VALUATIONS AND COMBINATORICS OF TRUNCATED EXPONENTIAL SUMS

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Abstract

A conjecture of G. McGarvey for the 2-adic valuation of the Schenker sums is established. These sums are $n!$ times the sum of the first $n+1$ terms of the series for e^n. A certain analytic expression for the p-adic valuation of these sums is provided for a class of primes. Some combinatorial interpretations (using rooted trees) are furnished for identities that arose along the way.

1. Introduction

Let $0 \neq x \in \mathbb{Q}$. The Fundamental Theorem of Arithmetic implies the prime factorization $|x| = \prod_p p^{n_p}$ where the product is over all primes and for some $n_p \in \mathbb{Z}$ (all but finitely many being zero). The *p-adic valuation* of x, denoted $\nu_p(x)$, is the exponent n_p in the power of p in the above factorization. For example, $\nu_2(2^k) = k$ and $\nu_2(2^k - 1) = 0$. By convention, $\nu_p(0) = +\infty$.

Given a sequence of positive integers a_n and a prime p, determining a closed form for the sequence of p-adic valuations $\nu_p(a_n)$ often presents interesting challenges. Legendre's classical formula for the factorials

$$\nu_p(n!) = \sum_{j=0}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor \tag{1.1}$$

appears in elementary textbooks. If $n \in \mathbb{N}$ is expanded in base p and $s_p(n)$ denotes the sum of its p-ary digits, then the alternative form

$$\nu_p(n!) = \frac{n - s_p(n)}{p-1} \tag{1.2}$$

follows directly from (1.1).

The presence of a compact formula, such as (1.2), facilitates the analysis of arithmetical properties of a given sequence a_n. For instance, it follows directly from (1.2) that

$$\nu_p\binom{2n}{n} = \frac{2s_p(n) - s_p(2n)}{p-1} \tag{1.3}$$

and in particular, for $p = 2$, this yields

$$\nu_2\binom{2n}{n} = s_2(n), \tag{1.4}$$

in view of $s_2(2n) = s_2(n)$. This provides an elementary proof that the central binomial coefficients $\binom{2n}{n}$ are always even, and exactly divisible by 2 if and only if n is a power of 2.

Introduce the sequence of positive integers

$$a_n = \sum_{k=0}^{n} \frac{n!}{k!} n^k. \tag{1.5}$$

One immediately recognizes that $\frac{a_n}{n!}$ equals the n^{th} partial sum of the exponential e^n. The sequence a_n appeared in a paper by S. Ramanujan [10] where he proposes the following problem:

Show that

$$\frac{1}{2}e^n = 1 + \frac{n}{1!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!}\theta, \tag{1.6}$$

for some θ *in the range between* $\frac{1}{2}$ *and* $\frac{1}{3}$.

The relation (1.6) may be expressed in the form

$$a_n = \frac{1}{2}n!e^n + (1-\theta)n^n. \tag{1.7}$$

The sequence $\{a_n\}$ resurfaced in Exercise 1.2.11.3.18 of [8] in an urn problem,

There are n balls in an urn. How many selections with replacement are made, on average, if we stop when we reach a ball already selected?

with answer a_n/n^n. In relation to this question, D. Knuth introduces the functions

$$Q(n) = 1 + \frac{n-1}{n} + \frac{(n-1)(n-2)}{n^2} + \cdots \text{ and } R(n) = 1 + \frac{n}{n+1} + \frac{n^2}{(n+1)(n+2)} + \cdots,$$

with $Q(n)+R(n) = n!e^n/n^n$. To derive asymptotics of the function $Q(n)$, Ramanujan resorts to the integral representation

$$Q(n) = \int_0^\infty e^{-x}\left(1+\frac{x}{n}\right)^{n-1} dx. \tag{1.8}$$

More details on an asymptotic analysis of the sequence a_n can be found in [2] and [5].

The sequence a_n is listed as A063170 on OEIS and the name *Schenker sum* is given to it. The comments there include the integral representation

$$a_n = \int_0^\infty e^{-x}(x+n)^n \, dx, \tag{1.9}$$

due to M. Somos and the following conjecture by G. McGarvey for the 2-adic valuation of a_n.

Conjecture 1.1. *For $n \in \mathbb{N}$, we have*

$$\nu_2(a_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n - s_2(n) & \text{if } n \text{ is even.} \end{cases} \tag{1.10}$$

A primary focus of this paper is to establish the above conjecture and extend the discussion to odd primes.

2. The Proof

The proof starts with an elementary observation.

Lemma 2.1. *Suppose $A(x)$ is a polynomial with integer coefficients. Assume every coefficient is divisible by r. Then, the integer*

$$\int_0^\infty A(x)e^{-x} \, dx \tag{2.1}$$

is divisible by r.

Proof. Write $A(x) = a_0 + a_1x + \cdots + a_nx^n$ and observe that

$$\int_0^\infty A(x)e^{-x} \, dx = \sum_{j=0}^{n} a_j \, j! \tag{2.2}$$

is clearly divisible by r. □

The previous result shows that if $A(x) \equiv B(x) \bmod r$, then

$$\int_0^\infty A(x)e^{-x}\,dx \equiv \int_0^\infty B(x)e^{-x}\,dx \bmod r. \tag{2.3}$$

For the proof of the conjecture, the integral representation (1.9) will be useful. The process consists of two cases based on the parity of n.

`Case 1`: Suppose n is odd, say $n = 2m+1$. Now write $n = 1 + 2n_1 + \cdots + 2^r n_r$ in base 2 and raise $n + x \equiv 1 + x \bmod 2$ to the n-th power to produce

$$(n+x)^n \equiv (x+1)\prod_{i=1}^{r}(1+x)^{2^{k_i}} \equiv (1+x)\prod_{i=1}^{r}(1+x^{2^{k_i}}) \equiv 1 + x + O(x^w) \bmod 2,$$

with $w \geq 2$. Fermat's little theorem was employed in the second congruence. Then

$$\begin{aligned} a_n &= \int_0^\infty (n+x)^n e^{-x}\,dx \\ &\equiv \int_0^\infty \left(1 + x + O(x^w)\right) e^{-x}\,dx \bmod 2 \\ &\equiv \int_0^\infty (1+x)e^{-x}\,dx = 2 \equiv 0 \bmod 2. \end{aligned}$$

It follows that a_n is even. But a_n is not divisible by 4. Indeed, if m is even

$$a_n \equiv \int_0^\infty (1+x)e^{-x}\,dx = 2 \equiv 2 \bmod 4 \tag{2.4}$$

and for m odd,

$$a_n \equiv \int_0^\infty (3+x)^3 e^{-x}\,dx = 78 \equiv 2 \bmod 4. \tag{2.5}$$

This proves the conjecture when n odd.

`Case 2`: Suppose n is even, say $n = 2m$. Then

$$\begin{aligned} a_{2m} &= \int_0^\infty (2m+x)^{2m} e^{-x}\,dx \qquad (2.6) \\ &= \sum_{k=0}^{2m} \binom{2m}{k} (2m)^{2m-k} \int_0^\infty x^k e^{-k}\,dx \\ &= \sum_{k=0}^{2m} \binom{2m}{k} (2m)^{2m-k}\, k!. \end{aligned}$$

Let t_k be the summand in the last sum. Then $2mt_{k+1} = (2m-k)t_k$ and if $j = 2m-k$, this becomes

$$2mt_{2m-j+1} = jt_{2m-j}. \tag{2.7}$$

This recurrence is now utilized in expressing the coefficients t_i in terms of t_{2m} and also in analyzing the 2-adic valuation of each term in the sum for a_{2m}. For example, $j = 1$ yields $t_{2m-1} = 2mt_{2m}$, therefore

$$\nu_2(t_{2m-1}) = 1 + \nu_2(m) + \nu_2(t_{2m}) > \nu_2(t_{2m}). \tag{2.8}$$

Similarly, $j = 2$ yields $t_{2m-2} = 2m^2 t_{2m}$ from which it follows that $\nu_2(t_{2m-2}) > \nu_2(t_{2m})$ and $j = 3$ gives the relation $4m^3 t_{2m} = 3t_{2m-3}$ and $\nu_2(t_{2m-3}) > \nu_2(t_{2m})$ is obtained. In general

Lemma 2.2. *For $1 \leq j \leq 2m$, the inequality $\nu_2(t_{2m-j}) > \nu_2(t_{2m})$ holds.*

Proof. Define $u_j = t_{2m-j}$. Then (2.7) gives

$$2mu_{j-1} = ju_j. \tag{2.9}$$

From here it follows that

$$u_j = \frac{2m}{j} u_{j-1} = \frac{2m}{j} \cdot \frac{2m}{j-1} u_{j-2} \tag{2.10}$$

and iterating produces

$$u_j = \frac{(2m)^j}{j!} t_{2m}. \tag{2.11}$$

Now write

$$j! = 2^{\nu_2(j!)} O_*(j) = 2^{j-s_2(j)} O_*(j), \tag{2.12}$$

with $O_*(j)$ representing an odd number, to obtain

$$O_*(j) u_j = 2^{s_2(j)} m^j t_{2m}. \tag{2.13}$$

This gives

$$\nu_2(u_j) = s_2(j) + j\nu_2(m) + \nu_2(t_{2m}) > \nu_2(t_{2m}), \tag{2.14}$$

completing the proof as required. □

Note 2.3. Lemma 2.2 implies $\nu_2(a_{2m}) = \nu_2(t_{2m}) = \nu_2(n!) = n - s_2(n)$. This completes the analysis of Case 2 and establishes Conjecture 1.1.

3. The p-Adic Valuations for p an Odd Prime

In view of the results established in the previous section, it is natural to consider the question of what happens when p is an odd prime, i.e., is there a simple expression for $\nu_p(a_n)$ when $p \neq 2$ is a prime? The present section gives partial answers to this problem.

Proposition 3.1. *Let p be an odd prime and assume $n = pm$ for some $m \in \mathbb{N}$. Then*

$$\nu_p(a_n) = \frac{n - s_p(n)}{p-1}. \tag{3.1}$$

Proof. Consider the integral expression

$$a_{pm} = \sum_{k=0}^{pm} \binom{pm}{k} (pm)^{pm-k} \int_0^\infty x^k e^{-x}\, dx = \sum_{k=0}^{pm} \binom{pm}{k} (pm)^{pm-k} k! \tag{3.2}$$

and let

$$t_{m,p}(k) = \binom{pm}{k} (pm)^{pm-k}\, k! \tag{3.3}$$

be the summand in (3.2). Observe that $t_{m,p}(mp) = (pm)!$. Pursuant, the case $p = 2$, suppose that

$$\nu_p(t_{m,p}(k)) > \nu_p(t_{m,p}(pm)) = \nu_p(n!). \tag{3.4}$$

Then

$$\nu_p(a_{pm}) = \nu_p(n!) = \frac{n - s_p(n)}{p-1}, \tag{3.5}$$

as claimed.

The proof of (3.4) begins with the computation of the ratio of two consecutive terms $t_{m,p}$ to produce the relation

$$pm\, t_{m,p}(k+1) = (pm-k) t_{m,p}(k). \tag{3.6}$$

The proof then proceeds as in the case $p = 2$. □

The next result is a crucial reduction towards the modular arithmetic employed in the computation of $\nu_p(a_n)$.

Proposition 3.2. *Let p be a prime and $n = pm + r$ with $0 < r < p$. Then $p|a_n$ if and only if $p|a_r$.*

Proof. The reduction

$$(x+n)^n \equiv (x+r)^{pm}\,(x+r)^r \equiv (x^{pm} + r^{pm})(x+r)^r \equiv (x^{pm} + r^m)(x+r)^r \bmod p,$$

is due the fact that p divides $\binom{pm}{k}$ for any $0 < k < pm$. This implies

$$\begin{aligned} a_n &= \int_0^\infty (x+n)^n e^{-x}\,dx \\ &\equiv \int_0^\infty (x^{pm}+r^m)(x+r)^r e^{-x}\,dx \\ &= \sum_{j=0}^{r} \binom{r}{j} r^{r-j} \int_0^\infty (x^{pm}+r^m)\,x^j e^{-x}\,dx \\ &= \sum_{j=0}^{r} \binom{r}{j} r^{r-j}\,[(pm+j)! + r^m j!] \\ &\equiv \sum_{j=0}^{r} \binom{r}{j} r^{m+r-j}\,j! \\ &\equiv \sum_{j=0}^{r} \binom{r}{j} r^{m+j}\,(r-j)! \\ &\equiv r^m \sum_{j=0}^{r} \frac{r!}{j!} r^j \\ &\equiv r^m a_r \bmod p. \end{aligned}$$

The assertion follows. □

Before embarking on the more general study, it is worthwhile to consider some toy examples (small primes). The reader will hopefully find these illustrative of the potential subtleties and obstacles.

Example 3.3. Let $p = 3$. Proposition 3.1 gives

$$\nu_3(a_{3n}) = \frac{1}{2}(3n - s_3(n)). \tag{3.7}$$

The remaining two cases are established by Proposition 3.2. Assume $n = 3m + r$ with $r = 1, 2$. Then $3|a_n$ if and only if $3|a_r$. Neither $a_1 = 2$ nor $a_2 = 10$ are divisible by 3, and therefore 3 does not divide a_n.

In summary,

$$\nu_3(a_n) = \begin{cases} \frac{1}{2}(n - s_3(n)) & \text{if } n \equiv 0 \bmod 3 \\ 0 & \text{if } n \not\equiv 0 \bmod 3. \end{cases} \tag{3.8}$$

Example 3.4. Let $p = 5$. This brings in the first difficult problem. Start with the simpler cases. Proposition 3.1 ensures that

$$\nu_5(a_{5n}) = \frac{1}{4}(n - s_5(n)). \tag{3.9}$$

By Proposition 3.2 and since none of the numbers $a_1 = 2$, $a_3 = 78$, $a_4 = 824$ is divisible by 5, the following holds

$$\nu_5(a_n) = 0 \quad \text{if } n \equiv 1,\, 3,\, 4 \bmod 5. \tag{3.10}$$

The remaining case $\nu_5(a_{5n+2})$ requires a closer look. A preliminary discussion is presented in the next section.

Example 3.5. Let $p = 7$. Because the first six numbers $a_1 = 2$, $a_3 = 78$, $a_4 = 824$, $a_5 = 10970$, $a_6 = 176112$ are not divisible by 7, it follows that

$$\nu_7(a_n) = \begin{cases} \frac{1}{6}(n - s_7(n)) & \text{if } n \equiv 0 \bmod 7 \\ 0 & \text{if } n \not\equiv 0 \bmod 7. \end{cases} \tag{3.11}$$

A direct computation of the values of a_j modulo 11 shows that a_j is not divisible by 11 for $1 \leq j < 11$. Therefore

$$\nu_{11}(a_n) = \begin{cases} \frac{1}{10}(n - s_{11}(n)) & \text{if } n \equiv 0 \bmod 11 \\ 0 & \text{if } n \not\equiv 0 \bmod 11. \end{cases} \tag{3.12}$$

The case $p = 13$ is similar to $p = 5$ since 13 divides $a_3 = 78$.

4. Schenker Primes

The results established in the previous sections determine the valuation $\nu_p(a_n)$ for a class of prime numbers. The primes not completely covered by those methods are fall under a special category as defined below.

Definition 4.1. A prime p is called a *Schenker prime* if p divides a_r for some value r in the range $1 \leq r \leq p - 1$.

The result is summarized in the next theorem.

Theorem 4.2. *Let p be a prime and assume that p is not a Schenker prime. Then*

$$\nu_p(a_n) = \begin{cases} \frac{1}{p-1}(n - s_p(n)) & \textit{if } n \equiv 0 \bmod p \\ 0 & \textit{if } n \not\equiv 0 \bmod p. \end{cases} \tag{4.1}$$

Example 4.3. The prime $p = 17$ is not a Schenker prime. The factorization of the numbers a_r, for $1 \leq r \leq 16$ is

$a_1 = 2$	$a_2 = 2 \cdot 5$
$a_3 = 2 \cdot 3 \cdot 13$	$a_4 = 2^3 \cdot 103$
$a_5 = 2 \cdot 5 \cdot 1097$	$a_6 = 2^4 \cdot 3^2 \cdot 1223$
$a_7 = 2 \cdot 5 \cdot 7 \cdot 41 \cdot 1153$	$a_8 = 2^7 \cdot 556403$
$a_9 = 2 \cdot 3^4 \cdot 149 \cdot 163 \cdot 439$	$a_{10} = 2^8 \cdot 5^2 \cdot 7281587$
$a_{11} = 2 \cdot 11 \cdot 9431 \cdot 6672571$	$a_{12} = 2^{10} \cdot 3^5 \cdot 5^3 \cdot 1443613$
$a_{13} = 2 \cdot 13 \cdot 179 \cdot 339211523363$	$a_{14} = 2^{11} \cdot 7^2 \cdot 595953719897$
$a_{15} = 2 \cdot 3^6 \cdot 5^3 \cdot 317 \cdot 13103$	$a_{16} = 2^{15} \cdot 13 \cdot 179 \cdot 116371 \cdot 11858447$

The prime $p = 17$ does not appear in any of these factorizations confirming that it is not a Schenker prime. In accord with Theorem 4.2, the 17-adic valuation of the sequence a_n is explicit:

$$\nu_{17}(a_n) = \begin{cases} \frac{1}{16}(n - s_{17}(n)) & \text{if } n \equiv 0 \bmod 17 \\ 0 & \text{if } n \not\equiv 0 \bmod 17. \end{cases} \tag{4.2}$$

Example 4.4. The prime 5 is a Schenker prime because 5 divides $a_2 = 10$. Similarly 37 is a Schenker prime since 37 divides a_{25}. The list of all Schenker primes up to 200 is

$$\{5, 13, 23, 31, 37, 41, 43, 47, 53, 59, 61, 71, 79, 101, 103, \\ 107, 109, 127, 137, 149, 157, 163, 173, 179, 181, 191, 197, 199\}. \tag{4.3}$$

Note 4.5. The valuation $\nu_5(a_n)$ is not obvious or as simple, so finding an analytic/explicit formula for it stands as an open question. *The description given below is purely experimental and no proofs are available at the moment.* The only rigorous result is Example 3.4, which determines the value of $\nu_5(a_n)$ except for indices congruent to 2 modulo 5.

The indices of the form $5n + 2$ are first divided according to the parity of n modulo 5. Symbolic computations show that

$$\nu_5(a_{5n+2}) = 1 \text{ for } n \not\equiv 2 \bmod 5. \tag{4.4}$$

Therefore it is now required to consider indices of the form

$$m_1 = 5(5n + 2) + 2 = 5^2 n + 5 \cdot 2 + 2. \tag{4.5}$$

Then it is observed that

$$\nu_5(a_{5^2 n+5\cdot 2+2}) = 2 \text{ for } n \not\equiv 0 \bmod 5, \tag{4.6}$$

leading to indices of the form

$$m_2 = 5^3 n + 5 \cdot 2 + 2. \tag{4.7}$$

Continuing this process, it is then observed that

$$\nu_5(a_{5^3 n+5\cdot 2+2}) = 3 \text{ for } n \not\equiv 4 \bmod 5, \tag{4.8}$$

leading to indices of the form

$$m_3 = 5^4 n + 5^3 \cdot 4 + 5^2 \cdot 0 + 5^1 \cdot 2 + 2, \tag{4.9}$$

and also

$$\nu_5(a_{5^4 n+5^3\cdot 2+5\cdot 2+2}) = 4 \text{ for } n \not\equiv 4 \bmod 5, \tag{4.10}$$

leading to

$$m_4 = 5^5 n + 5^4 \cdot 4 + 5^3 \cdot 4 + 5^2 \cdot 0 + 5^1 \cdot 2 + 2. \tag{4.11}$$

This process can be described in terms of the expansion of the index n in base 5 in the form

$$n = x_0 + x_1 \cdot 5 + x_2 \cdot 5^2 + x_3 \cdot 5^3 + x_4 \cdot 5^4 + \cdots \tag{4.12}$$

The results of Example 3.4 for $\nu_5(a_n)$ are

$$x_0 = \begin{cases} 0 & \nu_5(a_n) = \frac{1}{4}(n - s_5(n)) \\ 1,\, 3,\, 4 & \nu_5(a_n) = 0 \\ 2 & \nu_5(a_n) \text{ depends on } x_1. \end{cases} \tag{4.13}$$

The next steps are

$$x_0 = 2 \text{ and } x_1 = \begin{cases} \neq 2 & \nu_5(a_n) = 1 \\ 2 & \text{depends on } x_2, \end{cases} \tag{4.14}$$

and

$$x_0 = 2,\, x_1 = 2 \text{ and } x_2 = \begin{cases} \neq 0 & \nu_5(a_n) = 2 \\ 0 & \text{depends on } x_3, \end{cases} \tag{4.15}$$

and

$$x_0 = 2,\, x_1 = 2,\, x_2 = 0 \text{ and } x_3 = \begin{cases} \neq 4 & \nu_5(a_n) = 3 \\ 4 & \text{depends on } x_4. \end{cases} \tag{4.16}$$

The next conjecture has been verified numerically, for the prime $p = 5$, up to depth/level 10.

Conjecture 4.6. *Assume the valuation $\nu_5(a_n)$ is not determined by the first r digits of n; that is $x_0, x_1, \cdots, x_{r-1}$ do not determine $\nu_5(a_n)$. Then, among the 5 possible values for x_r, there is a single value for which the valuation is not determined by $x_0, x_1, \cdots, x_{r-1}, x_r$.*

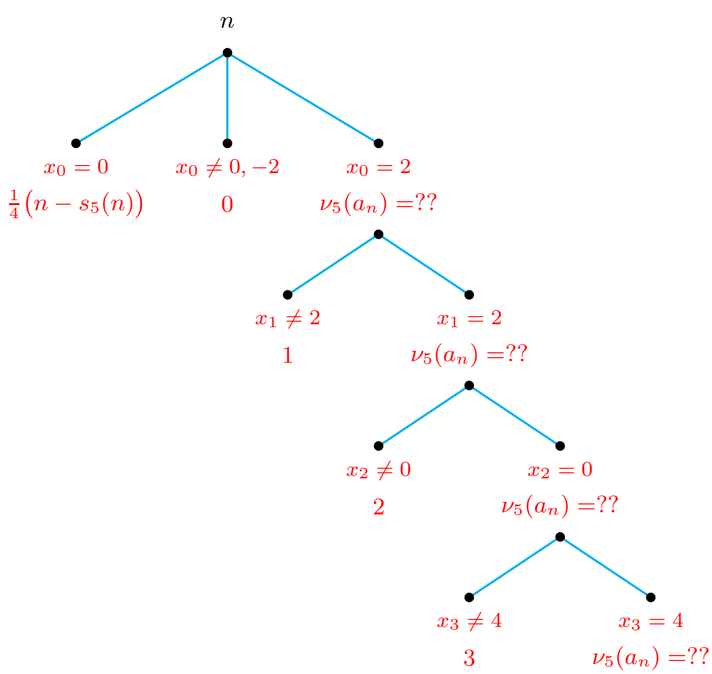
Note 4.7. Denote by d_j the j-th exceptional digit in Conjecture 4.6. The list of these digits begins with

$$d_0 = 2,\, d_1 = 2,\, d_2 = 0,\, d_3 = 4,\, d_4 = 4. \tag{4.17}$$

A similar conjecture has been proposed in [1] and [3] for the p-adic valuation of Stirling numbers of the second kind. Conjecture 4.6 is rephrased using *valuation trees.*

Tree construction. The tree starts with a top vertex v_0 labeled n that represents all of $\mathbb{N}$. This top vertex forms the *first level* of the tree. The expansion of n in base 5 in (4.12) is employed in the description of this tree.

From the top vertex, form the *second level* consisting of 5 vertices connected to v_0. Each vertex corresponds to a value of x_1 in the expansion of n in base 5. The figure shows three types of vertices: those with $x_0 = 0$ for which $\nu_5(a_n) = \frac{1}{4}(n - s_5(n))$ (shown to the left of the tree), those with $x_0 \neq 0, 2$ for which $\nu_5(a_n) = 0$ (shown at the center) and finally those vertices with $x_0 = 2$ for which the valuation $\nu_5(a_n)$ is not determined by x_0. In this form, each vertex represents a subset of $\mathbb{N}$ determined by some property of the digits x_i. Each vertex has a symbol indicating the type of digit x_i it represents (to be more precise all the properties determining this subset is obtained by reading the path from the top vertex to the vertex in question) and also the valuation $\nu_5(a_n)$ for those indices n associated to the vertex.



The valuation tree for $p = 5$

The discussion that follows excludes the vertex corresponding to $x_0 = 0$. The valuation for the indices corresponding to this vertex are determined by Proposition 3.1.

Definition 4.8. A vertex is called *terminal* if the valuation is the same for all indices associated to the vertex.

Example 4.9. All indices n associated to the vertex corresponding to $x_0 = 1$ have valuation $\nu_5(a_n) = 0$; that is $\nu_5(a_{5n+1}) = 0$. Therefore this vertex is terminal. On the other hand, if $n = 7$ then

$$\nu_5(a_7) = \nu_5(3309110) = 1 \tag{4.18}$$

and

$$\nu_5(a_{17}) = \nu_5(4845866591896268695010) = 3. \tag{4.19}$$

Both indices 7 and 17 are associated to the vertex with $x_0 = 2$ and they have different valuation. Therefore this is not a terminal vertex.

Note 4.10. The first level consists of the vertex with $x_0 = 0$, excluded from this discussion, the three vertices with $x_0 = 1, 3, 4$ (shown as one single vertex in the tree), and the vertex with $x_0 = 2$. This last vertex produces 5 new ones that form the second level. These five vertices correspond to indices with $x_0 = 2$ and $0 \leq x_1 \leq 4$. Each of them have a set of indices attached to them, for instance $x_1 = 2$ correspond to indices of the form $n = x_0 + 5x_1 + 5^2m = 2 + 5 \cdot 2 + 5^2m = 12 + 25m$. This describes the construction of the valuation tree: non-terminal vertices produce 5 new vertices at the next level.

Definition 4.11. The tree constructed above, extended naturally by simply replacing 5 by a prime p, is called *the valuation tree for* p.

The structure of this valuation tree described in the next conjecture generalizes Conjecture 4.6.

Conjecture 4.12. *Assume* p *is a Schenker prime. Then each level of the valuation tree for* p *contains a single non-terminal vertex.*

5. The Combinatorics of a_n

The arithmetic properties of the sequence a_n discussed in the earlier sections are based on the integral representation (1.9). In this section, the Abel-type identity Theorem 5.1 gives an alternative binomial representation of a_n. There is an extensive literature on Abel's identity and its numerous variants (see, for example, [9] and its references). Here, we give two short direct proofs, one analytic and one bijective.

Theorem 5.1. *The following identity provides two different formulation for the sequence* a_n*:*

$$\sum_{k=0}^{n} \frac{n!}{k!} n^k = \sum_{k=0}^{n} \binom{n}{k} k^k (n-k)^{n-k}. \tag{5.1}$$

Proof. Define

$$\begin{aligned} A_n(t) &= t\sum_{k=0}^{n}\binom{n}{k}(t+k)^{k-1}(n-k)^{n-k} \qquad (5.2)\\ B_n(t) &= \sum_{k=0}^{n}\frac{n!}{(n-k)!}(t+n)^{n-k},\\ C_n(t) &= \sum_{k=0}^{n}\binom{n}{k}(t+k)^{k}(n-k)^{n-k}. \end{aligned}$$

The relation $(t+k)^k = t(t+k)^{k-1} + k(t+k)^{k-1}$ gives

$$C_n(t) = A_n(t) + nC_{n-1}(t+1). \qquad (5.3)$$

The value

$$A_n(t) = (t+n)^n \qquad (5.4)$$

follows directly from Abel's identity

$$\sum_{k=0}^{n}\binom{n}{k}(t+k)^{k-1}(s-k)^{n-k} = \frac{(t+s)^n}{t}, \qquad (5.5)$$

Then

$$C_n(t) = (t+n)^n + nC_{n-1}(t+1), \qquad (5.6)$$

and it is easy to check that B_n also satisfies this recurrence. Since both B_n and C_n have the same initial conditions, it follows that $B_n(t) = C_n(t)$. The stated result now comes from $B_n(0) = C_n(0)$. □

Note 5.2. A nice proof of Abel's identity (5.5) appears in [4]. A nice combinatorial interpretation may be found in [6, 7] with the following picturesque formulation. Whereas the binomial identity $(t+s)^n = \sum_{k=0}^{n}\binom{n}{k}t^k s^{n-k}$ counts functions $f : [n] \to [t+s]$ by the number of elements that *map directly to* $[t]$, that is, by number of elements $i \in [n]$ for which $f(i) \in [t]$, (5.5) counts these same functions by the number of elements that *ultimately map to* $[s+1, s+t]$, that is, by number of elements $i \in [n]$ for which $\underbrace{f\circ f\circ \ldots \circ f}_{m}(i) \in [s+1, s+t]$ for some $m \geq 1$ (assuming $s \geq n$ so that all summands are nonnegative).

The identity $B_n(t) = C_n(t)$ implies another well known identity.

Corollary 5.3.

$$n! = \sum_{r=0}^{n}(-1)^r\binom{n}{r}(n-r)^n. \qquad (5.7)$$

Proof. Matching powers of t in $B_n(t-n) = C_n(t-n)$ gives

$$\frac{n!}{k!} = (-1)^k \sum_{r=k}^{n} (-1)^r \binom{n}{r}\binom{r}{k}(n-r)^{n-k} \tag{5.8}$$

and the special case $k = 0$ gives the result. □

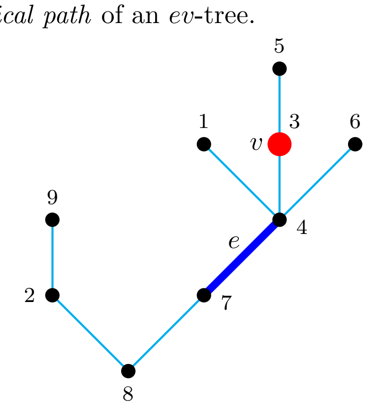
Note 5.4. An elementary combinatorial proof of (5.7) is obtained by counting all the $n!$ bijective functions on a set of n elements. The right-hand side employs the inclusion-exclusion principle by excluding maps according to the number of elements missed in the range.

Note 5.5. Theorem 5.1, after canceling some equal terms, is equivalent to the identity,

$$\sum_{k=2}^{n} \frac{n!}{(n-k)!} n^{n-k} = \sum_{k=1}^{n-1} \binom{n}{k} k^k (n-k)^{n-k} \tag{5.9}$$

for which we now give a combinatorial interpretation.

We will show that (5.9) counts a class of rooted trees in two different ways. Let us say a vertex in a rooted tree is a *descendant* of an edge in the tree if the path from the vertex to the root includes the edge. Define an *ev-tree* to be a rooted vertex-labeled tree on $[n]$ *with a highlighted edge* e *and a marked descendant* v *of* e, as illustrated below with $n = 9$. Call the (unique) path starting at edge e and ending at vertex v the *critical path* of an *ev*-tree.



An *ev*-tree

In the example, $e = 74$ (in blue) and $v = 3$ (in red). The descendants of e are $4, 1, 3, 6, 5$ and the critical path is $7 \to 4 \to 3$.

The left side of (5.9) counts *ev*-trees by the length k of the critical path as follows. Choose the k vertices that occur on the critical path—$\binom{n}{k}$ choices. Form a forest of trees on $[n]$ rooted at these k vertices—kn^{n-k-1} choices [11, Proposition 5.3.2]. Put a cycle structure on the k roots—$(k-1)!$ choices. Mark one of the vertices in the forest—n choices. Turn the cycle of roots into a path, $r_1 \to r_2 \to \cdots \to r_k$, by starting at the root of the tree containing the marked vertex. Ignoring the orientation of edges in this path, we now have a tree rooted at the marked vertex. Take e to be the edge $r_1 r_2$ and v to be the vertex r_k. This is the desired ev-tree and by construction, there are $\binom{n}{k} \cdot kn^{n-k-1} \cdot (k-1)! \cdot n = \frac{n!}{(n-k)!} n^{n-k}$ of them.

On the other hand, the right side of (5.9) counts ev-trees by the number k of descendants of e as follows. Choose the descendants of e—$\binom{n}{k}$ choices—and form a rooted tree on these vertices with one vertex colored blue—k^k choices, because Cayley's formula says there are k^{k-1} rooted trees. Similarly, form a rooted tree on the remaining $n-k$ vertices with one vertex colored blue—$(n-k)^{n-k}$ choices. Now join the two blue vertices with a blue edge and change the root of the first tree to a red vertex. The result will form an ev-tree by taking the blue edge as e and the red vertex as v.

Actually, identity (5.9) can be sharpened. Every term on the left side is obviously divisible by n and it is a fact, not quite so obvious, that every term on the right is also divisible by n. So we can divide by n to get another integer identity,

$$\sum_{k=2}^{n} \frac{(n-1)!}{(n-k)!} n^{n-k} = \sum_{k=1}^{n-1} \frac{1}{n} \binom{n}{k} k^k (n-k)^{n-k}. \tag{5.10}$$

Basically the same interpretation works for (5.10): simply observe that incrementing the vertex labels, $i \to i+1 \pmod{n}$, in an ev-tree leaves the statistics "number of descendants of the highlighted edge" and "length of the critical path" invariant, and partitions the class of ev-trees on $[n]$ into orbits, each of which has size n. So just pick out the ev-tree in each orbit whose root is, say, 1. Note that this argument provides a combinatorial proof that the summand on the right side of (5.10) is an integer.

Note 5.6. The sums in (5.10) give $(a_n)_{n\geq 1} = (1, 8, 78, 944, \ldots)$, A000435, "the sequence that started it all". A comment on A000435 by Geoffrey Critzer says that a_n, for $n > 1$, is the number of connected endofunctions on $[n]$ with no fixed points, that is, functions $f : [n] \to [n]$ with only one orbit of periodic points (connected) whose length is ≥ 2 (no fixed points). In fact, ev-trees with root 1 are just another way of looking at these endofunctions.

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