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ON DAVID'S 2-DIMENSIONAL CONTINUED FRACTION

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Abstract

In 1949 and 1950 two papers by M. David on modifications of the Jacobi algorithm appeared. Although the papers claim that it is easy to state sufficient conditions for the non-periodicity of cubic irrational numbers, the papers got just passing notice in Brentjes' book. Since the problem of periodicity of cubic irrationals is still open for the Jacobi algorithm, it seemed worthwhile to take a closer look at David's papers. It turned out that the result on cubic field with complex conjugates is correct (see Theorem 1) but the result on totally real cubic fields is not correct. Only a considerably weaker result is true (Theorem 2).

1. Introduction

We first describe one of David's algorithms ([3], [4]) in the framework of Schweiger's book ([7]). Let $x_1, x_2 \in \mathbb{R}$ such that $0 < x_1 \leq 1$, $0 \leq x_2 \leq 1$. Then we define

$$d + 1 = \lceil \frac{1}{x_1} \rceil, \; a = \lfloor \frac{x_2}{x_1} \rfloor$$

and the map T by

$$T(x_1, x_2) = (y_1, y_2) = \big(\frac{x_2}{x_1} - a, -\frac{1}{x_1} + d + 1\big)$$

which is equivalent to

$$\begin{pmatrix} 1 \\ y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -a & 1 \\ -1 & d+1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \end{pmatrix}.$$

Then we find $0 \leq a \leq d$, $1 \leq d$, and the equation $a = d$ implies $y_1 + y_2 \leq 1$. The inverse branches of the algorithm are given by the matrices

$$\beta(a, d) = \begin{pmatrix} d+1 & 0 & -1 \\ 1 & 0 & 0 \\ a & 1 & 0 \end{pmatrix} = \begin{pmatrix} B_0^{(1)} & -B_0^{(-1)} & -B_0^{(0)} \\ B_1^{(1)} & -B_1^{(-1)} & -B_1^{(0)} \\ B_2^{(1)} & -B_2^{(-1)} & -B_2^{(0)} \end{pmatrix}.$$

Matrix multiplication gives the recursion relations

$$B_j^{(n)} = (d_n+1)B_j^{(n-1)} - a_n B_j^{(n-2)} - B_j^{(n-3)},$$

$$a_n = a(T^{n-1}(x_1,x_2)), d_n = d(T^{n-1}(x_1,x_2)),\ j=0,1,2,\ n \geq 2.$$

We first state a useful proposition.

Lemma 1. *Let*

$$\prod_{j=1}^{n} \beta(a_j,d_j) = \begin{pmatrix} B_0^{(n)} & -B_0^{(n-2)} & -B_0^{(n-1)} \\ B_1^{(n)} & -B_1^{(n-2)} & -B_1^{(n-1)} \\ B_2^{(n)} & -B_2^{(n-2)} & -B_2^{(n-1)} \end{pmatrix}.$$

Then the relations $B_i^{(n)}B_0^{(n-1)} - B_i^{(n-1)}B_0^{(n)} \geq 0$ *and* $B_i^{(n)}B_0^{(n-2)} - B_i^{(n-2)}B_0^{(n)} \geq 0$ *hold for* $i=1,2$ *and* $n \geq 1$.

Proof. The proof follows from induction. □

Lemma 2. *The approximations satisfy the inequalities*

$$\frac{B_1^{(n)}}{B_0^{(n)}} \leq \frac{B_1^{(n+1)}}{B_0^{(n+1)}} \leq x_1,\ \frac{B_2^{(n)}}{B_0^{(n)}} \leq \frac{B_2^{(n+1)}}{B_0^{(n+1)}} \leq x_2.$$

Proof. We prove this easy lemma for $i=1$. Then

$$\frac{B_1^{(n+1)}}{B_0^{(n+1)}} - \frac{B_1^{(n)}}{B_0^{(n)}} = \frac{B_1^{(n+1)}B_0^{(n)} - B_1^{(n)}B_0^{(n+1)}}{B_0^{(n+1)}B_0^{(n)}} \geq 0.$$

Furthermore we find

$$x_1 - \frac{B_1^{(n)}}{B_0^{(n)}} = \frac{B_1^{(n)} - B_1^{(n-2)}y_1 - B_1^{(n-1)}y_2}{B_0^{(n)} - B_0^{(n-2)}y_1 - B_0^{(n-1)}y_2} - \frac{B_1^{(n)}}{B_0^{(n)}}$$

$$= \frac{(B_1^{(n)}B_0^{(n-2)} - B_0^{(n)}B_1^{(n-2)})y_1 + (B_0^{(n-1)}B_1^{(n)} - B_1^{(n-1)}B_0^{(n)})y_2}{B_0^{(n)}(B_0^{(n)} - B_0^{(n-2)}y_1 - B_0^{(n-1)}y_2)} \geq 0.$$

□

Assume that the algorithm is purely periodic: $T^p(x_1,x_2) = (x_1,x_2)$, say. Then $\phi(\lambda)$ denotes the characteristic polynomial of the periodicity matrix

$$\pi = \begin{pmatrix} B_0^{(p)} & -B_0^{(p-2)} & -B_0^{(p-1)} \\ B_1^{(p)} & -B_1^{(p-2)} & -B_1^{(p-1)} \\ B_2^{(p)} & -B_2^{(p-2)} & -B_2^{(p-1)} \end{pmatrix}.$$

As we will see David's claim that all eigenvalues are real is correct (Theorem 1). We denote by λ, λ' and λ'' the three eigenvalues ordered as $\lambda'' < \lambda' < \lambda$.

David claims that x_1 and x_2 are linearly independent numbers of a cubic field if the algorithm becomes periodic. This assertion is not true. Immediate counterexamples are given by the periodic algorithms

$$(x_1, x_2) = \left(\overline{\begin{matrix} d-1 \\ d \end{matrix}} \right).$$

The characteristic polynomial of $\beta(d-1, d)$ is $\phi(\lambda) = \lambda^3 - (d+1)\lambda^2 + (d-1)\lambda + 1$, which shows $\phi(1) = 0$. If we take $d = 1$ we also see that the algorithm is not convergent in all cases. The two points (g, g^2) and $(1, 1)$ which correspond to the eigenvalues $\lambda = G$ and $\lambda' = 1$ are invariant (here $G > 1$ satisfies $G^2 = G + 1$ and $Gg = 1$). Therefore the whole segment between (g, g^2) and $(1, 1)$ is invariant. Note, if $d \geq 2$ then the point $(1, d)$ lies outside the domain of T and therefore convergence is not affected. We also remark that $\lambda' = 1$ occurs in more complex situations. One example is the periodic algorithm

$$(x_1, x_2) = \left(\overline{\begin{matrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{matrix}} \right).$$

Its characteristic polynomial is given as $\phi(\lambda) = \lambda^3 - 11\lambda^2 + 11\lambda - 1$.

2. Cubic Numbers With Complex Conjugates

We consider the differences $D_i^{(n)} = B_0^{(n)} x_i - B_i^{(n)}$, $i = 1, 2$. The recursion relations translate into

$$D_i^{(n)} = (d_n + 1) D_i^{(n-1)} - a_n D_i^{(n-2)} - D_i^{(n-3)}.$$

On the other hand, we find for $(y_1, y_2) = T^{n-1}(x_1, x_2)$ the recursion

$$D_i^{(n)} = y_2 D_i^{(n-1)} + y_1 D_i^{(n-2)}.$$

This relation follows from induction replacing y_1 and y_2 by

$$y_1 = \frac{1}{d+1-z_2},\ y_2 = \frac{a+z_1}{d+1-z_2}$$

where $(z_1, z_2) = T(y_1, y_2) = T^n(x_1, x_2)$. For $n = 1$ this recursion amounts to

$$(d+1)x_1 - 1 = y_2 x_1,\ (d+1)x_2 - a = y_2 x_2 + y_1.$$

These relations are equivalent to the definition of the map T.

Following David's ideas we introduce the points W_n in $X_0X_1X_2$-space by

$$W_n = (B_0^{(n)}, B_1^{(n)}, B_2^{(n)}).$$

In ([4]) he considers the projections of the points W_n onto the X_1X_2-plane along the direction of $(1, x_1, x_2)$. Then we obtain the points

$$M_n = (-D_1^{(n)}, -D_2^{(n)}) = (-B_0^{(n)}x_i + B_1^{(n)}, -B_0^{(n)}x_i + B_2^{(n)}).$$

If the algorithm is periodic then the growth rate of $D_1^{(n)}$ and $D_2^{(n)}$ is governed by the second eigenvalue λ'. If $|\lambda'| < 1$ then $\lim_{n\to\infty} M_n = (0,0)$. If $|\lambda'| \geq 1$ no convergence occurs. This invalidates David's method in ([4]).

In fact, $\lambda' > 1$ occurs. An example is given by the periodic expansion

$$(x_1, x_2) = \left(\overline{\begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 3 \end{matrix}} \right).$$

The characteristic polynomial is $\phi(\lambda) = \lambda^3 - 7\lambda^2 + 9\lambda + 1$. Here $1 < \lambda' < 2$ and $5 < \lambda < 6$. A calculation gives

$$x_1 = \frac{2\lambda + 3}{3\lambda - 1}, x_2 = \frac{4}{\lambda + 1}.$$

Theorem 1. *If the cubic field K has complex conjugates then no pair $(x_1, x_2) \in K \times K$ has a periodic expansion.*

Proof. Let (x_1, x_2) have the period p. Then (x_1, x_2) is also periodic with period length $2p$. Therefore we can assume that $p \equiv 0 (\bmod 2)$. We write

$$(\alpha_1, \beta_1) = (x_1, x_2), (\alpha_2, \beta_2) = T(x_1, x_2), ..., (\alpha_p, \beta_p) = T^{p-1}(x_1, x_2).$$

If we introduce the auxiliary quantity $E_i^{(p+j)} = D_i^{(p+j+1)}$ the relations

$$D_i^{(p+j)} = \beta_{j+1} D_i^{(p+j-1)} + \alpha_{j+1} D_i^{(p+j-2)}$$

can be rewritten as the system

$$\begin{pmatrix} D_i^{(p+j-1)} \\ E_i^{(p+j-1)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_{j+1} & \beta_{j+1} \end{pmatrix} \begin{pmatrix} D_i^{(p+j-2)} \\ E_i^{(p+j-2)} \end{pmatrix}, 0 \leq j \leq p-1.$$

The matrix product

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_1 & \beta_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \alpha_p & \beta_p \end{pmatrix}$$

has the roots of the equation $X^2-(A+D)X+AD-BC=0$ as Floquet multipliers. Floquet multipliers govern the growth rate of the solutions of a system of homogeneous difference equations. We consider as an example the case $p=2$ and refer to [5] for an exposition of the theory. For simplicity we restrict to pure periodicity.

Let

$$D_i^{(1)} = \beta_1 D_i^{(0)} + \alpha_1 D_i^{(-1)}$$
$$D_i^{(2)} = \beta_2 D_i^{(1)} + \alpha_2 D_i^{(0)}$$

but

$$D_i^{(3)} = \beta_1 D_i^{(2)} + \alpha_1 D_i^{(1)}.$$

Then

$$D_i^{(3)} = \beta_1\beta_2 D_i^{(1)} + \alpha_1 D_i^{(1)} - \beta_1\alpha_2 D_i^{(0)}$$
$$= (\beta_1\beta_2+\alpha_1)D_i^{(1)} + \alpha_2(D_i^{(1)} - \alpha_1)D_i^{(-1)} = (\beta_1\beta_2+\alpha_1+\alpha_2)D_i^{(1)} - \alpha_1\alpha_2 D_i^{(-1)}.$$

Then the characteristic equation $X^2-(\beta_1\beta_2+\alpha_1+\alpha_2)X+\alpha_1\alpha_2=0$ is exactly the characteristic equation of the matrix product

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha_1 & \beta_1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ \alpha_2 & \beta_2 \end{pmatrix}.$$

In the general case these multipliers therefore concide with λ' and λ''. Since $AD-BC=\alpha_1\alpha_2...\alpha_p$ and $A+D \geq \alpha_1\alpha_3...\alpha_{p-1}+\alpha_2\alpha_4...\alpha_p$ we find

$$(A+D)^2-4(AD-BC) \geq (\alpha_1\alpha_3...\alpha_{p-1}-\alpha_2\alpha_4...\alpha_p)^2.$$

Therefore the multipliers are real numbers. $\square$

3. Cubic Numbers With Real Conjugates

We first make some remarks on periodicity. Let $\left(\overline{\begin{matrix} a_1 & a_2 & ... & a_p \\ d_1 & d_2 & ... & d_p \end{matrix}}\right)$ be a periodic admissible sequence. Then by Lemma 2

$$\lim_{g\to\infty}\left(\frac{B_1^{(gp)}}{B_0^{(gp)}}, \frac{B_2^{(gp)}}{B_0^{(gp)}}\right) =: (z_1, z_2)$$

exists. Then we find

$$\begin{pmatrix} B_0^{(p)} & -B_0^{(p-2)} & -B_0^{(p-1)} \\ B_1^{(p)} & -B_1^{(p-2)} & -B_1^{(p-1)} \\ B_2^{(p)} & -B_2^{(p-2)} & -B_2^{(p-1)} \end{pmatrix}\begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix} = \rho \begin{pmatrix} 1 \\ z_1 \\ z_2 \end{pmatrix}$$

for some eigenvalue $\rho \in \{\lambda, \lambda', \lambda''\}$. We introduce

$$\psi_1(t) = \frac{B_1^{(p-1)}t + B_1^{(p)}B_0^{(p-1)} - B_1^{(p-1)}B_0^{(p)}}{B_0^{(p-1)}t - B_1^{(p-1)}B_0^{(p-2)} + B_1^{(p-2)}B_0^{(p-1)}}$$

and

$$\psi_2(t) = \frac{B_2^{(p-2)}t + B_2^{(p)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p)}}{B_0^{(p-2)}t + B_2^{(p-1)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p-1)}}.$$

Lemma 3. We have $(z_1, z_2) = (\psi_1(\lambda), \psi_2(\lambda))$.

Proof. We first note that $B_0^{(gp)} - B_0^{(gp-2)}z_1 - B_0^{(gp-1)}z_2 = \rho^g$. The trace of the periodicity matrices gives the relation

$$B_0^{(gp)} - B_1^{(gp-2)} - B_2^{(gp-1)} = \lambda^g + (\lambda')^g + (\lambda'')^g.$$

Therefore $B_0^{(gp)} = a\lambda^g + ...$ with $a \neq 0$. The multiplication of the periodicity matrices gives

$$\frac{B_0^{((g+1)p)}}{B_0^{(gp)}} = B_0^{(p)} - B_0^{(p-2)}\frac{B_1^{(gp)}}{B_0^{(gp)}} - B_0^{(p-1)}\frac{B_2^{(gp)}}{B_0^{(gp)}}.$$

Letting $g \to \infty$, we obtain $\lambda = B_0^{(p)} - B_0^{(p-2)}z_1 - B_0^{(p-1)}z_2$. Hence, we find that $\rho = \lambda$. $\square$

No result on uniqueness or convergence of this algorithm has been published. The algorithm mentioned in the introduction, $\overline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$, provides a counterexample, namely $\lambda = G$ and $\lambda' = 1$, which gives

$$(g, g^2) = (\psi_1(G), \psi(G)),\ (1, 1) = (\psi_1(1), \psi_2(1)).$$

Probably, this is the only counterexample within purely periodic algorithms. If the periodicity matrix consists of non-negative elements, uniqueness for periodic algorithms would be provided. For the well-known Jacobi-Perron algorithm, the relation

$$A_0^{(pg+3)} + A_0^{(pg+1)}\psi_1(\lambda') + A_0^{(pg+2)}\psi_2(\lambda') = (\lambda')^g$$

is incompatible with $(\psi_1(\lambda'), \psi_2(\lambda')) \in [0,1]^2$ (see [6],[7], and [8]). Therefore we assume the following: if (x_1, x_2) is periodic then $(x_1, x_2) = (\psi_1(\lambda), \psi_2(\lambda))$.

Theorem 2: *Let K be a totally real cubic number field. If the algorithm of a pair $(x_1, x_2) \in K \times K$ has a purely periodic expansion then*

$$(x_1 - x_1')(x_2 - x_2') > 0,\ (x_1 - x_1'')(x_2 - x_2'') < 0.$$

Proof. Using our first lemma we see that $\psi_1(t)$ is a decreasing function with the pole

$$\mu = \frac{B_1^{(p-1)}B_0^{(p-2)} - B_1^{(p-2)}B_0^{(p-1)}}{B_0^{(p-1)}}.$$

But also $\psi_2(t)$ is decreasing. If we calculate its derivative we have to prove

$$B_2^{(p-2)}(B_2^{(p-1)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p-1)}) < B_0^{(p-2)}(B_2^{(p)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p)}).$$

Since

$$\begin{aligned} B_2^{(p)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p)} &= (d_p+1)(B_2^{(p-1)}B_0^{(p-2)} - B_2^{(p-1)}B_0^{(p-2)}) \\ &\qquad +B_2^{(p-2)}B_0^{(p-3)} - B_2^{(p-3)}B_0^{(p-2)} \\ &\geq 2(B_2^{(p-1)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p-1)}) \end{aligned}$$

and $B_0^{(p-2)} \geq B_2^{(p-2)}$, this is true.

We now assume again that $p \equiv 0 (\bmod 2)$. Then the characteristic polynomial has three real roots: $0 < \lambda'' < \lambda' < \lambda$, which satisfy $\lambda\lambda'\lambda'' = 1$. Therefore we obtain

$$0 < x_2 < x_2' < x_2''.$$

We now use the following result about difference equations. If for periodic algorithms the quantities $B_i(np)$ are linear combinations of λ^n, $(\lambda')^n$, and $(\lambda'')^n$ then the quantities $B_i^{(np-1)}B_0^{(np-2)} - B_i^{(np-2)}B_0^{(np-1)}$ (and similar expressions) are linear expressions of the products $(\lambda\lambda')^n$, $(\lambda'\lambda'')^n$, and $(\lambda''\lambda)^n$.

Now let us replace, for a moment, the period length p by np and consider

$$\mu_n = \frac{B_1^{(np-1)}B_0^{(np-2)} - B_1^{(np-2)}B_0^{(np-1)}}{B_0^{(np-1)}}.$$

The periodicity of the algorithm implies $\mu_n = c(\lambda')^n + o(1)$. Therefore, we obtain $(\lambda'')^n < \mu_n$ for n big enough. Then $x_1'' < 0$ and we see that in fact $0 < \lambda'' < \mu$.

Now, we again use the equations

$$B_0^{(p)} - B_0^{(p-2)}x_1 - B_0^{(p-1)}x_2 = \lambda$$

$$B_1^{(p)} - B_1^{(p-2)}x_1 - B_1^{(p-1)}x_2 = \lambda x_1$$

$$B_2^{(p)} - B_2^{(p-2)}x_1 - B_2^{(p-1)}x_2 = \lambda x_2.$$

We use these relations to find the equation for x_1 in the form

$$A_3x^3 + A_2x^2 + A_1x + A_0 = 0.$$

We have

$$A_0 = B_1^{(p-1)}(B_2^{(p)}B_1^{(p-1)} - B_1^{(p)}B_2^{(p-1)}) + B_1^{(p)}(B_1^{(p)}B_0^{(p-1)} - B_1^{(p-1)}B_0^{(p)})$$

$$A_3 = B_0^{(p-1)}(B_2^{(p-1)}B_0^{(p-2)} - B_2^{(p-2)}B_0^{(p-1)}) + B_0^{(p-2)}(B_1^{(p-1)}B_0^{(p-2)} - B_0^{(p-1)}B_1^{(p-2)}).$$

Then the lemma shows that $A_3 > 0$. Furthermore we have the relations

$$\begin{aligned} &B_2^{(n)}B_1^{(n-1)} - B_1^{(n)}B_2^{(n-1)} \\ &\quad = a_{n-1}(B_2^{(n-1)}B_1^{(n-2)} - B_2^{(n-2)}B_1^{(n-1)}) + (B_2^{(n-1)}B_1^{(n-3)} - B_2^{(n-3)}B_1^{(n-1)}) \end{aligned}$$

and

$$\begin{aligned} &B_2^{(n)}B_1^{(n-2)} - B_1^{(n)}B_2^{(n-2)} \\ &= (d_{n-1}+1)(B_2^{(n-1)}B_1^{(n-2)} - B_2^{(n-2)}B_1^{(n-1)}) + (B_2^{(n-2)}B_1^{(n-3)} - B_2^{(n-3)}B_1^{(n-2)}). \end{aligned}$$

Since $B_2^{(2)}B_1^{(1)} - B_1^{(2)}B_2^{(1)} = 1$ and $B_2^{(2)}B_1^{(0)} - B_1^{(2)}B_2^{(0)} = 0$, we obtain $A_0 > 0$. Therefore $x_1x_1'x_1'' < 0$ and $x_1x_1' > 0$. This implies $\mu < \lambda'$ and so $x_1' > x_1$. $\square$

David uses in his first paper ([3]) the vector products $P_n = -W_{n-1} \wedge W_n$. He assumes that the direction of these vectors tends to $(1, x_1', x_2')$ or $(1, x_1'', x_2'')$. However this direction must be an eigenvector of the transposed matrix π^t of the periodicity matrix π.

Remark. Unfortunately, Theorem 2 is of no great value since the property stated in Theorem 2 can be destroyed by the pre-period. Take the pair

$$(x_1, x_2) = \begin{pmatrix} \overline{0 \quad 0} \\ 1 \quad 2 \end{pmatrix} = (\frac{\lambda}{2\lambda - 1}, \frac{1}{\lambda})$$

where $\lambda > 1$ satisfies $\lambda^3 - 6\lambda^2 + 5\lambda - 1 = 0$. Then $\frac{1}{5} < \lambda'' < \lambda' < 1 < \lambda$ and $\lambda \sim 5$. We choose

$$\alpha = \frac{1}{5 - x_2} = \frac{\lambda}{5\lambda - 1}$$

and

$$\beta = \frac{3 + x_1}{5 - x_2} = \frac{-\lambda^2 + 9\lambda - 2}{5\lambda - 1} = \frac{-6\lambda^2 + 29\lambda + 28}{29}.$$

Since $f(t) = \frac{t}{5t-1}$ is decreasing we get $0 < \alpha < \alpha' < \alpha''$. But $\beta < \beta'$ and $\beta < \beta''$ are provided by the inequalities

$$-6\lambda^2 + 29\lambda < -6\lambda'^2 + 29\lambda',\ -6\lambda^2 + 29\lambda < -6\lambda''^2 + 29\lambda''$$

$$29(\lambda - \lambda') < 6(\lambda^2 - \lambda'^2),\ 29(\lambda - \lambda') < 6(\lambda^2 - \lambda''^2)$$
$$29 < 6(\lambda + \lambda'),\ 29 < 6(\lambda + \lambda'').$$

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