TWO-COLOR BABYLON

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Abstract
We solve the game of Babylon when played with chips of two colors, giving a winning strategy for the second player in all previously unsolved cases.

1. Introduction

The game of Babylon begins with n chips of various colors, spread out as "stacks" of height 1. Players Alice and Bob alternate combining two stacks which have the same number of chips or the same color on top (or both). The last player to make a legal move wins.

The game was introduced in 2003 by French designer Bruno Faidutti [2] and is available in commercial form, with 12 chips in four colors, three of each color. (Of course, the game can be played with ordinary poker chips.) Computer-aided analysis [1, 3, 5] has shown that with best play the aforementioned initial configuration of the commercial game is a win for the second player.

The simplicity and attractiveness of Babylon has led researchers to try to solve the game in general, and in particular the cases where the tokens are of only two colors. Goadrich and Schlatter [4] show that the first player (Alice) can always win if there are only one or two chips of the minority color, or if n is odd; they conjectured that Bob can win in the remaining, most difficult cases, when n is even and there are three or more chips of the minority color. We prove their conjecture, completing the winning-player classification for two-color Babylon. Our strategy for Bob is explicit and easy to implement in practice.

2. General Observations

Alice wins any game of Babylon if the total number of moves is odd; Bob, if even. Since the number of stacks decreases by one at each player's turn, the number of moves is at most $n-1$. In two-color Babylon, the only other possibility is that there are $n-2$ moves, ending with two stacks of different height and color (the color of a stack is naturally defined to be the color of its top chip; the colors of its other chips cannot affect the game any more).

The one-color game is of course trivial, with Alice winning when n is even and Bob when n is odd. If n is even and there are only one or two chips of the minority color (which, continuing the convention of [4], is red), Alice wins easily by obliterating the red chips; if there's only one she covers it with a blue chip, and if there are two she stacks them and at her next move covers them with the blue 2-stack that Bob was forced to make at his first move.

If n is odd and there are one or two red chips, Alice needs to keep them alive; this is easily done by stacking red on blue (if there is one red chip) or stacking the two red chips, then stacking the result on Bob's blue 2-stack. Thereafter if Bob ever threatens the red stack by building a blue one of the same height, Alice piles her red stack on top of the challenging blue one. Thus the size of the red stack is always a power of 2, and at least 4. There is never any danger that Alice will herself be forced to make a blue stack of the same height as the red one, because this can only happen if all blue stacks are exactly half the height of the red one; but that is impossible because n is odd.

Thus, Alice wins all two-color games that begin with one or two red chips. When n is odd and there are three or more red chips, Alice again needs to preserve both colors; and again, she can easily do so, as shown in [4]. Thus the critical remaining case is when there are three or more red chips and n is even. This time it is Bob who needs to keep both colors alive. The difficulty of this case, relative to the odd-n case, is twofold: (1) Bob lacks the power of first move, and (2) Bob must take care not to create, or permit Alice to create, a stack of height $n/2$ that is the only stack of its color.

We will show that Bob can overcome these difficulties and win in all cases, completing the classification for two-color Babylon.

3. Notation

We use the symbol b to denote a blue chip, r a red chip; x is a chip of either color and y a chip of the *other* color; z stands for a chip of any color. We will be using strings of the above symbols to describe stacks, with the left-most symbol standing for the chip on top. A *hill* is a stack of height at least two. When there is no possibility

of confusion we often denote a hill by the capitalized form of the notation for its color, thus we could write R instead of rbb and X instead of $xxyz$.

In this notation a *move* by either player consists of appending one string (stack) to another, always with the restriction that the two stacks were of the same color or the same height.

Since the other cases are solved, we will be concerned exclusively with games in which there are initially an even number $2m$ of chips, each its own stack, and at least three of the minority color (red). The parity of the number of stacks at any state of the game tells us who is next to move; Alice moves next from any even state, Bob from any odd state.

A state will be termed *safe* if it leads to a win by Bob, with best play. Equivalently, we may say that an even state in which Alice cannot move (i.e., a two-stack state consisting of red and blue hills of different heights) is safe, that an even state is safe if any move of Alice produces a safe state, and an odd state is safe if there is some move by Bob leading to another safe state. Our objective is to show that the initial state is safe.

4. Main Results

Theorem 1. *Two-color Babylon with p chips of one color and q of the other, $1 \leq p \leq q$, is a win for the second player if and only if $p+q$ is even and $p \geq 3$.*

Proof. As noted in the introduction, owing to the results in [4] we need only provide a winning strategy for Bob when $p+q = 2m$ and $p \geq 3$. We will show that Bob can win by creating even states in which there is at most one hill of each color, until an "end game" is reached in which he has to be a bit more careful.

A *singleton* is a stack that is not a hill, that is, one consisting of just one chip. A state with j red singletons, k blue singletons, red hills of heights $u_1, u_2, \ldots$ and blue hills of heights $v_1, v_2, \ldots$ will be denoted by $\langle j, k; u_1, u_2, \ldots; v_1, v_2, \ldots \rangle$. The initial configuration is thus $\langle p, q; ; \rangle$.

Even states with exactly one hill of each color, each of even height, and at least four singletons of each color, will be called *target* states. Thus, a target state with j red singletons, k blue singletons, a red hill of height $2u$ and a blue hill of height $2v$ will be denoted by $\langle j, k; 2u; 2v \rangle$.

Alice's options at a target state, and our notations for them, are:

1. place a singleton on top of another singleton: xz
2. place a singleton on top of the hill of its color (or vice-versa): xX
3. place one hill on top of the other (provided they have the same height): XY.

Bob normally answers xz with xzX, xX with xX and XY with yy, always creating a new even state in which there is again exactly one hill of each color,

each of even height. We call this kind of play by Bob the *even-hill-strategy*. In particular, $\langle j, k; 2u; 2v \rangle$ transforms to either $\langle j{-}2, k; 2u{+}2; 2v \rangle$, $\langle j, k{-}2; 2u; 2v{+}2 \rangle$, $\langle j{-}1, k{-}1; 2u{+}2; 2v \rangle$, $\langle j{-}1, k{-}1; 2u; 2v{+}2 \rangle$, $\langle j{-}2, k; 2; 2v{+}2u \rangle$, or $\langle j, k{-}2; 2u{+}2v; 2 \rangle$. Since the number of singletons of one or both colors may have been reduced below four, the new state might not be a target state.

Observe that if $p, q \geq 6$ then Bob can create a target state at his first move: if Alice opens with xx he replies with yy, and if she opens with xy he replies with yx. Either way the result is target state $\langle p{-}2, q{-}2; 2, 2 \rangle$.

Before proceeding further it will be useful to understand what happens when there is only one stack left of one of the colors.

Lemma 2. *Any even state with just one stack X of color x and at least one of color y is safe, provided the height of X exceeds m.*

Proof. Nothing can ever be combined with X, so both colors survive and Bob wins. $\square$

Lemma 3. *Any odd state with just one stack X of color x is safe, provided the height u of X is less than m, not all y-stacks are of height $u/2$, and at most two are of height u.*

Proof. We proceed by induction on the (even) number t of y-stacks. If there are two y-stacks of height u, Bob stacks them; if one, he stacks it with any other y-stack. All y-stacks of height u are thus killed. In all other cases, Bob makes the largest y-stack possible of height $u' \neq u$. (Since the y-stacks are not all of height $u/2$, either the tallest two or the shortest two will combine to make a stack of height $\neq u$; thus u' is well-defined.) Now Alice cannot change the height of X, nor can she suddenly make three y-stacks of height u. Can she cause all the y-stacks to become of height $u/2$? No: if there had been a y-stack of height $u/2$ before we would now have $u' > u/2$. So we are reduced to the cases where Alice has created a first or second y-stack of height $u/2$. Since Alice leaves an even number of y stacks, and $u + u/2 + u/2 = 2u < 2m = n$, there must be a stack of height different than $u/2$ after her move. $\square$

Let us return to the proof of the main theorem. Bob's strategy is to continue applying the even-hill-strategy as long as it leaves a target state for Alice, i.e., it leaves one hill of each color, each of even height, and at least four singletons in each of the colors. Now consider the first position S faced by Bob such that playing from S according to the even-hill-strategy would leave less than four singletons in one of the colors. If following the even-hill-strategy would yield a position with 3 singletons in one color and at least 3 in the other color (which happens if $S = \langle 3, k; 2u, 2; 2v \rangle$ or $S = \langle 3, k; 2u; 2v, 2 \rangle$ with $k \geq 3$, Alice having just created a stack of height 2, or

if $S = \langle 5, k; 0; 2v \rangle$, Alice having just played BR), then Bob wins by the following lemma.

Lemma 4. *Any even state of the form $\langle 3, k; 2u; 2v \rangle$ with $k \geq 3$ and $u + v \geq 3$ is safe.*

Proof. We analyze all possible replies by Alice. In each case Bob must be careful not to allow Alice to create a solo stack of height m. His moves must thus take into account the value of $2u$ (of concern when at or just below m) and the relative values of $2u$ and $2v$ (of concern when they are both around $m/2$).

  (i) Alice replies rb: By Lemma 5 Bob wins by rbR unless $2u + 4 = m$ and $k > 3$. If the latter is the case, then Alice threatens to create a single red stack containing half the chips. But Bob can reply with br, and it is now easy for him to make sure that both colors survive; see Lemmas 2 and 3.

 (ii) Alice replies br: Arguing as in the previous case, Bob can play brB unless $2u + 2 = m$ and $k > 3$. Note that in the latter case $u > 1$ must hold for $u + v \geq 3$. Bob can then play br, threatening rR in his next move which would finish the game by Lemma 3. Alice can try to prevent this by rb, but then $brrb$ wins for Bob by Lemma 3 again.

(iii) Alice replies rr: If $2u + 3 = m$ then br is an easy win for Bob. If $u = v$ then BR is easily seen to be safe using Lemma 3. If neither of the above is the case, then rrr is safe as it allows Bob to create a single red stack of height $2u + 3 \neq m$ on his next move.

(iv) Alice replies rR: If $2u + 3 = m$, or $2u + 3 = m - 2$ (the latter possibility is dangerous if and only if $v = 1$) then br leads to an easy win for Bob. If $2u + 3 = m + 2$ then Bob replies rR. Otherwise rr wins.

 (v) Alice replies RB: If the height of the new hill is $m-3$ or $m-2$ then br is safe. If it is $m-1$ then rr is safe. The other cases are easy.

(vi) Alice replies BR: Bob easily wins by rr by making sure that red survives.

(vii) Alice replies bB or bb: We have $k > 3$ without loss of generality, for otherwise we can switch colors. Thus in both cases, bbB creates a position of the same form as the original one since k is odd and so at least 5, and we proceed by induction.

This completes the proof of the lemma. $\square$

   Thus, it only remains to consider positions S from which the even-hill-strategy would yield a position with exactly 2 singletons in one color and S has not been

preceeded by a position with exactly 3 singletons in that color, for we have already seen that such positions are won by Bob. Since Alice's last state was a target state, such an S must have one of the following forms (or its red-blue reflection):

1. $\langle 3, k; 2u+1; 2v \rangle$ with $k \geq 4$ (if Alice just played rR);

2. $\langle 2, k; 2u, 2; 2v \rangle$ with $k \geq 4$ and even (if Alice just played rr);

3. $\langle 4, k; 0; 2v \rangle$ with $k \geq 4$ and even (if Alice just played BR).

These cases will be easily solved using the following lemma.

Lemma 5. *Any even state of the form $\langle 2, 2s; 2u; 2v \rangle$ with $u, v \geq 1$ is safe unless $2u + 2 = m$ and $s > 1$.*

Proof. We analyze all possible moves of Alice from this state.

(i) Alice plays BR: (Provided $u = v$.) Bob plays rr and can make sure that the rr stack survives to the end by immediately obliterating any blue stack of size 2 Alice creates.

(ii) Alice plays RB: If the height $2u+2v$ of the new stack X is greater than m or less than $m - 2$ then Bob plays rr and wins easily by Lemma 2 or Lemma 3. Otherwise we have to be slightly more careful to avoid a red stack of height m: if $2u+2v = m$ play rR, if $2u+2v = m-1$ then play rr, and if $2u+2v = m-2$ play br and counter rb with $brrb$.

(iii) Alice plays bb: We may assume $s = 1$ without loss of generality, since otherwise bbB leads to a state of the same form. If $2v + 2 \neq m$ then bbB wins. If $2v + 2 = m$, in which case $u = v$ holds, we win by BR.

(iv) Alice plays bB: As before, we may assume $s = 1$ without loss of generality. If $2v + 2 \neq m$ then bbB wins as above. If $2v + 2 = m$ then rb wins.

(v) Alice plays br: If $2u = m - 1$, then $u > 1$ because we have at least 8 chips in total. It is now easy to see that br wins. If $2u \neq m - 1$ then rR wins by Lemma 2 or Lemma 3.

(vi) Alice plays rb: Bob easily wins by playing rrb, unless $2u = m - 3$ in which case he can safely play br.

(vii) Alice plays rR: Bob easily wins by playing rrb, unless $2u = m - 2$ in which case he can safely play br.

(viii) Alice plays rr: This is the only case in which, under certain circumstances, Bob can lose. Indeed, if $2u + 2 = m$, then Alice threatens to combine all red chips in a stack of height m, winning the game. The only way for Bob to

defend is to capture the red hill of height $2u$ with the blue hill, and he can do so if and only if $u = v$ which implies $s = 1$.

If $2u + 2 \neq m$ then playing rrR reaches a safe position by Lemma 2 or 3.

This completes the proof of the lemma.   □

We can now analyze the positions 1-3 above. For position 1 note that, by Lemma 5, rR is safe unless $2u + 4 = m$. If the latter is the case, then br will easily allow Bob to create a single red stack of height different than m, winning by Lemmas 2 and 3. Position 2 is very similar and we leave the details to the reader. Finally, in position 3 Bob can play rr and win by Lemma 5 since $m > 4$.

This completes the proof in all cases when $p \geq 6$, and so Bob can create a target state at his first move as already mentioned. It remains to consider the simple cases when $p = 3, 4$ or 5.

When $p = 5$, Bob has to deviate from his general strategy only slightly. If Alice opens with bb or rb then he replies with rb and bb respectively. This creates a target state unless the number of blue chips q is also 5. But in the latter case Lemma 5 applies, with blue being the minority color now. If Alice opens with br then rr wins by Lemma 5 again. Finally, if Alice opens with rr then rr will quickly allow Bob to form a single red stack, winning by Lemma 2 or Lemma 3.

When $p = 4$ then Bob can play his first move according to his even-hill strategy for the general case unless $q = 4$; this immediately reaches a position $\langle 2, 2s; 2; 2 \rangle$, which is a win by Lemma 5. If $q = 4$ then Bob can force the position $\langle 1, 3; 2; 2 \rangle$ after the first move, and it is easy to win that position.

Finally, the case $p = 3$ is an easy excersise left for the entertainment of the reader.   □

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