.$$

In order to derive the necessary condition, we evaluate Equation 3 at $a = n, c = 1/n$ to get:

$$\begin{aligned} 0 &\equiv nm^2 - mn^2 - 2mn + n \pmod{n^2} \\ &\equiv n(m-1)^2 \pmod{n^2} \end{aligned}$$

and thus $n|(m-1)^2$. This completes the proof.

3. Quadratic Packing Polynomials on $I(\alpha)$ With α an Integer

If $n \in \mathbb{N}$ and $P(x, y)$ is a quadratic packing polynomial on $S(n)$, then by Theorem 2, it must be of the form:

$$P(x, y) = (n/2)x^2 + (d + n/2)x + ey + f.$$

Furthermore, we have:

$$\begin{aligned} d &= P(1, 0) - P(0, 0) - n \in \mathbb{Z} \\ e &= P(1, 1) - P(1, 0) \in \mathbb{Z} \\ f &= P(0, 0) \in \mathbb{N}. \end{aligned}$$

We see that $e \neq 0$, since otherwise $P(1,0) = P(1,1)$. Furthermore, we have a correspondence between quadratic packing polynomials with positive e and those with negative e, given by precomposition with the linear transformation:

$$T_n = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix}.$$

Besides the class of packing polynomials derived in the introduction for integral α, we have the following additional packing polynomials on $S(3)$ and $S(4)$.

Proposition 4. *The following is a packing polynomial on $S(3)$:*

$$P(x,y) = (3/2)x^2 - (7/2)x + 3y + 2.$$

Proof. It is easy to check the following residue properties:

$x \pmod 3$	$P(x,y) \pmod 3$
0	2
1	0
2	1

.

We can also check that:

$$P(0,0) = 2$$
$$P(1,0) = 0$$
$$P(2,0) = 1.$$

Therefore, in order to prove that $P(x,y)$ is a packing polynomial, we only need that $P(m,3m)+3 = P(m+3,0)$ for all $m \in \mathbb{N}$. This holds by easy calculation. □

Corollary 1. *The following is a packing polynomial on $S(3)$:*

$$Q(x,y) = (3/2)x^2 + (11/2)x - 3y + 2.$$

Proof. This holds by precomposition with T_3. □

Proposition 5. *The following is a packing polynomial on $S(4)$:*

$$P(x,y) = 2x^2 - 3x + 2y + 1.$$

Proof. We have the following residue properties:

$x \pmod 2$	$P(x,y) \pmod 2$
0	1
1	0

.

We also see that:

$$P(0,0) = 1$$
$$P(1,0) = 0.$$

Therefore, in order to show that $P(x, y)$ is a packing polynomial, we need only show that $P(m, 4m)+2 = P(m+2, 0)$ for all $m \in \mathbb{N}$. This holds by straightforward calculation. □

Corollary 2. *The following is a packing polynomial on $S(4)$:*

$$Q(x, y) = 2x^2 + x - 2y + 1.$$

Proof. This holds by precomposition by T_4. □

The next theorem shows that the above polynomials, along with those described in the introduction, are the only quadratic packing polynomials on integral sectors.

Theorem 3. *Let n be a positive integer. If $n \notin \{3, 4\}$, then there are exactly two quadratic packing polynomials on $S(n)$. When $n \in \{3, 4\}$, there are four.*

Proof. We may assume that $n > 1$, since when $n = 1$, quadratic packing polynomials are in bijection with those on $S(\infty)$ by precomposition with U_1, see [3]. We may also assume that $e > 0$, by the correspondence mentioned earlier.

We now prove the following useful property regarding the possible primes dividing e.

Proposition 6. *If prime $p|e$, then $p|n$.*

Proof. Write $e = me'$ such that $(m, n) = 1$, and any prime dividing e' also divides n.

Clearly $(d, e') = 1$, otherwise we have some prime p which divides n, e, and d, and thus for any $(x, y) \in I(n)$:

$$\begin{aligned} P(x, y) &\equiv \frac{n(x^2 + x)}{2} + dx + ey + f \pmod p \\ &\equiv f \pmod p \end{aligned}$$

implying that $P|_{I(n)}$ cannot be surjective.

By a straightforward calculation, we can see that for any $k \in \mathbb{N}$:

$$P(x, k) = P(x + e', 0)$$

when $x = \frac{-2d+2mk-n-e'n}{2n}$. It is easy to check that $x \in \mathbb{Z}$ for the appropriate choice of k modulo n. In particular, we let:

$$k \equiv dm^{-1} \pmod n$$

if n odd, or

$$k \equiv m^{-1}\left(d + \frac{n+ne'}{2}\right) \pmod n$$

if n is even.

For large enough k, $x > 0$ and hence the second point is in $I(n)$. We therefore contradict injectivity unless the first point is not in $I(n)$, in which case we must have $k > nx$ and thus $2(m-1)k < 2d + n + e'n$. This can only hold for arbitrarily large k if $m = 1$, and thus $e = e'$. □

Corollary 3. *(d,e) = 1.*

Proof. We showed in the proof of Proposition 6 that $(d, e') = 1$ and $e = e'$. □

We now limit the possible values of e.

Proposition 7. *If* $e > 0$*, then* $e = 1, 2,$ *or* 3.

Proof. Suppose $e \geq 4$. Then the values $0, 1, 2,$ and 3 must occur along the x-axis, otherwise choosing a point with one smaller y-coordinate gives a point in $I(n)$ on which $P(x, y)$ is negative.

Let $P(x)$ denote $P(x, 0) = (n/2)x^2 + (d + n/2)x + f$, and let $\beta = \frac{-n-2d}{2n}$ be the point in $\mathbb{R}$ on which $P(x)$ achieves its minimum. We see that $\beta > 0$, otherwise we must have:

$$\begin{aligned}
P(0) &= f = 0 \\
P(1) &= n + d + f = 1 \\
P(2) &= 3n + 2d + f = 2,
\end{aligned}$$

since $P(x)$ is strictly increasing on $\mathbb{N}$. This can only be satisfied when $n = 0$, which is a contradiction.

Since $\beta > 0$, we have that $\lfloor \beta \rfloor \geq 0$, and thus for all $x \in \mathbb{Z}$,

$$P(x) \geq \min\{P(\lfloor \beta \rfloor), P(\lceil \beta \rceil)\} \geq 0.$$

We see that $\beta \notin \mathbb{Z}$, since otherwise we have that $\beta \geq 1$ and $P(\beta + 1) = P(\beta - 1)$, contradicting the injectivity of $P : \mathbb{N} \to \mathbb{N}$.

Expanding the domain, we have that $P : \mathbb{Z} \to \mathbb{N}$ achieves values $0, 1, 2,$ and 3.

Proposition 8. $P : \mathbb{Z} \to \mathbb{N}$ *is injective.*

Proof. Suppose $x, y \in \mathbb{Z}$ are distinct elements such that $P(x) = P(y)$. Then $y = 2\beta - x$, and thus $2\beta \in \mathbb{Z}$. Since $\beta \notin \mathbb{Z}$, this must be an odd integer. But $2\beta = -1 - \frac{2d}{n}$, and thus we have that $\frac{2d}{n}$ is an even integer, and therefore $n | d$.

Since $e \geq 4$, there is some prime $p | e$. By Proposition 6, $p | n$ and thus $p | d$. But $(e, d) = 1$, so we have a contradiction. □

We may assume without loss of generality that:

$$P(0) = f = 0$$
$$P(1) = n + d + f = 1$$

by possibly precomposing with a translation or reflection. Therefore we have:

$$P(x) = (n/2)x^2 + (1 - n/2)x.$$

Clearly the next smallest values of $P(x)$ must occur when $x = -1, 2$, and thus we have:

$$P(-1) = n - 1 = 2$$
$$P(2) = 2 + n = 3.$$

These cannot both hold, and thus we have arrived at a contradiction. □

Now we evaluate the quadratic packing polynomials that arise for each possible value of e.

`Case 1:` $e = 1$.

We have that $P(x, y)$ is of the form:

$$P(x, y) = (n/2)x^2 + (d + n/2)x + y + f.$$

When we evaluate $P(x, y)$ along each J_m, we obtain a sequence of consecutive numbers. Therefore if we have distinct $m_0, m_1 \in \mathbb{N}$ and $(x_0, y_0) \in J_{m_0}$, $(x_1, y_1) \in J_{m_1}$ such that $P(x_0, y_0) < P(x_1, y_1)$, then by injectivity of P, we have that $P(x, y) < P(x', y')$ for all $(x, y) \in J_{n_0}$ and $(x', y') \in J_{n_1}$. We denote this condition by $J_{n_0} < J_{n_1}$.

A simple calculation shows that for large enough m, $P(m+1, n(m+1)) > P(m, 0)$, and therefore $J_m < J_{m+1}$. In order for $P(x, y)$ to be surjective, we need that $P(m, nm) + 1 = P(m + 1, 0)$ for large m. This can only happen when $d = 1 - n$, and thus we have:

$$P(x, y) = \frac{n}{2}x^2 + (1 - n/2)x + y + f.$$

The minimum of $P(x, 0)$ on $\mathbb{R}$ occurs at $\beta = \frac{n-2}{2n}$. Since $0 \leq \beta \leq 1$, we have that either $P(0, 0) = 0$ or $P(1, 0) = 0$. But $P(0, 0) = f$ and $P(1, 0) = 1 + f$, and thus the minimum on $I(n)$ occurs at $(0, 0)$. Therefore $f = 0$, and our final polynomial is:

$$P(x, y) = \frac{n}{2}x^2 + (1 - n/2) + y.$$

`Case 2:` $e = 2$.

$P(x, y)$ is of the form:

$$P(x,y) = \frac{n}{2}x^2 + (d + n/2)x + 2y + f.$$

By Proposition 6 we have that n is even, and by Corollary 3 d is odd.

We see that:

$$\begin{aligned} P(x,y) - f &\equiv (n/2)x^2 + (d + n/2)x \pmod 2 \\ &\equiv (n/2)(x^2 + x) + dx \pmod 2 \\ &\equiv x \pmod 2 \end{aligned}$$

and thus $P(x, y) \equiv f \pmod 2$ if and only if $2|x$.

Evaluating $P(x, y)$ along each J_m, we see that the values form a sequence of consecutive numbers equivalent modulo 2. Therefore we have that $J_m < J_{m'}$ or $J_m > J_{m'}$ if $m \equiv m' \pmod 2$.

As before, we have that for large enough m, $P(m, 0) < P(m + 2, n(m + 2))$ and thus $J_m < J_{m+2}$. In order for $P(x, y)$ to be surjective, we need that $P(m, nm)+2 = P(m + 2, 0)$ for large enough m. This can only happen when $d = 1 - \frac{3n}{2}$, and thus we have:

$$P(x, y) = (n/2)x^2 + (1 - n)x + 2y + f.$$

The minimum of $P(x, 0)$ occurs when $x = \beta = \frac{n-1}{n}$, and thus $P(x, 0) = 0$ for either $x = 0$ or 1. $P(0, 0) = f$ and $P(1, 0) = 1 - \frac{n}{2} + f$. Since $2|n$, we have that $\frac{n}{2} \geq 1$, and thus $P(1, 0) \leq f$. Therefore we have that $P(1, 0) = 0$, $f = \frac{n}{2} - 1$, and:

$$P(x, y) = (n/2)x^2 + (1 - n)x + 2y + \frac{n}{2} - 1.$$

The next smallest value of $P(x, y)$ occurs at $(0, 0)$, and thus we have that $\frac{n}{2} - 1 = 1$ and $n = 4$. Our final polynomial on $I(4)$ is therefore:

$$P(x, y) = 2x^2 - 3x + 2y + 1.$$

`Case 3:` $e = 3$.

$P(x, y)$ is of the form:

$$P(x,y) = \frac{n}{2}x^2 + (d + n/2)x + 3y + f,$$

with $3|n$ and $3 \nmid d$, by Proposition 6 and Corollary 3.

We see that $P(x, y) \equiv f \pmod 3$ if and only if $3|x$:

$$\begin{aligned} P(x,y) - f &\equiv (n/2)x^2 + (d + n/2)x + 3y \pmod 3 \\ &\equiv (n/2)(x^2 + x) + dx \pmod 3 \\ &\equiv dx \pmod 3. \end{aligned}$$

Since evaluating $P(x, y)$ along each column gives a sequence of consecutive integers module 3, for distinct $m, m' \in \mathbb{N}$ we have $J_{3m} < J_{3m'}$ or $J_{3m} > J_{3m'}$. A short calculation shows that for large enough m, $P(3m, 0) < P(3(m+1), 3(m+1)n)$, and thus $J_{3m} < J_{3(m+1)}$.

In order for $P(x, y)$ to cover all positive integers congruent to f modulo 3, we need that for large m, $P(3m, 3mn) + 3 = P(3(m + 1), 0)$. This can only happen when $d = 1 - 2n$, and thus we have:

$$P(x, y) = (n/2)x^2 + (1 - (3/2)n)x + 3y + f.$$

The minimum value of $P(x, 0)$ on $\mathbb{R}$ occurs at $\beta = 3/2 - 1/n$. Since $3|n$, we have that $1/n < 1/2$, and thus $1 < \beta < 2$. Therefore the minimum of $P(x, y)$ on $I(n)$ must occur at $(1, 0)$ or $(2, 0)$. We see that:

$$P(1, 0) = 1 - n + f$$
$$P(2, 0) = 2 - n + f$$

and thus $P(1, 0) = 0$ and $P(2, 0) = 1$. We have $f = n - 1$, and our polynomial is:

$$P(x, y) = (n/2)x^2 + (1 - (3/2)n)x + 3y + n - 1.$$

We know that $P(x, y)$ must take the value 2 along the x-axis. The next smallest value of $P(x, 0)$ with $x \in \mathbb{N}$ clearly occurs when $x = 0$, and thus we have $P(0, 0) = n - 1 = 2$, and $n = 3$. Therefore our final packing polynomial is:

$$P(x, y) = (3/2)x^2 - (7/2)x + 3y + 2$$

on $S(3)$.

This completes the proof. □

4. Future Directions

So far I have only found sectors for which there are an even number of quadratic packing polynomials. It seems that there may be an odd number for $\alpha = 9/4$ or $9/7$, but so far I do not have a quick method for determining the linear part of a quadratic packing polynomials for non-integral α, and thus the necessary casework is tedious.

It is also unclear whether the necessary condition described in this paper is also sufficient. There are quadratic packing polynomials for $\alpha = 4/3, 9/4$ and $9/7$, and I have yet to find $n, m \in \mathbb{N}$ relatively prime with $n|(m - 1)^2$ such that there are no quadratic packing polynomials on $S(n/m)$.

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