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ON INTERPOLATING POWER SERIES

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Abstract

We derive a simple error estimate for equally spaced, polynomial interpolation of power series that does not require the uniform bounds on derivatives of the Cauchy remainder. The key steps are expressing Newton coefficients in terms of Stirling numbers $S(i,\, j)$ of the second kind and applying the concavity of $\ln S(i,\, j)$.

1. Introduction

Let f be in $C[a,b]$ and let $P_n f$ denote the unique polynomial of degree at most n that interpolates f at $n+1$ equally spaced nodes. By a classical result [3, Theorem 4.3.1] of interpolation theory, $P_n f$ converges uniformly to f if f can be extended analytically to a certain region of the complex plane that contains the interior of a lemniscate formed from disks centered at a and b with radii greater than $b-a$. We show that more is true for real functions: if f is a Taylor series about a or b, then there is a simple bound on the uniform error that implies uniform convergence when the derived series f' is in $C[a,b]$. We expand the Newton coefficients of $P_n f$ in terms of Stirling numbers $S(i,\, j)$ of the second kind and use the well known log-concavity property of L. H. Harper and E. H. Lieb ([7], [9]):

$$\textit{for each } \ i > 1\,, \ \textit{the ratio } \frac{S(i, j+1)}{S(i,j)} \textit{ is strictly decreasing.} \tag{1}$$

2. The Main Result

By the transformation $x \mapsto \frac{x-a}{b-a}$, we may assume that the underlying interval is $[0,1]$. An analog of Abel's partial summation formula [1, vol. I, Theorem 10.16] will be used in our arguments:

$$\sum_{i=m}^{n} a_i b_i = \left(\sum_{i=m}^{n} a_i\right) b_m + \sum_{j=m}^{n-1}\left(\sum_{i=j+1}^{n} a_i\right)(b_{j+1} - b_j)\,. \tag{2}$$

For example, by Abel's theorem [2, p. 325] a power series $\sum_{i\geq 0} a_i x^i$ is in $C[0,1]$ if and only if the series $\sum a_i$ converges. In this case, since for fixed $x < 1$, $b_i = x^i$ decreases monotonically to zero, we have the error estimate

$$\left\|\sum_{i>n} a_i x^i\right\|_\infty \leq 2\epsilon_n(<a_i>)$$

by (2), where

$$\epsilon_n(<a_i>) := \max\left\{\left|\sum_{i>k} a_i\right| : k \geq n\right\}$$

converges to zero. A similar estimate that is a refinement of [5, Theorem 1] holds for polynomial interpolation:

Theorem 1. *Let f be in $C[0,1]$ and let $P_n f$ be its interpolating polynomial at the nodes $0, 1/n, 2/n, \ldots, 1$. If either $f(x)$ or the reflection $f(1-x)$ of $f(x)$ about $x = \frac{1}{2}$ is represented by a power series $\sum a_i x^i$ such that $\sum a_i$ converges, then $P_n f$* satisfies

$$\|f - P_n f\|_\infty \leq (2n+1)\epsilon_n(<a_i>). \tag{3}$$

In particular, $P_n f$ converges to f uniformly whenever $\sum i a_i$ converges.

Proof. Stirling numbers of the second kind are the coefficients in the formula that converts powers to binomial coefficients: for $i > 0$

$$x^i = \sum_{j=1}^{i} S(i,j) j! \binom{x}{j} \tag{4}$$

where $j!\,\binom{x}{j} = x(x-1)\cdots(x-j+1)$ is a factorial polynomial. Since $x^i = x\;x^{i-1}$ and $\binom{x}{j} = \binom{x-1}{j-1} + \binom{x-1}{j}$, we have the usual recurrence relation

$$S(i,j) = S(i-1,j-1) + jS(i-1,j)$$

with

$$S(i,1) = S(i,i) = 1 \text{ and } S(i,j) = 0 \text{ for } j > i.$$

Hence (4) may be easily solved for $S(i,\,j)$ by induction on row i:

$$S(i,j) = \frac{1}{j!}\sum_{k=1}^{j}(-1)^{j-k}k^i\binom{j}{k}. \tag{5}$$

By Lagrange's form [1, vol. II, Theorem 15.2] for P_nf, we have that for linear combinations, $P_n(\alpha f + \beta g) = \alpha P_nf + \beta P_ng$; and if $f_m \longrightarrow f$ pointwise, then $P_nf_m \longrightarrow P_nf$. Moreover by (4), for $i > 0$,

$$P_nx^i = \sum_{j=1}^{n}\frac{S(i,j)}{n^i}j!\binom{nx}{j} = \sum_{j=1}^{n}\frac{S(i,j)}{n^{i-j}}x\left(x-\frac{1}{n}\right)\cdots\left(x-\frac{j-1}{n}\right)$$

in Newton's form [1, vol. II, Theorem 15.5]. Therefore, if $\sum a_i$ converges, then

$$P_n\left(\sum_{i\ge 0}a_ix^i\right) = a_0 + \sum_{j=1}^{n}\left(\sum_{i>0}\frac{S(i,j)}{n^i}a_i\right)j!\binom{nx}{j}.$$

Suppose first that $f(x) = \sum a_ix^i$ where $\sum a_i$ converges. If $f_n(x) := \sum_{i=0}^{n}a_ix^i$, then $f_n = P_nf_n$ by uniqueness and

$$\begin{aligned}\|f - P_nf\|_\infty &\le \|f - f_n\|_\infty + \|P_n(f-f_n)\|_\infty\\ &\le 2\epsilon_n(<\!a_i\!>) + \left\|\sum_{j=1}^{n}\left(\sum_{i>n}\frac{S(i,j)}{n^i}a_i\right)j!\binom{nx}{j}\right\|_\infty.\end{aligned}$$

Clearly $\left|\binom{nx}{j}\right| \le \binom{n}{j}$ if $j-1 \le nx \le n$. Suppose that $k-1 \le nx < k$ for some integer k in $[1, j-1]$. Then $j!\left|\binom{nx}{j}\right| = (\Pi_{i=0}^{k-1}|nx-i|)(\Pi_{i=k}^{j-1}|nx-i|) \le k!(j-k)!$ where $k! \le \frac{n!}{(n-k)!}$ and $(j-k)! \le \frac{(n-k)!}{(n-j)!}$. It follows that $j!\left|\binom{nx}{j}\right| \le j!\binom{n}{j}$ for all x in $[0,1]$. Thus, if

$$b_{ij} := \frac{S(i,j)}{n^i}j!\binom{n}{j},$$

then

$$\|f - P_nf\|_\infty \le 2\epsilon_n(<\!a_i\!>) + \sum_{j=1}^{n}\left|\sum_{i>n}a_ib_{ij}\right| \tag{6}$$

where $< b_{ij} : j = 1, \ldots, n >$ is a probability vector for each $i > n$: $b_{ij} \geq 0$, and by (5) and the identity $\binom{n}{j}\binom{j}{k} = \binom{n}{k}\binom{n-k}{j-k}$,

$$\sum_{j=1}^{n} b_{ij} = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{i} \left[\sum_{j=1}^{n} (-1)^{j-k} \binom{n}{j}\binom{j}{k}\right] = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^{i} \binom{n}{k} \delta_{nk} = 1$$

where δ_{nk} is the Kronecker delta.

Consider the term $j = n$ in (6). If $n = 1$, then $b_{in} = 1$. And if $n > 1$, then the sequence b_{in} $(i > n)$ increases to 1 since

i. $b_{i+1,n} > b_{in}$ if and only if $S(i+1, n) > n\, S(i, n)$. But $S(i+1, n) = S(i, n-1) + n\, S(i, n) > n\, S(i, n)$.

ii. By (5),

$$b_{in} = \sum_{k=1}^{n} (-1)^{n-k} \left(\frac{k}{n}\right)^{i} \binom{n}{k} \longrightarrow 1.$$

Therefore by (2), for all n,

$$\left| \sum_{i>n} a_i b_{in} \right| \leq \epsilon_n(<a_i>). \tag{7}$$

Next, suppose that $j < n$. We show that $\{b_{ij} : i > n\}$ is unimodal: There exists some integer $i(j) > n$ such that for $i > n$,

$$b_{i+1,j} \leq b_{ij} \text{ ifandonlyif } i \geq i(j). \tag{8}$$

As above, (8) is equivalent to

$$S(i+1, j) \leq nS(i, j) \text{ ifandonlyif } i \geq i(j).$$

Clearly $i(1) = n + 1$ so assume that $j > 1$. We first verify that the sequence ln $S(i, j)$ $(i \geq j)$ is concave, i.e., the ratio $\frac{S(i+1,j)}{S(i,j)}$ is strictly decreasing. It is easy to check the case $j = 2$ with (5). Then for $j > 2$, the following inequalities are equivalent:

$$\frac{S(i+2, j)}{S(i+1, j)} = \frac{S(i+1, j-1)}{S(i+1, j)} + j < \frac{S(i+1, j)}{S(i, j)} = \frac{S(i, j-1)}{S(i, j)} + j$$

$$\frac{S(i+1, j-1)}{S(i+1, j)} = \frac{S(i, j-2) + (j-1)S(i, j-1)}{S(i, j-1) + jS(i, j)} < \frac{S(i, j-1)}{S(i, j)}$$

$$\frac{S(i, j)}{S(i, j-1)} - \frac{S(i, j)}{S(i, j-2)} < \frac{S(i, j-1)}{S(i, j-2)}.$$

The last inequality is true by (1). Thus by (5), the sequence $\frac{S(i+1,j)/j^i}{S(i,j)/j^i}$ strictly decreases to $j < n$, and therefore (8) holds.

Also by (5),

$$b_{ij} = \sum_{k=1}^{j} (-1)^{j-k} \left(\frac{k}{n}\right)^{i} \binom{n}{j}\binom{j}{k}$$

converges to zero. Hence by (2), for $j = 1, \ldots, n-1$,

$$\left|\sum_{i>n} a_i b_{ij}\right| \leq 2 b_{i(j),j}\,\epsilon_n(<a_i>) \leq 2\epsilon_n(<a_i>)$$

and (3) now follows from (6) and (7).

Finally suppose that $g(x) := f(1-x) = \sum_{i\geq 0} a_i x^i$ where $\sum a_i$ converges. The polynomial $(P_n f)(1-x)$ is of degree at most n and interpolates g at the nodes $0, \frac{1}{n}, \frac{2}{n}, \ldots, 1$. Hence by uniqueness, $(P_n f)(1-x) = (P_n g)(x)$ and we have that for x in $[0,1]$,

$$|\ f(x) - (P_n f)(x)| \ \leq\ \max\{|g(x) - (P_n g)(x)| : 0 \leq \mathrm{x} \leq 1\} = \|g - P_n g\|_\infty .$$

Thus (3) is a consequence of the first case.

Assume now that $\sum i a_i$ converges. By (2), $\sum a_i$ converges and

$$\begin{aligned}\epsilon_n(<a_i>) &= \max\left\{\left|\sum_{i>k} \frac{1}{i}(i a_i)\right| : k \geq n\right\}\\ &\leq \max\left\{\frac{2}{k+1}\epsilon_k(<i a_i>) : k \geq n\right\} \leq \frac{2}{n+1}\epsilon_n(<i a_i>).\end{aligned}$$

Hence

$$(2n+1)\epsilon_n(<a_i>) \leq 4\epsilon_n(<i a_i>) - \epsilon_n(<a_i>)$$

and therefore $P_n f \longrightarrow f$ uniformly by (3). □

3. Examples

Runge's famous example $f(x) = \frac{1}{x^2+1}$ on $[-5,5]$ is *not* the uniform limit of $P_n f$ ([4], [8, Sec. 3.4]). However, Theorem 1 implies uniform convergence on $[0,1]$:

Example 2. For $f(x) = \frac{1}{x^2+1}$ in $C[0,1]$, we have that

$$f(1-x) = \frac{1}{2i}\left[\frac{1}{x-(1+i)} - \frac{1}{x-(1-i)}\right]$$

$$= \frac{i}{4}\left[(1-i)\sum_{k\geq 0}\left(\frac{x}{1+i}\right)^k - (1+i)\sum_{k\geq 0}\left(\frac{x}{1-i}\right)^k\right] = \sum_{k\geq 0}\frac{\sin[(k+1)\pi/4]}{2^{(k+1)/2}}x^k.$$

Therefore,

$$\|f - P_n f\|_\infty \le \frac{2n+1}{(\sqrt{2}-1)(\sqrt{2})^{n+1}} \longrightarrow 0.$$

Uniform convergence of $P_n f$ holds for functions f in the large class of Taylor series about 0 or 1 with absolutely summable coefficients [6]. This also follows from the proof above since if $\sum_{i>n}|a_i| < \infty$, then in (6),

$$\sum_{j=1}^{n}\left|\sum_{i>n} a_i b_{ij}\right| \le \sum_{i>n}|a_i|\left(\sum_{j=1}^{n} b_{ij}\right) = \sum_{i>n}|a_i|\,.$$

Moreover, $\|f - P_n f\|_\infty \le 2\sum_{i>n}|a_i|$ in this case since $\left\|\sum_{i>n} a_i x^i\right\|_\infty \le \sum_{i>n}|a_i|$. Thus we have an improvement for Example 2:

$$\|f - P_n f\|_\infty \le \frac{1}{(\sqrt{2}-1)(\sqrt{2})^{n-1}}.$$

Example 3. The coefficients of $f(x) = \sum_{i\geq 2}\frac{(-1)^i x^i}{i\ln i}$ are not absolutely summable by the integral test. However, f and f' are in $C[0,1]$ by the alternating series test and therefore

$$\|f - P_n f\|_\infty \le \frac{2n+1}{(n+1)\ln(n+1)}.$$

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