ON THE MIKI AND MATIYASEVICH IDENTITIES FOR BERNOULLI NUMBERS

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Abstract

In this paper, we study the well-known Miki and Matiyasevich convolution identities for Bernoulli numbers and deduce analogues of their identities for numbers related to Bernoulli numbers.

1. Introduction

The Bernoulli numbers B_n appear in many areas of mathematics, most notably in number theory, the calculus of finite differences and asymptotic analysis with important applications. They can be defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi). \tag{1.1}$$

One can easily see that $B_{2k+1} = 0$ and $(-1)^{k+1}B_{2k} > 0$ for all $k \geq 1$. Let $\mathbb{Z}_p$ be the ring of p-adic integers with p a prime. The von Staudt-Clausen theorem asserts that $B_n \in \mathbb{Z}_p$ if $p - 1 \nmid n$, and $pB_n \in \mathbb{Z}_p$, more precisely $pB_n \equiv -1 \pmod{p}$ if $p - 1 \mid n$.

Various types of linear and nonlinear recurrence relations for Bernoulli numbers have been studied for a long time. We can find a large number of formulas in the classical books [17, 18] and in [12]. For special type recurrence and reciprocity relations for these numbers, see, e,g., [1, 2, 3, 8, 11].

As one of many convolution identities for Bernoulli numbers, Miki [16] proved in 1978 the following curious identity based on p-adic arguments.

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Proposition 1.1 (Miki, 1978). *For $n \geq 4$,*

$$\sum_{i=2}^{n-2} \frac{B_i}{i} \frac{B_{n-i}}{n-i} - \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i}{i} \frac{B_{n-i}}{n-i} = 2H_n \frac{B_n}{n}, \tag{1.2}$$

where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is the nth harmonic number.

Since $\frac{1}{i(n-i)} = \frac{1}{n}\left(\frac{1}{i} + \frac{1}{n-i}\right)$, we may rewrite (1.2) as the form

$$\sum_{i=2}^{n-2} \frac{B_i B_{n-i}}{i} - \sum_{i=2}^{n-2} \binom{n}{i} \frac{B_i B_{n-i}}{i} = H_n B_n. \tag{1.3}$$

In 2005, Miki's identity was extended by Gessel [10] to the Bernoulli polynomials $B_n(\lambda)$ $(n \geq 0)$ defined by

$$\frac{xe^{\lambda x}}{e^x - 1} = \sum_{n=0}^{\infty} B_n(\lambda) \frac{x^n}{n!} \quad (|x| < 2\pi).$$

Indeed, he proved the following

Proposition 1.2 (Gessel, 2005). *For $n \geq 1$,*

$$\sum_{i=1}^{n-1} \frac{B_i(\lambda) B_{n-i}(\lambda)}{i(n-i)} - \frac{2}{n} \sum_{i=0}^{n-1} \binom{n}{i} \frac{B_i(\lambda) B_{n-i}}{n-i} = 2H_{n-1} \frac{B_n(\lambda)}{n} + B_{n-1}(\lambda). \tag{1.4}$$

Since $B_n(0) = B_n$, identity (1.2) is given as a special case of (1.4) where $\lambda = 0$. Also the case $\lambda = 1/2$ reduces to Faber and Pandharipande's identity shown in [7] (see also [6] observing it from a quite different viewpoint) for the numbers defined by $\overline{B}_n = ((1 - 2^{n-1})/2^{n-1}) B_n$. Indeed, we have

$$\overline{B}_n = \frac{1 - 2^{n-1}}{2^{n-1}} B_n = 2\left(\frac{1}{2}\right)^n B_n - B_n = B_n\left(\frac{1}{2}\right).$$

On the other hand, Matiyasevich [15] discovered the following good companion identity for (1.2) with the aid of computer software system "*Mathematica*":

Proposition 1.3 (Matiyasevich, 1997). *For $n \geq 4$,*

$$(n+2) \sum_{i=2}^{n-2} B_i B_{n-i} - 2 \sum_{i=2}^{n-2} \binom{n+2}{i} B_i B_{n-i} = n(n+1) B_n. \tag{1.5}$$

There are several kinds of proofs of (1.2), (1.4) and (1.5) using some tools from combinatorics, contour integrals, p-adic analysis and others, however we now pay particular attention to Crabb's short and intelligible proof of (1.4) given in [5]. To prove (1.4), he used a certain functional equation related to the generating function of $B_n(\lambda)$ stated above.

This paper is concerned with the Miki and Matiyasevich convolution identities. In Section 2, as a preliminary we first explain an umbral notation and later introduce Euler-type identities and some required functional identities. In Section 3, we rederive (1.2) and (1.5) with elementary and shorter proofs based on essentially the same idea as Crabb's. In Section 4, we deal with analogues of their convolution identities for the numbers $B'_n = (1-2^n)B_n$ defined by the generating function

$$\frac{x}{e^x+1} = \sum_{n=0}^{\infty} B'_n \frac{x^n}{n!} \quad (|x| < \pi). \tag{1.6}$$

We note that these numbers B'_n are closely related to the Genocchi numbers defined by $G_n = 2(1-2^n)B_n = 2B'_n$ $(n \geq 1)$. Therefore, all the results concerning B'_n given below can be expressed in terms of Genocchi numbers.

2. Preliminary

In this section, we prepare some matters which will be needed in the later sections.

Given two sequences $S = \{S_n\}_{n\geq 0}$ and $T = \{T_n\}_{n\geq 0}$ of numbers or functions, we use the following umbral notation (for more details on umbral calculus, see [9, 19, 20, 21]). We now define for any $u, v \in \mathbb{R}$,

$$(uS+vT)^0 = S_0T_0, \quad (uS+vT)^n = \sum_{i=0}^{n} \binom{n}{i} u^i v^{n-i} S_i T_{n-i} \quad (n \geq 1).$$

In other words, we expand $(uS+vT)^n$ in full by means of the binomial theorem and replace S^i and T^i by S_i and T_i $(i = 0, 1, ..., n)$, respectively. For example, we may write the most basic identity which is usually attributed to Euler as, considering the sequence $B = \{B_n\}_{n\geq 0}$,

$$(B+B)^0 = (B_0)^2 = 1, \quad (B+B)^n = (1-n)B_n - nB_{n-1} \quad (n \geq 1). \tag{2.1}$$

If $S(x)$ and $T(x)$ are the exponential generating functions of the sequences S and T, respectively, then we have

$$S(ux)T(vx) = \sum_{n=0}^{\infty} (uS+vT)^n \frac{x^n}{n!},$$

and hence the following derivative expression can be given:

$$(uS+vT)^n = \left[\frac{d^n}{dx^n} S(ux)T(vx)\right]_{x=0}.$$

As we mentioned above, we use the notation $B'_n = (1-2^n)B_n$ $(n \geq 0)$. It is easy to construct the following functional equations from the generating functions (1.1) and (1.6) of B_n and B'_n, respectively:

$$\frac{x}{e^x-1}\cdot\frac{x}{e^x+1} = \frac{x}{2}\left(\frac{x}{e^x-1}-\frac{x}{e^x+1}\right) = \frac{x}{2}\cdot\frac{2x}{e^{2x}-1},$$
$$\frac{x}{e^x+1}\cdot\frac{x}{e^x+1} = (x-1)\frac{x}{e^x+1} + x\left(\frac{d}{dx}\frac{x}{e^x+1}\right).$$

Using these, we can immediately produce analogues of Euler's identity (2.1) for the sequences $B = \{B_n\}_{n\geq 0}$ and $B' = \{B'_n\}_{n\geq 0}$ by the derivative method.

Proposition 2.1. *For $n \geq 1$, we have*

$$(B+B')^n = \frac{n}{2}\left(B_{n-1} - B'_{n-1}\right) = 2^{n-2}nB_{n-1}, \tag{2.2}$$
$$(B'+B')^n = nB'_{n-1} + (n-1)B'_n. \tag{2.3}$$

We will observe again these identities from a different direction in Section 4.

For $\alpha, \beta \in \mathbb{R}$, we present the following three types of rational function identities:

(a) $$\frac{1}{X^\alpha-1}\cdot\frac{1}{X^\beta-1} = \frac{1}{X^{\alpha+\beta}-1}\left(1+\frac{1}{X^\alpha-1}+\frac{1}{X^\beta-1}\right) \quad (\alpha, \beta, \alpha+\beta \neq 0),$$

(b) $$\frac{1}{X^\alpha+1}\cdot\frac{1}{X^\beta+1} = \frac{1}{X^{\alpha+\beta}-1}\left(1-\frac{1}{X^\alpha+1}-\frac{1}{X^\beta+1}\right) \quad (\alpha+\beta \neq 0),$$

(c) $$\frac{1}{X^\alpha-1}\cdot\frac{1}{X^\beta+1} = \frac{1}{X^{\alpha+\beta}+1}\left(1+\frac{1}{X^\alpha-1}-\frac{1}{X^\beta+1}\right) \quad (\alpha \neq 0).$$

These identities can be easily confirmed by direct calculations. In the following sections, we utilize them to construct some required functional equations related to the generating functions (1.1) and (1.6) of B_n and B'_n, respectively.

3. The Miki and Matiyasevich Identities

In this section, we first present an elementary and shorter proof of Proposition 1.1 by applying a certain functional equation constructed using (a), which is a slightly modified version of Crabb's proof. Subsequently, we give a very short proof of Proposition 1.3 and later we discuss other types of convolution identities.

In what follows, we assume that $n \geq 4$ and n is even. Otherwise, both sides of (1.2) vanish because $B_{2k+1} = 0$ for all $k \geq 1$ and so it is meaningless.

Proof of Proposition 1.1. Put $\alpha = t$, $\beta = 1-t$ and $X = e^x$ in (a), and multiply both sides by $t(1-t)x^2$. Then we establish the functional equation

$$\begin{aligned}&\frac{tx}{e^{tx}-1}\cdot\frac{(1-t)x}{e^{(1-t)x}-1}\\=&\frac{x}{e^x-1}\left(t(1-t)x+(1-t)\frac{tx}{e^{tx}-1}+t\frac{(1-t)x}{e^{(1-t)x}-1}\right).\end{aligned}\tag{3.1}$$

Differentiate (3.1) n-times with respect to x by applying Leibniz's rule and put $x=0$. Then we get

$$(tB+(1-t)B)^n=t(1-t)nB_{n-1}+(1-t)(B+tB)^n+t(B+(1-t)B)^n,$$

namely,

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}B_iB_{n-i}-t(1-t)nB_{n-1}\\&\quad=(1-t)\sum_{i=0}^{n}\binom{n}{i}t^{n-i}B_iB_{n-i}+t\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}B_iB_{n-i}.\end{aligned}\tag{3.2}$$

If we gather the terms involving $B_n(=B_0B_n)$, then, noting that $B_{n-1}=0$ for an even $n\geq 4$, above (3.2) becomes

$$\begin{aligned}&\sum_{i=2}^{n-2}\binom{n}{i}t^i(1-t)^{n-i}B_iB_{n-i}-\left(1-t^{n+1}-(1-t)^{n+1}\right)B_n\\&\quad=\sum_{i=2}^{n-2}\binom{n}{i}\left((1-t)t^{n-i}+t(1-t)^{n-i}\right)B_iB_{n-i}.\end{aligned}\tag{3.3}$$

Divide (3.3) by $t(1-t)$ and integrate it between 0 and 1 with respect to t. Then, making use of the easily shown formulas

$$\begin{aligned}&\int_0^1 t^m(1-t)^k dt=\frac{m!k!}{(m+k+1)!}\quad(m,k\geq 0),\\&\frac{1}{2}\int_0^1\frac{1-t^{m+1}-(1-t)^{m+1}}{t(1-t)}dt=\int_0^1\frac{1-t^m}{1-t}dt=H_m\quad(m\geq 1),\end{aligned}\tag{3.4}$$

we obtain, since $\binom{n}{i}\frac{(i-1)!(n-1-i)!}{(n-1)!}=\frac{n}{i(n-i)}=\frac{1}{i}+\frac{1}{n-i}$,

$$n\sum_{i=2}^{n-2}\frac{B_iB_{n-i}}{i(n-i)}-2H_nB_n=2\sum_{i=2}^{n-2}\frac{1}{n-i}\binom{n}{i}B_iB_{n-i}=n\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}}{i(n-i)}.$$

This is exactly the same as (1.2) if we divide both sides by n. □

It is also possible to deduce (1.2) by dividing (3.3) by $1-t$ and integrating between 0 and 1 with respect to t. Indeed, if we divide (3.3) by $1-t$, then we have

$$\sum_{i=2}^{n-2}\binom{n}{i}t^i(1-t)^{n-1-i}B_iB_{n-i}-\left(\frac{1-t^{n+1}}{1-t}-(1-t)^n\right)B_n$$
$$=\sum_{i=2}^{n-2}\binom{n}{i}\left(t^{n-i}+t(1-t)^{n-1-i}\right)B_iB_{n-i}.$$

Integrating this between 0 and 1, we get, using (3.4),

$$\sum_{i=2}^{n-2}\frac{B_iB_{n-i}}{n-i}+\left(\frac{1}{n+1}-H_{n+1}\right)B_n=\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}}{n-i},$$

which is equivalent to (1.3) (hence to (1.2)), since $H_{n+1}=H_n+1/(n+1)$.

As a further application of (3.3), we can easily reestablish Matiyasevich's identity (1.5) with a simpler and shorter proof.

Proof of Proposition 1.3. Integrating directly both sides of (3.3) from 0 to 1, we obtain, using (3.4),

$$\frac{1}{n+1}\sum_{i=2}^{n-2}B_iB_{n-i}-\frac{n}{n+2}B_n=2\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}}{(n+1-i)(n+2-i)}.$$

Multiply both sides by $(n+1)(n+2)$ to obtain (1.5). □

The following identities are also easy consequences from (3.3).

Proposition 3.1. *For $n\geq 4$, we have*

$$\sum_{i=2}^{n-2}\binom{n}{i}B_iB_{n-i}=-(n+1)B_n \quad \text{(Euler)}, \tag{3.5}$$

$$\sum_{i=2}^{n-2}\binom{n}{i}iB_iB_{n-i}=-\binom{n+1}{2}B_n, \tag{3.6}$$

$$\sum_{i=2}^{n-2}\binom{n}{i}2^iB_iB_{n-i}=-(n+2^n)B_n, \tag{3.7}$$

$$\sum_{i=2}^{n-2}\binom{n}{i}3^i(2+2^{n-i})B_iB_{n-i}=-(3^{n+1}+2^n+3n-1)B_n. \tag{3.8}$$

Proof. For Euler's identity (3.5), we have only to divide (3.3) by t and put $t=0$. For (3.6), differentiate both sides of (3.3) with respect to t only once and put $t=1/2$. Then we get

$$\sum_{i=2}^{n-2}\binom{n}{i}(2i-n)B_iB_{n-i}=0,$$

and hence (3.6) is given by using (3.5). Next, putting $t = 1/2$ in (3.3) and multiplying it by 2^n, we have

$$\sum_{i=2}^{n-2} \binom{n}{i} B_i B_{n-i} + (1-2^n)B_n = \sum_{i=2}^{n-2} \binom{n}{i} 2^i B_i B_{n-i},$$

which gives (3.7) by using (3.6). Similarly, if we put $t = 1/3$ in (3.3) and multiply it by 3^{n+1}, then

$$3\sum_{i=2}^{n-2} \binom{n}{i} 2^{n-i} B_i B_{n-i} - (3^{n+1} - 1 - 2^{n+1})B_n = \sum_{i=2}^{n-2} \binom{n}{i} 3^i (2 + 2^{n-i}) B_i B_{n-i},$$

which leads to (3.8) using (3.7). This completes the proof. □

The identities (3.7) and (3.8) can also be obtained by putting $t = 2, 3$ in (3.3).

4. Analogues of Propositions 1.1 and 1.3

In this section, we study analogues of the Miki and Matiyasevich convolution identities for the numbers $B'_n = (1-2^n)B_n$ by the same arguments as that performed in Section 3.

We first prove the following analogues of Miki's identity (1.2):

Proposition 4.1. *For $n \geq 4$, we have*

$$\sum_{i=2}^{n-2} \frac{B'_i B'_{n-i}}{i} + \sum_{i=2}^{n-2} \binom{n}{i} \frac{B'_i B_{n-i}}{i} = -\frac{B'_n}{n}, \tag{4.1}$$

$$\sum_{i=2}^{n-2} \frac{B_i B'_{n-i}}{i(n-i)} - \sum_{i=2}^{n-2} \binom{n-1}{i} \frac{2^i B_i B'_{n-i}}{i(n-i)} = H_{n-1} \frac{B'_n}{n}. \tag{4.2}$$

Proof. We first prove (4.1). Put $\alpha = t$, $\beta = 1-t$ and $X = e^x$ in (b) and multiply it by $t(1-t)x^2$. Then we get the functional equation

$$\begin{aligned} &\frac{tx}{e^{tx}+1} \cdot \frac{(1-t)x}{e^{(1-t)x}+1} \\ =&\frac{x}{e^x-1}\left(t(1-t)x - (1-t)\frac{tx}{e^{tx}+1} - t\frac{(1-t)x}{e^{(1-t)x}+1}\right). \end{aligned} \tag{4.3}$$

If we differentiate (4.3) n-times with respect to x and put $x = 0$, then we obtain for the sequences $B = \{B_n\}_{n\geq 0}$ and $B' = \{B'_n\}_{n\geq 0}$,

$$(tB' + (1-t)B')^n = t(1-t)nB_{n-1} - (1-t)(B+tB')^n - t(B+(1-t)B')^n,$$

namely,

$$\sum_{i=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}B_i'B_{n-i}' - t(1-t)nB_{n-1}$$
$$= -\sum_{i=0}^{n}\binom{n}{i}\left((1-t)t^{n-i} + t(1-t)^{n-i}\right)B_iB_{n-i}'.$$

Considering the obvious facts $B_0 = 1$, $B_0' = 0$ and $B_{n-1} = B_{n-1}' = 0$ for an even $n \geq 4$, this identity implies

$$\begin{aligned}&\sum_{i=2}^{n-2}\binom{n}{i}t^i(1-t)^{n-i}B_i'B_{n-i}' + \left((1-t)t^n + t(1-t)^n\right)B_n' \\ &= -\sum_{i=2}^{n-2}\binom{n}{i}\left((1-t)t^{n-i} + t(1-t)^{n-i}\right)B_iB_{n-i}'.\end{aligned} \tag{4.4}$$

Dividing (4.4) by $t(1-t)$, we have

$$\sum_{i=2}^{n-2}\binom{n}{i}t^{i-1}(1-t)^{n-1-i}B_i'B_{n-i}' + \left(t^{n-1} + (1-t)^{n-1}\right)B_n'$$
$$= -\sum_{i=2}^{n-2}\binom{n}{i}\left(t^{n-1-i} + (1-t)^{n-1-i}\right)B_iB_{n-i}'.$$

Integrating this between 0 and 1 with respect to t and dividing it by n, we deduce, using the first formula in (3.4),

$$n\sum_{i=2}^{n-2}\frac{B_i'B_{n-i}'}{i(n-i)} + \frac{2}{n}B_n' = -2\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}'}{n-i},$$

which gives, since $\frac{1}{i(n-i)} = \frac{1}{n}\left(\frac{1}{i} + \frac{1}{n-i}\right)$,

$$2\sum_{i=2}^{n-2}\frac{B_i'B_{n-i}'}{n-i} + \frac{2}{n}B_n' = -2\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}'}{n-i}.$$

This implies (4.1) if we divide this by 2 and replace $n-i$ by i.

For the proof of (4.2), we put $\alpha = t$, $\beta = 1-t$ and $X = e^x$ in (c) and multiply it by $t(1-t)x^2$ to get the functional equation

$$\begin{aligned}&\frac{tx}{e^{tx}-1}\cdot\frac{(1-t)x}{e^{(1-t)x}+1} \\ =&\frac{x}{e^x+1}\left(t(1-t)x + (1-t)\frac{tx}{e^{tx}-1} - t\frac{(1-t)x}{e^{(1-t)x}+1}\right).\end{aligned} \tag{4.5}$$

Differentiating (4.5) n times with respect to x and putting $x = 0$, we obtain

$$(tB + (1-t)B')^n = t(1-t)nB'_{n-1} + (1-t)(B' + tB)^n - t(B' + (1-t)B')^n,$$

which implies

$$\sum_{i=0}^{n}\binom{n}{i}t^i(1-t)^{n-i}B_iB'_{n-i} - t(1-t)nB'_{n-1}$$
$$= (1-t)\sum_{i=0}^{n}\binom{n}{i}t^{n-i}B'_iB_{n-i} - t\sum_{i=0}^{n}\binom{n}{i}(1-t)^{n-i}B'_iB'_{n-i}.$$

Since $B_0 = 1$, $B'_0 = 0$ and $B_{n-1} = B'_{n-1} = 0$, we can write this as

$$\begin{aligned}&\sum_{i=2}^{n-2}\binom{n}{i}t^i(1-t)^{n-i}B_iB'_{n-i} - ((1-t) - (1-t)^n)B'_n\\ &=(1-t)\sum_{i=2}^{n-2}\binom{n}{i}t^{n-i}B'_iB_{n-i} - t\sum_{i=2}^{n-2}\binom{n}{i}(1-t)^{n-i}B'_iB'_{n-i}.\end{aligned} \tag{4.6}$$

Dividing (4.6) by $t(1-t)$, we have

$$\sum_{i=2}^{n-2}\binom{n}{i}t^{i-1}(1-t)^{n-1-i}B_iB'_{n-i} - \left(\frac{1-(1-t)^{n-1}}{t}\right)B'_n$$
$$= \sum_{i=2}^{n-2}\binom{n}{i}t^{n-1-i}B'_iB_{n-i} - \sum_{i=2}^{n-2}\binom{n}{i}(1-t)^{n-1-i}B'_iB'_{n-i}.$$

Similarly to the above, integrating between 0 and 1 with respect to t, we have, using both formulas in (3.4),

$$\begin{aligned}n\sum_{i=2}^{n-2}\frac{B_iB'_{n-i}}{i(n-i)} - H_{n-1}B'_n &= \sum_{i=2}^{n-2}\binom{n}{i}\frac{B'_i(B_{n-i} - B'_{n-i})}{n-i}\\ &= \sum_{i=2}^{n-2}\binom{n}{i}\frac{2^{n-i}B'_iB_{n-i}}{n-i},\end{aligned}$$

which yields (4.2) dividing by n and replacing $n-i$ by i on the right-hand side. □

Next, we deduce analogues of Matiyasevich's identity (1.5) by making again use of (4.4) and (4.6).

Proposition 4.2. *For $n \geq 4$, we have*

$$(n+2)\sum_{i=2}^{n-2}B'_iB'_{n-i} + 2\sum_{i=2}^{n-2}\binom{n+2}{i}B_iB'_{n-i} = -2B'_n, \tag{4.7}$$

$$(n+2)\sum_{i=2}^{n-2}B_iB'_{n-i} - \sum_{i=2}^{n-2}\binom{n+2}{i}2^{n-i}B'_iB_{n-i} = \frac{(n-1)(n+2)}{2}B'_n. \tag{4.8}$$

Proof. Directly integrating (4.4) between 0 and 1 with respect to t, we have

$$\begin{aligned}\frac{1}{n+1}\sum_{i=2}^{n-2}B_i'B_{n-i}' &+ \frac{2}{(n+1)(n+2)}B_n' \\ &= -2\sum_{i=2}^{n-2}\binom{n}{i}\frac{B_iB_{n-i}'}{(n+1-i)(n+2-i)}.\end{aligned} \tag{4.9}$$

By the same arguments as done above, we obtain from (4.6)

$$\begin{aligned}\frac{1}{n+1}\sum_{i=2}^{n-2}B_iB_{n-i}' &- \left(\frac{1}{2}-\frac{1}{n+1}\right)B_n' \\ &= \sum_{i=2}^{n-2}\binom{n}{i}\frac{B_i'(B_{n-i}-B_{n-i}')}{(n+1-i)(n+2-i)} \\ &= \frac{1}{(n+1)(n+2)}\sum_{i=2}^{n-2}\binom{n+2}{i}2^{n-i}B_i'B_{n-i}.\end{aligned} \tag{4.10}$$

Multiplying (4.9) and (4.10) by $(n+1)(n+2)$, we get the identities indicated. □

Euler-type identities (2.2) and (2.3) in Proposition 2.1 can also be deduced from (4.4) and (4.6), respectively. Indeed, if we divide (4.4) and (4.6) by t (or $1-t$) and put $t=0$ (or $t=1$), then we have for $n \geq 4$,

$$\sum_{i=2}^{n-2}\binom{n}{i}B_i'B_{n-i} = -B_n', \tag{4.11}$$

$$\sum_{i=2}^{n-2}\binom{n}{i}B_i'B_{n-i}' = (n-1)B_n', \tag{4.12}$$

which are equivalent to (2.2) and (2.3), respectively. Since $B_{n-i}' = (1-2^{n-i})B_{n-i}$, subtracting (4.12) from (4.11), we get

$$\sum_{i=2}^{n-2}\binom{n}{i}2^{n-i}B_i'B_{n-i} = -nB_n'.$$

As easily seen, it is also possible to derive this identity multiplying (3.5) by 2^n and subtracting it from (3.7).

At the end of this paper, we would like to mention that many interesting identities related to Bernoulli, Euler and other polynomials obtained by using umbral calculus and p-adic integral on $\mathbb{Z}_p$ can be found in [4, 13, 14].

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