ON AN INCOMPLETE ARGUMENT OF ERDŐS ON THE IRRATIONALITY OF LAMBERT SERIES

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Abstract

We show that the Lambert series $f(x) = \sum d(n)x^n$ is irrational at $x = 1/b$ for negative integers $b < -1$ using an elementary proof that finishes an incomplete proof of Erdős.

1. Introduction

Chowla [5] conjectured that the functions

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} \qquad \text{and} \qquad g(x) = \sum_{n=1}^{\infty} \frac{x^n}{1-x^n}(-1)^{n+1}$$

are irrational at all rational values of x satisfying $|x| < 1$. For such x the above functions may be rewritten as

$$f(x) = \sum_{n=1}^{\infty} d(n)x^n \qquad \text{and} \qquad g(x) = \frac{1}{4}\sum_{n=1}^{\infty} r(n)x^n,$$

where $d(n)$ is the number of divisors of n and $r(n)$ is the number of representations of n as a sum of two squares.

Erdős [6] proved that for any integer $b > 1$, the value $f(1/b)$ is irrational. He did so by showing that $f(1/b)$ written in base b contains arbitrary long strings of 0's without terminating on 0's completely. If we take $b < -1$ to be a negative integer, then Erdős' methods show by the same method that $f(1/b)$ in base $|b|$ contains arbitrary long strings of 0's; however, Erdős claims without proof that showing it will not terminate on 0's can be done using similar methods. It is not clear what method Erdős intended, and in later papers (including his review of similar irrationality results [7]) Erdős only refers to proving the case of positive b.

Since then, several proofs have been offered for the irrationality of the $b < -1$ case and far more general theorems besides. Borwein [3] is credited for proving the first major generalizations of these results, using Padeé approximants to show that all the numbers

$$\sum_{n=1}^{\infty} \frac{1}{q^n - r}, \qquad q \in \mathbb{N}_{\geq 2}, \quad r \in \mathbb{Q} \setminus \{0\}, \quad r \neq -q^m \text{ for any } m \geq 1$$

are irrational. Bundschuh and Väänänen [4], using a result of Bézivin [2] on the linear independence over $\mathbb{Q}$ of the infinite product

$$E_q(z) = \prod_{n=1}^{\infty} (1 + zq^{-j})$$

and its derivative, provide a quick second proof of Borwein's theorem. Other results can often be found in the literature under the term of the q-analogue of the logarithm or, simply, the q-logarithm.

However, all of these results are proved using entirely different techniques than those Erdős used and leave open the question of whether his method could have finished the proof.

Erdős' method can be extended to the following stronger result with a virtually identical proof.

Theorem 1.1. *Let $b > 1$ be a positive integer and $\mathcal{A}$ be any finite set of non-negative integers. Then for any sequence $\{a_n\}_{n=1}^{\infty}$ taking values in $\mathcal{A}$ such that the sequence does not end on repeated 0's, we have that*

$$\sum_{n=1}^{\infty} d(n) \frac{a_n}{b^n}$$

is irrational.

Theorem 1.1 has the following curious corollary. Let $a_n(x)$ be the nth base b digit of a number x in $(0, 1)$. (If x has two base b expansions, then we chose the one which does not end on repeated 0's.) Then the map

$$x = \sum_{n=1}^{\infty} \frac{a_n(x)}{b^n} \longmapsto \sum_{n=1}^{\infty} d(n) \frac{a_n(x)}{b^n}$$

has its image in $\mathbb{R} \backslash \mathbb{Q}$ and is also continuous at all x that do not have a representation as a finite base b expansion.

We could replace the condition that a_n be in the finite set $\mathcal{A}$ with a restriction that $0 \leq a_n \leq \phi(n)$ for some sufficiently slowly growing integer-valued function ϕ. It would be interesting to know what the fastest growing ϕ for which the Theorem 1.1 holds would be.

In this paper, we will prove the following extension of Theorem 1.1.

Theorem 1.2. *Let $b > 1$ be a positive integer and $\mathcal{A}$ be any finite set of integers* that does not contain 0. *Then for any sequence of $\{a_n\}_{n=1}^{\infty}$ taking values in $\mathcal{A}$, we have that*

$$\sum_{n=1}^{\infty} d(n)\frac{a_n}{b^n}$$

is irrational.

The new ingredient to extend Erdős' method is finding arbitrarily long strings of zeros *that are known to be preceded by a non-zero number*, and to find these strings arbitrarily far into the base $|b|$ expansion.

In particular, by taking $a_n = (-1)^n$, this proves that $f(1/b)$ is irrational for negative integers $b < 1$ as well, completing Erdős' proof.

Theorem 1.2 has parallels with a result of Tachiya [8], which showed that if $\{a_n\}$ is a sequence of rational numbers of period 2 (not identically zero), then the function

$$f(q) = \sum_{n=1}^{\infty} a_n \frac{1}{1-q^n}$$

is irrational for integers q with $|q| > 1$.

2. Proof of Theorem 1.2

We will require a result mentioned by Alford, Granville, and Pomerance [1, p. 705]. The function $\pi(N; d, a)$ equals the number of primes up to N that are congruent to a modulo d.

Proposition 2.1. *Let $0 < \delta < 5/12$. Then there exist positive integers N_0 and $\overline{\mathcal{D}}$ dependent only on δ, such that the bound*

$$\pi(N; d, a) \geq \frac{N}{2\varphi(d)\log N}$$

holds for all $N > N_0$; for all moduli d with $1 \leq d \leq N^{\delta}$, except, possibly for those d that are multiples of some element in $\mathcal{D}(N)$, a set of at most $\overline{\mathcal{D}}$ different integers that all exceed $\log N$*; and for all integers a that are relatively prime to d.*

We begin our proof much as Erdős did his. Let $b \geq 2$ be a fixed positive integer, let $\mathcal{A}$ be a finite set of integers that does not contain 0, and let N be a large positive integer that is allowed to vary. Define k in terms of N by

$$k = k(N) := \lfloor (\log N)^{1/10} \rfloor.$$

Let j_0 be a fixed integer, independent of N, so that $2\max_{a\in\mathcal{A}} |a|/b^{j_0} < 1$.

Let $0 < \delta < 5/12$ be some sufficiently small fixed constant, and let $N_0 = N_0(\delta)$ and $\overline{\mathcal{D}} = \overline{\mathcal{D}}(\delta)$ be the corresponding constants from Proposition 2.1. Let $N_1 > N_0$ be large enough so that for any $N > N_1$, the interval $((\log N)^2, 2(\log N)^2)$ cotains at least $u + \overline{\mathcal{D}}$ primes, where $u = u(N) = k(k+1)/2$. In addition, for such $N > N_1$, let $\mathcal{D}(N)$ be the set of exceptional moduli from Proposition 2.1. Since we assume that δ is constant, $|\mathcal{D}(N)| \leq \overline{\mathcal{D}}$ is bounded.

For each D in $\mathcal{D}(N)$, let $\tilde{p}_D$ denote the smallest prime strictly greater than $(\log N)^2$ that divides D, if such a prime exists, and then let $p_1 < p_2 < \cdots < p_u$ be the smallest u primes strictly greater than $(\log N)^2$ that are not equal to $\tilde{p}_D$ for any $D \in \mathcal{D}(N)$; by assumption on N, we have that each such p_i is less than $2(\log N)^2$. Finally, let

$$A := \prod_{i=1}^{j_0(j_0+1)/2} p_i^b \prod_{i=(j_0+1)(j_0+2)/2+1}^{u} p_i^b,$$

so that, in particular, A is not a multiple of any D in $\mathcal{D}(N)$; moreover, provided N is sufficiently large, we have

$$A < (2(\log N)^2)^{bk(k+1)/2} \leq N^\delta.$$

By the Chinese remainder theorem, there exists an integer r, with $0 \leq r \leq A-1$, such that

$$r + j \equiv \prod_{i=j(j+1)/2+1}^{(j+1)(j+2)/2} p_i^{b-1} \pmod{\prod_{i=j(j+1)/2+1}^{(j+1)(j+2)/2} p_i^b}, \qquad 0 \leq j \leq k-1, j \neq j_0.$$

(The exception $j \neq j_0$ marks the key difference between this proof and Erdős'.) Since all the p_i's are bounded below by $(\log N)^2$, we have that r necessarily tends to infinity as N does, although possibly much slower.

With this value of r, for any integer of the form

$$r + mA, \qquad 0 \leq m < \lfloor N/A \rfloor,$$

we have that

$$d(r + mA + j) \equiv 0 \pmod{b^{j+1}}, \qquad 0 \leq j < k,\ j \neq j_0 \tag{1}$$

by the multiplicitivity of $d(\cdot)$. Moreover, $r + j_0$ is relatively prime to A, since each p dividing A also divides some $r + j$, with $0 \leq j < k$, $j \neq j_0$; the largest j can be is $k \leq (\log N)^{1/10}$, but all primes dividing A are at least $(\log N)^2$.

We can also apply Proposition 2.1 to see that

$$\pi(N, A, r + j_0) \geq \frac{N}{2\varphi(A)\log N}. \tag{2}$$

Erdős in [6] also proved the following result, which we give here without reproof. (While our construction of A and r are different from Erdős', they satisfy all the requirements for Erdős' proof technique to still hold.)

Lemma 2.2. *With A, r, b, and k all as above, the number of $m < \lfloor N/A \rfloor$ such that*

$$\sum_{n>r+k+mA} d(n)\frac{1}{b^n} > \frac{1}{b^{r+k/2+mA}}$$

is less than

$$\frac{10cN(\log N)^2}{A2^{k/4}}$$

for some constant c independent of all variables.

Regardless of how large c is, we have, for sufficently large N, that

$$\frac{N}{2\varphi(A)\log N} \geq \frac{10cN(\log N)^2}{A2^{k/4}}.$$

Therefore, by combining Lemma 2.2 with (1) and (2), we see that for sufficiently large N there exists some $m_0 < \lfloor N/A \rfloor$ such that

$$b^{j+1} | d(r + m_0A + j), \qquad 0 \leq j < k, j \neq j_0, \tag{3}$$

$$r + m_0A + j_0 \text{ is prime}, \tag{4}$$

and

$$\sum_{n>r+k+m_0A} \left| d(n)\frac{a_n}{b^n} \right| \leq \frac{\max_{a\in\mathcal{A}} |a|}{b^{r+k/2+mA}}. \tag{5}$$

Now consider a particular sequence (a_n) with each $a_n \in \mathcal{A}$ together with the sum

$$\sum_{n=1}^{\infty} d(n)\frac{a_n}{b^n}.$$

By (3), the partial sum

$$\sum_{\substack{n\leq r+k+m_0A \\ n\neq r+j_0+m_0A}} d(n)\frac{a_n}{b^n},$$

when written in base b, has its last non-zero digit in the $(r - 1 + m_0A)$th place or earlier.[1] In addition, by (5), the partial sum

$$\sum_{n>r+k+m_0A} d(n)\frac{a_n}{b^n}$$

[1]Here we switch back to the convention that finite expansions are assumed to end on an infinite string of zeros.

when written in base b has its first non-zero digit in the

$$(r + k/2 + m_0 A - \lceil \log_b \max_{a \in \mathcal{A}} |a| \rceil)\text{th}$$

place or later. The number

$$d(r + j_0 + m_0 A) \frac{a_{r+j_0+m_0 A}}{b^{r+j_0+m_0 A}} = \frac{2a_{r+j_0+m_0 A}}{b^{r+j_0+m_0 A}}$$

when written in base b has its non-zero digits only between the $(r + m_0 A)$th and $(r + j_0 + m_0 A)$th place, and it has at least one such non-zero digit. Thus the full sum has a string of at least $k/2 + O(1)$ zeroes immediately preceded by a non-zero digit starting somewhere between the $(r + m_0 A)$th and $(r + j_0 + m_0 A)$th place.

So as N increases to infinity, we can find arbitrarily long strings of 0's (this corresponds to k increasing to infinity) immediately preceded by a non-zero digit, and we find these strings arbitrarily far out in the expansion (since r also tends to infinity). The base b digits cannot therefore be periodic and hence the sum is irrational. This completes the proof.

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