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A NEW PROOF OF WINQUIST'S IDENTITY

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Abstract

Winquist's identity plays a vital role in the proof of Ramanujan's congruence $p(11n+6) \equiv 0 \pmod{11}$. In this paper, we give a new proof of Winquist's identity.

1. Introduction

In 1969, L. Winquist [15] found an elementary proof of the congruence $p(11n+6) \equiv 0 \pmod{11}$, which was first stated by Ramanujan in [12], where $p(n)$ is the number of partitions of the positive integer n. A certain identity, later named Winquist's identity, played an essential role in his proof.

Later, L. Carlitz and M. V. Subbarao [4] and M. D. Hirschhorn [8] discovered four-parameter generalizations of Winquist's identity. By multiplying two pairs of quintuple product identities and adding them, S.-Y. Kang [9] gave another proof of Winquist's identity. Recently, new proofs have been given by P. Hammond, R. Lewis and Z.-G. Liu [7], H. H. Chan, Z.-G. Liu and S. T. Ng [5], and S. Kongsiriwong and Z.-G. Liu [10]. Winquist's identity was generalized to affine root systems by I. Macdonald in [11], and a proof of Macdonald's identities for infinite families of root systems was given by D. Stanton [13].

In this paper, we give a new proof of Winquist's identity. In [14], K. Venkatachaliengar gave a proof of the quintuple product identity using a similar method. Venkatachaliengar's work is included in S. Cooper's comprehensive survey [6].

We use the standard notation for q-products, defining

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \qquad |q|<1.$$

The Jacobi triple product identity in its analytical form is given by [2, p. 10].

Theorem 1. *For $z \neq 0$, $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} z^n q^{n^2} = (-zq; q^2)_\infty (-q/z; q^2)_\infty (q^2; q^2)_\infty. \tag{1}$$

2. Proof of Winquist's Identity

Theorem 2. (Winquist's Identity) *For any nonzero complex numbers a, b and for $|q| < 1$,*

$$\sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (-1)^{m+n} q^{f(m,n)} (a^{-3m} b^{-3n} - a^{-3m} b^{3n+1} - a^{-3n+1} b^{-3m-1} + a^{3n+2} b^{-3m-1})$$
$$= (q;q)_\infty^2 (a;q)_\infty (a^{-1}q;q)_\infty (b;q)_\infty (b^{-1}q;q)_\infty (ab;q)_\infty \times (a^{-1}b^{-1}q;q)_\infty (ab^{-1};q)_\infty (a^{-1}bq;q)_\infty, \tag{2}$$

where $f(m,n) = \frac{3m^2+3n^2+3m+n}{2}$

Proof. We begin with the left-hand side of (2) and denote it by $L(a,b)$. By Jacobi's triple product identity (1),

$$\begin{aligned} L(a,b) &= \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3m^2+3m}{2}} a^{-3m} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} \left(b^{-3n} - b^{3n+1}\right) \\ &\quad - \frac{a}{b} \sum_{m=-\infty}^{\infty} (-1)^m q^{\frac{3m^2+3m}{2}} b^{-3m} \sum_{n=-\infty}^{\infty} (-1)^n q^{\frac{3n^2+n}{2}} \left(a^{-3n} - a^{3n+1}\right) \\ &= \left(\frac{q^3}{a^3}; q^3\right)_\infty (a^3;q^3)_\infty (q^3;q^3)_\infty \\ &\quad \times \left[(\frac{q^2}{b^3}; q^3)_\infty (b^3 q; q^3)_\infty (q^3;q^3)_\infty - b(\frac{q}{b^3}; q^3)_\infty (b^3 q^2; q^3)_\infty (q^3;q^3)_\infty \right] \\ &\quad - \frac{a}{b} (\frac{q^3}{b^3}; q^3)_\infty (b^3;q^3)_\infty (q^3;q^3)_\infty \\ &\quad \times \left[(\frac{q^2}{a^3}; q^3)_\infty (a^3 q; q^3)_\infty (q^3;q^3)_\infty - a(\frac{q}{a^3}; q^3)_\infty (a^3 q^2; q^3)_\infty (q^3;q^3)_\infty \right]. \end{aligned} \tag{3}$$

We can write (3) as

$$L(a,b) = g(a)h(b) - \frac{a}{b} g(b)h(a), \tag{4}$$

where

$$g(z) := (\frac{q^3}{z^3}; q^3)_\infty (z^3; q^3)_\infty (q^3; q^3)_\infty,$$
$$h(z) := (\frac{q^2}{z^3}; q^3)_\infty (z^3 q; q^3)_\infty (q^3; q^3)_\infty - z(\frac{q}{z^3}; q^3)_\infty (z^3 q^2; q^3)_\infty (q^3; q^3)_\infty.$$

From the definition of $L(a, b)$, it is easy to show that

$$\frac{L(aq, b)}{L(a, b)} = -\frac{1}{a^3}. \tag{5}$$

Next we show that $L(a, b)$ is zero when a, b, ab, or a/b is an integral power of q. We consider the following cases.

`Case 1.` $a = q^m$ or $b = q^m$, where m is integer. For the case $a = q^m$, by the functional equation (5), we only need to consider the case $a = 1$. Since $g(1) = h(1) = 0$, we have $L(1, b) = g(1)h(b) - \frac{1}{b} g(b)h(1) = 0$. The proof is similar for the case $b = q^m$.

`Case 2.`$ab = q^m$, where m is integer. As above, we only need to consider the case $ab = 1$. We have

$$\frac{g(a)}{g(1/a)} = -a^3, \quad \frac{h(a)}{h(1/a)} = -a.$$

So $L(a, b) = L(a, 1/a) = g(a)h(1/a) - a^2 g(1/a)h(a) = 0$.

`Case 3.` $a/b = q^m$, where m is integer. We only need to consider the case $a/b = 1$. We have $L(a, b) = L(a, a) = g(a)h(a) - g(a)h(a) = 0$, so $L(a, b)$ vanishes whenever a, b, ab, or a/b is an integral power of q (the zeros of $L(a, b)$ are not necessarily simple, and it is possible for $L(a, b)$ to have other zeros).

We construct another function

$$R(a, b) = (a^{-1}q; q)_\infty (b; q)_\infty (b^{-1}q; q)_\infty (ab; q)_\infty (a^{-1}b^{-1}q; q)_\infty (ab^{-1}; q)_\infty (a^{-1}bq; q)_\infty. \tag{6}$$

It is easy to see that $R(a, b)$ is zero precisely when a, b, ab, or a/b is an integral power of q, all the zeros of $R(a, b)$ are simple, and

$$\frac{R(aq, b)}{R(a, b)} = -\frac{1}{a^3}\ .$$

We denote the domain of both $L(a, b)$ and $R(a, b)$ by A, where

$$A = \{(a, b) : a, b \in \mathbb{C},\ a \neq 0,\ b \neq 0\}.$$

Let $B = \{(a, b) : (a, b) \in A$, where a or b is an integral power of q. Define, for $a, b \in A \setminus B$,

$$Q(a, b) = \frac{L(a, b)}{R(a, b)}\ . \tag{7}$$

Note that $Q(a, b)$ is analytic for $0 < |a| < \infty$ for each fixed b and satisfies the functional equation $Q(aq, b) = Q(a, b)$. We denote the Laurent series for $Q(a, b)$ by

$$Q(a,b) = \sum_{n=-\infty}^{\infty} a_n(b)a^n \ . \tag{8}$$

Since $Q(aq, b) = Q(a, b)$, (8) implies $\sum_{n=-\infty}^{\infty} a_n(b)(1-q^n)a^n = 0$. We have $a_n(b) = 0$ for $n \neq 0$. Thus $Q(a, b) = a_0(b)$ is independent of a. From (4),

$$L(a, b) = (-a/b)L(b, a). \tag{9}$$

From (6), it is easy to verify that

$$R(a, b) = (-a/b)R(b, a). \tag{10}$$

By (7), (9), and (10), we have $Q(a, b) = Q(b, a)$. By the symmetry of $Q(a, b)$ in a and b, $Q(a, b)$ is also independent of b. Thus $Q(a, b)$ is a constant.

Let ω =exp$(2\pi i/3)$. For any complex number x, $(1-x)(1-x\omega)(1-x\omega^2) = 1-x^3$. Let a and q be complex number with $|q| < 1$. We have

$$(a; q)_\infty(a\omega; q)_\infty(a\omega^2; q)_\infty = (a^3; q^3)_\infty.$$

If $b = \omega$ in $L(a, b)$ and $R(a, b)$, we find that

$$L(a,\omega) = (1-\omega)(q;q)_\infty(\frac{q^3}{a^3};q^3)_\infty(a^3;q^3)_\infty(q^3;q^3)_\infty,$$
$$R(a,\omega) = (1-\omega)(\frac{q^3}{a^3};q^3)_\infty(a^3;q^3)_\infty(q^3;q^3)_\infty/(q;q)_\infty.$$

Thus, $Q(a, \omega) = (q; q)_\infty^2$. We conclude that $Q(a, b) = (q; q)_\infty^2$ for arbitrary nonzero complex numbers $(a, b) \in A \setminus B$. We also have $L(a, b) = R(a, b) = 0$ for $(a, b) \in B$. So $L(a, b) = (q; q)_\infty^2 R(a, b)$ for any nonzero complex numbers a and b, and this completes the proof. □

Certain other theta function identities can be derived by using the foregoing analysis. For example, let $a = q^{\frac{1}{3}}$ and $b = -1$, then from (2) and (3), respectively, we deduce a theta function identity due to Ramanujan [1, pp. 48–49]:

$$\psi(q) = f(q^3, q^6) + q\psi(q^9).$$

We can also verify that the constant value of $Q(a, b)$ is $(q; q)_\infty^2$ by choosing $(a, b) = (q^{\frac{1}{3}}, -1)$ in the proof of Winquist's identity.

This method can also be applied to prove many theta function identities, for example, an analogue of Winquist's identity found by the author in [3].

Theorem 3. *For a, b nonzero and for $|q| < 1$,*

$$\begin{aligned}(a;q)_\infty(a^{-1}q;q)_\infty(b;q)_\infty(b^{-1}q;q)_\infty(ab;q)_\infty(a^{-1}b^{-1}q;q)_\infty(q;q)_\infty^2 \\ = (-ab^{-1}q;q^2)_\infty(-a^{-1}bq;q^2)_\infty(q^2;q^2)_\infty \\ \times \big[(-a^3b^3q;q^6)_\infty(-a^{-3}b^{-3}q^5;q^6)_\infty(q^6;q^6)_\infty \\ - a^2b^2(-a^3b^3q^5;q^6)_\infty(-a^{-3}b^{-3}q;q^6)_\infty(q^6;q^6)_\infty\big] \\ + (-ab^{-1}q^2;q^2)_\infty(-a^{-1}b;q^2)_\infty(q^2;q^2)_\infty \\ \times \big[a^2b(-a^3b^3q^4;q^6)_\infty(-a^{-3}b^{-3}q^2;q^6)_\infty(q^6;q^6)_\infty \\ - a(-a^3b^3q^2;q^6)_\infty(-a^{-3}b^{-3}q^4;q^6)_\infty(q^6;q^6)_\infty\big]. \qquad (11)\end{aligned}$$

The proof of Theorem 3 is similar to the proof of Winquist's identity.

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References

[1] B. C. Berndt, *Ramanujan's Notebooks*, Part III, Springer-Verlag, New York, 1991.

[2] B. C. Berndt, *Number Theory in the Spirit of Ramanujan*, AMS, Providence, RI, 2006.

[3] Z. Cao, *Integer matrix covering systems and product identities for theta functions*, Int. Math. Res. Not. IMRN 2010; doi: 10.1093/imrn/rnq253.

[4] L. Carlitz and M. V. Subbarao, *On a combinatorial identity of Winquist and its generalization*, Duke Math. J. **39** (1972), 165–172.

[5] H. H. Chan, Z.-G. Liu and S. T. Ng, *Elliptic functions and the quintuple, Hirschhorn and Winquist product identities*, Int. J. Number Theory **1** (2005), 33–43.

[6] S. Cooper, *The quintuple product identity*, Int. J. Number Theory **2** (2006), 115–161.

[7] P. Hammond, R. Lewis and Z.-G. Liu, Hirschhorn's identities, *Bull. Austral. Math. Soc.* **60** (1) (1999), 73–80.

[8] M. D. Hirschhorn, *A generalisation of Winquist's identity and a conjecture of Ramanujan*, J. Indian Math. Soc. **51**(1986), 49–55.

[9] S.-Y. Kang, *A new proof of Winquist's identity*, J. Combin Theory (A) **78** 1997, 313–318.

[10] S. Kongsiriwong and Z.-G. Liu, *Uniform proofs of q-series-product identities*, Result. Math. **44** (2003), 312–339.

[11] I. Macdonald, *Affine root systems and Dedekind's η-function*, Inv. Math. **15** (1972), 91–143.

[12] S. Ramanujan, *Some properties of $p(n)$, the number of partitions of n*, Proc. Cambridge Philos. Soc. **19** (1919), 207–210.

[13] D. Stanton (ed.), *An elementary approach to the Macdonald identities*, q-Series and Partitions, IMA Volumes in Math. and its Applications, Springer, New York, 1989, 139–149.

[14] K. Venkatachaliengar, *Development of Elliptic Functions according to Ramanujan*, Technical Report 2, Madurai Kamaraj University, Madurai, 1988.

[15] L. Winquist, *An elementary proof of $p(11n+6) \equiv 0$ (mod 11)*, J. Combin. Theory **6** (1969), 56–59.