SELMER'S MULTIPLICATIVE ALGORITHM

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Abstract

The behavior of the multiplicative acceleration of Selmer's algorithm is widely unknown and no general result on convergence has been detected yet. Solely for its 2-dimensional, periodic expansions exist some results on convergence and approximation due to Fritz Schweiger. In this paper we show that periodic expansions of any dimension do in fact converge and that the coordinates of the limit points are rational functions of the largest eigenvalue of the periodicity matrix.

1. Introduction

While simple continued fractions are quite popular and, as a matter of fact, well-known throughout the world of mathematics, their multidimensional generalizations are rather unknown. But, there are at least two ways to approach a theory of multidimensional continued fractions (MCFs). One is more of a geometric nature, while another concentrates on multidimensional continued fractions which can be described by a set of $(n+1) \times (n+1)$-matrices. The latter set includes amongst others the Jacobi-Perron algorithm, as well as algorithms of Brun and Selmer. Each of these algorithms generalizes the matrices $\begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ associated with continued fractions to higher dimensions in order to achieve an equivalent of Lagrange's Theorem, but for cubic or higher roots. Interestingly, it is still unclear whether any one of these algorithms really meets this requirement.

Despite the lack of such a theorem, there are a lot of discoveries that have been made concerning periodicity and approximation characteristics of multidimensional continued fractions. For further information, the books of Brentjes [2] and Schweiger [11], both named Multidimensional Continued Fractions, are highly recommended.

This paper is organized as follows. Section 1.1 is devoted to the basic definition of a fibred system that links multidimensional continued fractions to its simpler counterpart, the continued fractions. The concept of cylinders, introduced in Section

1.2, helps us to understand why the multiplicative version of Selmer's algorithm is so special among other algorithms, like that of Jacobi-Perron, Brun or even Selmer's subtractive algorithm. In Section 2 we introduce Selmer's algorithm in its subtractive version (SSA) and show that it depicts a fibred system. We conclude the section with an example for which the SSA eventually becomes periodic - an example for which it is already known that both Brun's and Jacobi-Perron's algorithms become periodic. After the introduction of Selmer's multiplicative algorithm (MSA) in Section 3.1, we show in Section 3.2 that it defines a fibred system, as well. In Section 3.3 we apply the concept of cylinders to the MSA and illustrate it graphically for the 2-dimensional MSA. After a short section on the matrices of the MSA, the main result of this paper, Theorem 11, which shows convergence of the periodic MSA, is presented in Section 3.5. It is followed by Lemma 12, which is essential for the proof of the weak convergence of the MSA. In the end, after a periodic example for the MSA, we emphasize in Section 3.5.3 the importance of Theorem 11 for the theory of multidimensional continued fractions and give a short discussion of future directions in the final Section 4.

1.1. Fibred Systems

In this section we first introduce the notion of a fibred system and illustrate its characteristics in the context of continued fractions. Thereby, we are able to define multidimensional continued fractions by a set of matrices on such a fibred system.

Definition 1. Let B be a set and $T\colon B \to B$ be a map. The pair (B, T) is called a *fibred system* if the following conditions are satisfied:

1. There is a finite or countable set I (called the digit set).
2. There is a map $k\colon B \to I$. Then the sets $B(i) = k^{-1}\{i\} = \{x \in B : k(x) = i\}$ form a partition of B, and hence $\bigcup_{i\in I} B(i) = B$.
3. The restriction of T to any $B(i)$ is an injective map.

(The partition $\{B(i) : i \in I\}$ is called the time-1-partition.)

Example. We consider the continued fraction algorithm. It is well-known that for each irrational $x \in [0, 1]$ there exists a corresponding continued fraction expansion of the form

$$x = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}},$$

where we restrict $a_1, a_2, \ldots$ to be positive integers. Now, we set $B := [0, 1[$ and obtain for $x \neq 0$ the map

$$\begin{aligned} T\colon B &\to B \\ x &\mapsto \frac{1}{x} - a(x), \quad a(x) := \lfloor x^{-1} \rfloor. \end{aligned}$$

For $x = 0$, however, we arrive at a stop and thus get $Tx = 0$. Further, by setting $I := \mathbb{N}_0$ we obtain the map $i\colon B \to I$ defined by

$$i\colon x \mapsto a(x).$$

Then the set $I := \mathbb{N}_0$ is countable, while the sets $B(i)$ form a partition of B. Now we set $y := Tx = \frac{1}{x} - a(x)$ and restrict T to one of the sets $B(i), i \in I$. Then for $x_1, x_2 \in B(i)$ with $Tx_1 = Tx_2$ we immediately obtain $x_1 = x_2$, since $a(x_1) = a(x_2) = i$. Hence, the pair (B, T) indicates a fibred system.

With the knowledge of fibred systems we are now able to define multidimensional continued fractions on the aforementioned systems by a set of matrices.

Definition 2. The fibred system (B, T) is called a *(multidimensional) continued fraction* if

1. B is a subset of $\mathbb{R}^n$.
2. For every digit $k \in I$ there is an invertible matrix $\alpha = \alpha(k) = ((A_{ij}))$, $0 \leq i, j \leq n$, such that $y = Tx$, $x \in B(k)$, is given as

$$y_i = \frac{A_{i0} + \sum_{j=1}^{n} A_{ij} x_j}{A_{00} + \sum_{j=1}^{n} A_{0j} x_j}.$$

In particular, we are interested in the inverse matrix of such a multidimensional continued fraction, as with this inverse we obtain an expression of x via its image under the map T.

Definition 3. If (B, T) is a (multidimensional) continued fraction, then we denote the inverse matrix of $\alpha(k)$ by $\beta(k) = ((B_{ij}))$, $0 \leq i, j \leq n$. Then we define for $s \geq 1$:

$$\beta(k_1, \ldots, k_s) := \beta(k_1) \ldots \beta(k_s) = ((B_{ij}^{(s)})), \quad 0 \leq i, j \leq n,$$

where $B_{ij}^{(1)} = B_{ij}$. Then $y = T^s x$ is equivalent to

$$x_i = \frac{B_{i0}^{(s)} + \sum_{j=1}^{n} B_{ij}^{(s)} y_j}{B_{00}^{(s)} + \sum_{j=1}^{n} B_{0j}^{(s)} y_j}, \quad 1 \leq i \leq n.$$

Example. We consider again the continued fraction algorithm restricted to the set $B := [0, 1[$. Then with help of a lemma in [7] there exists a $\xi_n \in \mathbb{R}$ for all $n \in \mathbb{N}_0$ such that for appropriate $x := [0, a_1, \ldots, a_n] \in B$ we have

$$x = [0, a_1, \ldots, a_n] = \frac{p_n}{q_n} \quad \text{and} \quad x = [0, a_1, \ldots, a_n, \xi_n] = \frac{p_n + p_{n-1}y}{q_n + q_{n-1}y},$$

where $y := T^n x$. According to the continued fraction algorithm, $\xi_n \neq 0$ is always satisfied and hence $y = \xi_n^{-1}$. Now we turn to the matrices associated with the continued fraction. Since $0 \leq x \leq 1$, we have $a_0 = 0$ and thus

$$\begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

which makes the continued fraction algorithm restricted to the set $B := [0, 1[$ a (multidimensional) continued fraction in the sense of Definition (2) and (3).

1.2. The Concept of Cylinders

In the previous section we saw that the digit set I of a fibred system causes a partition of the set B into the subsets $B(i), i \in I$; that is,

$$\bigcup_{i \in I} B(i) = B.$$

Here, we did not require that T restricted to $B(i)$, $i \in I$, is a surjective map. Therefore a partition of $TB(i)$, $i \in I$, could be of interest as well and we have eventually arrived at the concept of cylinders.

Definition 4. The *cylinder of rank s*, defined by the digits $i_1, \ldots, i_s$, is the set

$$\begin{aligned} B(i_1, \ldots, i_s) : &= B(i_1) \cap T^{-1}B(i_2) \cap \ldots \cap T^{-(s-1)}B(i_s) \\ &= \{x : i_1(x) = i_1, \ldots, i_s(x) = i_s\}. \end{aligned}$$

Such a cylinder $B(i_1, \ldots, i_s)$ is called proper (or full) if $T^s B(i_1, \ldots, i_s) = B$.

Proposition 5. *All cylinders of arbitrary rank $s \in \mathbb{N}$ are full if all cylinders of rank 1 are full.*

Proof. Assume that all cylinders of rank 1 are full. Then we get

$$TB(i) = B$$

for all $i \in I$. The rest follows from Definition 4 of cylinders, since for $(i_1, \ldots, i_s) \in I^s$ we obtain $TB(i_1, \ldots, i_s) = B(i_2, \ldots, i_s)$. □

2. Selmer's Algorithm

In 1961, in connection with Brun's algorithm, Selmer published a variation of its subtractive version called Selmer's subtractive algorithm (SSA). But instead of subtracting the second largest initial value from the largest, like Brun did, he chose to subtract the smallest initial value from the largest. This may at first seem to be a marginal deviation, but it actually implicates a fundamental change. That is, at some point in the expansion one inevitably ends up in the absorbing set $D := B(n-1) \cup B(n)$.

2.1. Subtractive Version

Let $\Delta^{n+1} := \{b = (b_0, b_1, \dots, b_n) : b_0 \geq b_1 \geq \cdots \geq b_n \geq 0\}$ and define

$$\sigma b := (b_0 - b_n, b_1, \dots, b_n).$$

There is an index $i = i(b)$, $0 \leq i \leq n$, such that

$$\pi\sigma b := (b_1, b_2, \dots, b_i, b_0 - b_n, \dots, b_n) \in \Delta^{n+1}.$$

Further, let

$$B^n := \{(x_1, \dots, x_n) : \ 1 \geq x_1 \geq \cdots \geq x_n \geq 0\}.$$

Then with the help of the projection $p : \Delta^{n+1} \to B^n$ defined by

$$p(b_0, b_1, \dots, b_n) = \left(\frac{b_1}{b_0}, \dots, \frac{b_n}{b_0}\right),$$

we obtain the map $T : B^n \to B^n$ which makes the diagram

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{\pi\sigma} & \Delta^{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B^n & \xrightarrow[T]{} & B^n \end{array}$$

commutative.

2.2. Fibred System and Absorbing Set

In this section we prove that the SSA intrinsically represents a fibred system. Shortly afterwards, we point to the absorbing set of Selmer's algorithm. As a matter of fact, the restriction of the SSA to this absorbing set coincides with an algorithm of Mönkemeyer. Hence, for further references we point to [6], [8] and [11].

Proposition 6. *The partition* $B(j) := \left\{x \in B^n : i(p^{-1}x) = j\right\}, \quad 0 \leq j \leq n,$ *makes* (B^n, T) *a fibred system.*

Proof. We set $y := Tx$ and calculate the map T as follows:

$\boxed{j=0}$

$$y_k = \frac{x_k}{1-x_n}, \quad 1 \le k \le n, \qquad\qquad x_k = \frac{y_k}{1+y_n}, \quad 1 \le k \le n.$$

$\boxed{1 \le j \le n-1}$

$$\begin{aligned} y_k &= \frac{x_{k+1}}{x_1}, \quad 1 \le k \le j-1, & x_1 &= \frac{1}{y_j + y_n},\\ y_j &= \frac{1-x_n}{x_1}, & x_k &= \frac{y_{k-1}}{y_j+y_n}, \quad 2 \le k \le j,\\ y_k &= \frac{x_k}{x_1}, \quad j+1 \le k \le n, & x_k &= \frac{y_k}{y_j+y_n}, \quad j+1 \le k \le n. \end{aligned}$$

$\boxed{j=n}$

$$\begin{aligned} y_k &= \frac{x_{k+1}}{x_1}, \quad 1 \le k \le n-1, & x_1 &= \frac{1}{y_{n-1} + y_n},\\ y_n &= \frac{1-x_n}{x_1}, & x_k &= \frac{y_{k-1}}{y_{n-1}+y_n}, \quad 2 \le k \le n. \end{aligned}$$

In summary, we obtain:

1. The digit set $I := \{0, \ldots, n\}$ is a countable set.
2. The map $i\colon B^n \to I$ defined by $i\colon x \mapsto i(p^{-1}x)$ causes a partition of B^n, since evidently $\bigcup_{j\in I} B(j) = B^n$.
3. If we restrict T to any $B(j)$, then we obtain for $y^{'} = y^{''}$ and $0 \le j \le n$:

$\boxed{j=0}$ As $y_n^{'} = y_n^{''}$, it follows that $x_n^{'} = x_n^{''}$ and thus $x^{'} = x^{''}$.

$\boxed{1 \le j \le n-1}$ As $y_n^{'} = y_n^{''}$ and $y_j^{'} = y_j^{''}$, we obtain $x_1^{'} = x_1^{''}$ and hence $x^{'} = x^{''}$.

$\boxed{j=n}$ As $y_n^{'} = y_n^{''}$ and $y_{n-1}^{'} = y_{n-1}^{''}$, we get $x_1^{'} = x_1^{''}$ and therefore $x^{'} = x^{''}$.

Consequently, the restriction of T to any $B(j)$ is an injective map. □

Theorem 7. (Absorbing Set) *Let* $D := \{x \in B^n : x_{n-1} + x_n \ge 1\}$*. Then* D *is an absorbing set, i.e.,*

1. *we have* $TD = D$, *and*

2. *for almost every* $x \in B^n$ *there is an* $N = N(x)$, *such that* $T^N x \in D$.

Proof. A proof of this theorem can be found in [11, p. 55] and verifies that $D = B(n-1) \cap B(n)$. $\square$

2.3. A Case of Periodicity

As the question of periodicity is by far the most interesting one, we state an example of a periodic SSA.

Definition 8. The multidimensional continued fraction of x is called *periodic* if there are numbers $m \geq 0$, $p \geq 1$ such that $T^{m+p}x = T^m x$.

Example. (SSA) We consider the tuple $x := (x_1, x_2) = \left(\sqrt[3]{4} - 1, \sqrt[3]{2} - 1\right)$ and apply the SSA. A straightforward calculation then shows that $T^{31}x = Tx$,

$$\begin{aligned}
Tx &= \left(\frac{\sqrt[3]{4}-1}{2-\sqrt[3]{2}}, \frac{\sqrt[3]{2}-1}{2-\sqrt[3]{2}}\right)\\
T^2x &= \left(\frac{3-2\sqrt[3]{2}}{\sqrt[3]{4}-1}, \frac{\sqrt[3]{2}-1}{\sqrt[3]{4}-1}\right)\\
&\vdots\\
T^{30}x &= \left(\frac{54-29\sqrt[3]{2}-11\sqrt[3]{4}}{30\sqrt[3]{4}+13\sqrt[3]{2}-64}, \frac{24\sqrt[3]{2}+3\sqrt[3]{4}-35}{30\sqrt[3]{4}+13\sqrt[3]{2}-64}\right)\\
T^{31}x &= \left(\frac{27\sqrt[3]{4}-11\sqrt[3]{2}-29}{54-29\sqrt[3]{2}-11\sqrt[3]{4}}, \frac{24\sqrt[3]{2}+3\sqrt[3]{4}-35}{54-29\sqrt[3]{2}-11\sqrt[3]{4}}\right).
\end{aligned}$$

3. Multiplicative Algorithms

While for Brun's algorithm the multiplicative version really causes an acceleration of expansions, the multiplicative version of Selmer's algorithm does not. In addition, none of its cylinders are full. It is, thus, more adequate to describe it as a mere division algorithm in order to avoid any confusion. Under these circumstances it seems unlikely that the algorithm provides convergent expansions or approximations that are competitive in the field of multidimensional continued fractions. However, it does.

3.1. Selmer's Division Algorithm

Let $\Delta^{n+1} := \{b = (b_0, b_1, \dots, b_n) : b_0 \geq b_1 \geq \dots \geq b_n \geq 0\}$. Then we define

$$\delta b := (b_0 - kb_n, b_1, \dots, b_n),\ k := \left[\frac{b_0}{b_n}\right].$$

Since $b_n \geq b_0 - kb_n$ we get $\pi\colon \Delta^{n+1} \to \Delta^{n+1}$ defined by $\pi\delta b := (b_1, \dots, b_n, b_0 - kb_n)$. Now let $B^n := \{(x_1, \dots, x_n) : 1 \geq x_1 \geq \dots \geq x_n \geq 0\}$. With the help of the projection $p\colon \Delta^{n+1} \to B^n$, defined by

$$p(b_0, b_1, \dots, b_n) = \left(\frac{b_1}{b_0}, \dots, \frac{b_n}{b_0}\right),$$

we finally get the bottom map

$$S(x_1, \dots, x_n) = \left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}, \frac{1 - kx_n}{x_1}\right),$$

which makes the diagramm

$$\begin{array}{ccc} \Delta^{n+1} & \xrightarrow{\pi\sigma} & \Delta^{n+1} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ B^n & \xrightarrow[S]{} & B^n \end{array}$$

commutative.

Hence the multiplicative version of Selmer's algorithm (MSA) is given by

$$\begin{aligned} &S\colon B^n \to B^n \\ &S(x_1, \dots, x_n) = \left(\frac{x_2}{x_1}, \dots, \frac{x_n}{x_1}, \frac{1 - kx_n}{x_1}\right). \end{aligned}$$

3.2. The Fibred System

In this section we simply prove that the MSA represents a fibred system.

Proposition 9. *The partition*

$$B(k) := \left\{x \in B^n : \frac{1}{k+1} < x_n \leq \frac{1}{k}\right\}, \quad k = 1, 2, \dots$$

makes (B^n, S) a fibred system.

Proof. If $x_n = 0$, we simply restrict to $B^{n-1} := \{(x_1, \dots, x_{n-1}) : 1 \geq x_1 \geq \dots \geq x_{n-1} \geq 0\}$. Then from $k := [x_n^{-1}]$ we immediately get $k \in \mathbb{N}$ and thus:

1. The digit set $I := \mathbb{N}$ is a countable set.
2. The map $i\colon B^n \to I$, with $i\colon x \mapsto k := [x_n^{-1}]$, amounts to a partition of B^n, since evidently $\bigcup_{k\in I} B(k) = B^n$.
3. If we restrict T to $B(k)$, then for $y^{'} = y^{''}$ (and thus $y_n^{'} = y_n^{''}$, $y_{n-1}^{'} = y_{n-1}^{''}$) we obtain $x_1^{'} = x_1^{''}$, and hence $x^{'} = x^{''}$. Thus, the restriction of T to $B(k)$ is an injective map for all $k \in \mathbb{N}$.

$\square$

3.3. Cylinders and Time-1-Partition

Since $k = [x_n^{-1}]$ the pair (B^n, S) is a fibred system with cells

$$B(k) := \left\{ x \in B^n : \frac{1}{k+1} < x_n \leq \frac{1}{k} \right\}, \quad k = 1, 2, \ldots .$$

These $B(k)$ denote cylinders of rank 1 and indicate, in our case, convex sets with vertices

$$\begin{array}{l} \left(1, \ldots, 1, \frac{1}{k}\right), \left(1, \ldots, 1, \frac{1}{k+1}\right) \\ \left(1, \ldots, \frac{1}{k}, \frac{1}{k}\right), \left(1, \ldots, \frac{1}{k+1}, \frac{1}{k+1}\right) \\ \quad \vdots \\ \left(\frac{1}{k}, \ldots, \frac{1}{k}\right), \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right) \end{array}$$

depending only on $k \in \mathbb{N}$. By the way, we obtain $S(\frac{1}{k+1}, \ldots, \frac{1}{k+1}) = (1, \ldots, 1)$ for all cylinders $B(k)$, $k \in \mathbb{N}$.

In order to achieve more clarity we restrict our attention to the 2-dimensional case. Hence, we consider the set $B^2 = \{1 \geq x_1 \geq x_2 \geq 0\}$. Since $1 \geq x_1 \geq x_2$ und $\frac{1}{k} \geq x_2 > \frac{1}{k+1}$, we obtain the convex set $B(k)$ with vertices

$$\left(1, \frac{1}{k}\right), \left(1, \frac{1}{k+1}\right), \left(\frac{1}{k}, \frac{1}{k}\right), \left(\frac{1}{k+1}, \frac{1}{k+1}\right).$$

These cells $B(k)$ form a partition of the set B^2 that can be easily illustrated by means of Figure (1).

Clearly, none of the cylinders $B(k)$ are full, as they are mapped under S onto the convex set with vertices

$$\left(\frac{1}{k}, 0\right), \left(\frac{1}{k+1}, \frac{1}{k+1}\right), (1, 0), (1, 1).$$

Hence, $SB(k) \subset SB(k+1)$ for all $k \in \mathbb{N}$ and additionally $SB(k)$ is not a union of cylinders of rank 1.

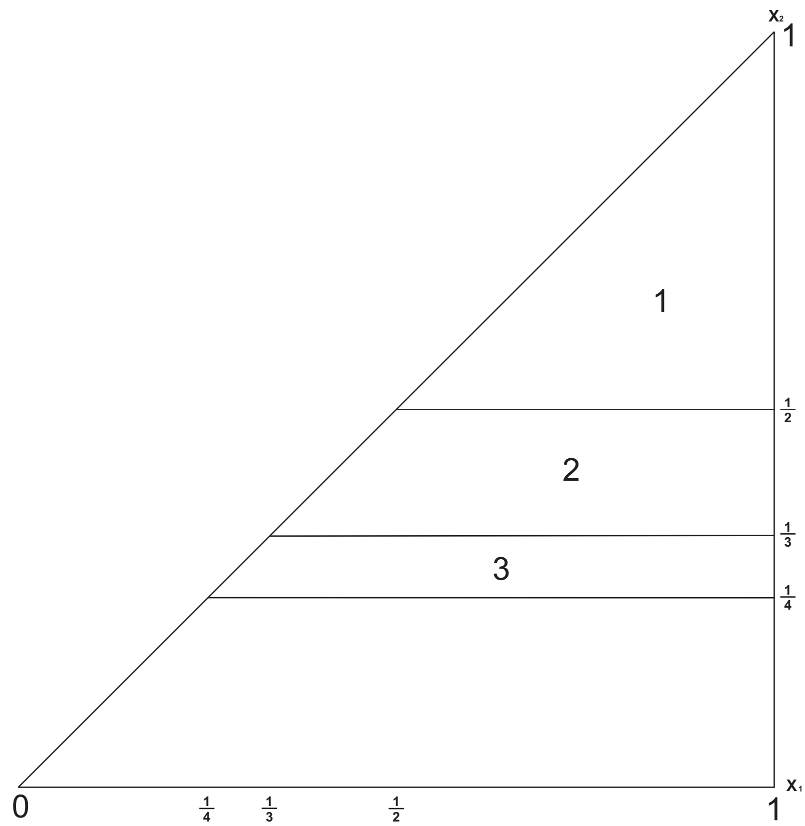

Figure 1: The time-1-partition of the set B^2 by the 2-dimensional MSA, where $k \in \mathbb{N}$ indicates the associated cylinder $B(k)$.

3.4. The Matrices

Notice that if we set $y := Sx$, then

$$\begin{aligned} x_1 &= \frac{1}{ky_{n-1} + y_n} \\ x_i &= \frac{y_{i-1}}{ky_{n-1} + y_n}, \quad 2 \leq i \leq n. \end{aligned}$$

Thus, according to the definition of a multidimensional continued fraction, the associated $(n+1) \times (n+1)-$matrices of the MSA are given by

$$\beta(k) := \begin{pmatrix} 0 & \dots & k & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

and in dimension $n = 3$ by
$$\begin{pmatrix} 0 & 0 & k & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Expanding the determinant along the last column leads to

$$\det \beta(k) = (-1)^n \det \mathbb{1} = \pm 1. \tag{1}$$

Now we define the matrices $\beta^{(s)}(k_1, \ldots, k_s)$, in common notation, as

$$\beta^{(s)}(k_1, \ldots, k_s) := \beta(k_1) \ldots \beta(k_s) = \begin{pmatrix} B_0^{(s-n+1)} & \ldots & B_0^{(s-1)} & B_0^{(s)} & B_0^{(s-n)} \\ \vdots & \vdots & & & \vdots \\ B_n^{(s-n+1)} & \ldots & B_n^{(s-1)} & B_n^{(s)} & B_n^{(s-n)} \end{pmatrix}.$$

Hence for $s \geq 0$ we obtain the relation

$$B_i^{(s+1)} = k_{s+1} B_i^{(s-n+1)} + B_i^{(s-n)}, \quad i = 0, \ldots, n,$$

where $\beta^{(0)}$ denotes the unit matrix. If we set $y = S^s x$ and $k_i = k(S^{i-1})$, $1 \leq i \leq s$, then we find that

$$x_i = \frac{B_i^{(s-n+1)} + y_1 B_i^{(s-n+2)} + \ldots + y_{n-1} B_i^{(s)} + y_n B_i^{(s-n)}}{B_0^{(s-n+1)} + y_1 B_0^{(s-n+2)} + \ldots + y_{n-1} B_0^{(s)} + y_n B_0^{(s-n)}}, \quad i = 1, \ldots, n. \tag{2}$$

3.5. Periodic Expansions for Selmer's Division Algorithm

In this section we eventually prove the convergence of the periodic MSA and quote a simple example of periodicity. Since the proof allows us to apply a variety of other theorems to Selmer's multiplicative algorithm, we subsequently mention a few of them. However for completeness and further information we refer to [11].

3.5.1. Weak Convergence of the Periodic MSA

Definition 10. The multidimensional continued fraction is *weakly convergent* if for every $x \in B$ we have

$$\lim_{s \to \infty} \left(\frac{B_{10}^{(s)}}{B_{00}^{(s)}}, \ldots, \frac{B_{n0}^{(s)}}{B_{00}^{(s)}} \right) = x.$$

If for every g with $0 \leq g \leq n$ we get

$$\lim_{s\to\infty}\left(\frac{B_{1g}^{(s)}}{B_{0g}^{(s)}},\ldots,\frac{B_{ng}^{(s)}}{B_{0g}^{(s)}}\right)=x,$$

for every $x \in B$ we call the multidimensional continued fraction *uniformly weakly convergent.*

Due to the special form of the matrices of the MSA (see Section 3.4), where in the course of an expansion each n-th column passes through the whole matrix before it drops out, Equation (3) already ensures uniformly weak convergence of the MSA.

Theorem 11. *Assume that the algorithm of* $x = (x_1, \ldots, x_n)$ *eventually becomes periodic with period length* p. *Then*

$$\lim_{s\to\infty}\left(\frac{B_{1}^{(s)}}{B_{0}^{(s)}},\ldots,\frac{B_{n}^{(s)}}{B_{0}^{(s)}}\right)=x. \tag{3}$$

Hence, the periodic, multiplicative algorithm of Selmer is weakly convergent and even uniformly weakly convergent.

Proof. As a preperiod does not affect the convergence of an expansion, we can clearly assume that the expansion is purely periodic with period length p.

Let M denote the matrix of the periodic expansion of length p, i.e.,

$$M := \beta^{(p)}(k_1,\ldots,k_p) = \begin{pmatrix} B_0^{(p-n+1)} & \ldots & B_0^{(p-1)} & B_0^{(p)} & B_0^{(p-n)} \\ \vdots & \vdots & & & \\ B_n^{(p-n+1)} & \ldots & B_n^{(p-1)} & B_n^{(p)} & B_n^{(p-n)} \end{pmatrix};$$

and let M^k the matrix of the periodic expansion of length kp so that

$$M^k = \begin{pmatrix} B_0^{(kp-n+1)} & \ldots & B_0^{(kp-1)} & B_0^{(kp)} & B_0^{(kp-n)} \\ \vdots & \vdots & & & \\ B_n^{(kp-n+1)} & \ldots & B_n^{(kp-1)} & B_n^{(kp)} & B_n^{(kp-n)} \end{pmatrix}.$$

The characteristic polynomial of M can be written as

$$\chi_M(t) := \det(t\mathbb{1} - M) = t^{n+1} - b_n t^n - \ldots - b_1 t - b_0,$$

where $b_0 = (-1)^{n-1}\det(M)$, and we denote its eigenvalues by $\rho_0, \rho_1, \ldots, \rho_n$.

Note that in case that the matrix M has entries equal to 0, then due to the periodicity of the expansion we can use Lemma (12) (which is stated with its proof at the end of the argument) in order to get a natural number $m(n)$, such that M^m

is a positive matrix for all $m \geq m(n)$ and we could continue with the period length mp. Hence, without loss of generality, we can assume that M is a positive matrix and that the period length remains at p.

Due to the periodicity of the expansion, M is a positive matrix and we can thus apply the Perron-Frobenius Theorem [11, page 112], which allows us to arrange the eigenvalues so that $\rho_0 > |\rho_1| \geq \ldots \geq |\rho_n|, \quad \rho_0 > 1$, where ρ_0 is a simple and positive root of $\chi_M(t)$. Furthermore, with the help of the famous Caley-Hamilton Theorem we obtain $M^{n+1} - b_n M^n - \ldots - b_1 M - b_0 = 0$. By multiplying this expression by $M^k \beta(k_1, \ldots, k_j)$, we see that for the entries of the matrices the relations

$$B_i^{((k+n+1)p+j)} - b_n B_i^{((k+n)p+j)} - \ldots - b_1 B_i^{((k+1)p+j)} - b_0 B_i^{(kp+j)} = 0$$

hold for $0 \leq i \leq n$ and $0 \leq j < p$. Applying Theorem 41 of [11, page 114] we obtain the general solution

$$\begin{aligned} B_i^{(kp+j)} &= d(i,j)\rho_0^k \\ &+ b_{10}(i,j)\binom{k}{0}\rho_1^k + \ldots + b_{1,m_1-1}(i,j)\binom{k}{m_1-1}\rho_1^{k-m_1+1} \\ &+ \ldots \\ &+ b_{s0}(i,j)\binom{k}{0}\rho_s^k + \ldots + b_{s,m_s-1}(i,j)\binom{k}{m_s-1}\rho_s^{k-m_s+1}, \end{aligned} \tag{4}$$

where $m_1, \ldots, m_s$ are the multiplicities of the roots $\rho_1, \ldots, \rho_s$, which satisfy $1 + m_1 + \ldots + m_s = n + 1$. If the start values $B_i^{(kp+j)}$ are given for all $k = 0, \ldots, n$, then the solution sequence $(B_i^{(kp+j)})$, $k \geq 1$, is uniquely determined.

Now, we will consider the terms in Equation (4) more precisely. Clearly from $\rho_0 > |\rho_1| \geq \cdots \geq |\rho_n|$ and $\rho_0 > 1$ follows

$$\lim_{k\to\infty} \frac{\rho_\gamma^{k-\mu}}{\rho_0^k} = 0, \quad 1 \leq \gamma \leq s, \quad 0 \leq \mu \leq m_\gamma - 1.$$

Furthermore, for all γ, $1 \leq \gamma \leq s$, and $\varepsilon > 0$, the relations

$$\begin{aligned} \binom{k+1}{\mu} &= \binom{k}{\mu}\frac{k+1}{k+1-\mu}, \quad 1 \leq \mu \leq m_\gamma - 1 \\ (1+\varepsilon)^{k+1} &= (1+\varepsilon)^k(1+\varepsilon) \end{aligned}$$

hold for all $k \in \mathbb{N}$, and it follows that

$$\begin{aligned} \frac{k+1}{k+1-\mu} &\xrightarrow{k\to\infty} 1, \quad 1 \leq \mu \leq m_\gamma - 1 \\ (1+\varepsilon) &> 1. \end{aligned}$$

In summary, there exists a $k^{'}$, such that for all $k \geq k^{'}(\varepsilon)$ the inequality $\frac{k+1}{k+1-\mu} < (1+\varepsilon)$ holds, and consequently with increasing k, $k \geq k^{'}(\varepsilon)$, the term $(1+\varepsilon)^k$ grows faster than $\binom{k}{\mu}$. Thus

$$\lim_{k\to\infty} \binom{k}{\mu} (1+\varepsilon)^{-k} = 0 \text{ for all } \mu \text{ with } 1 \leq \mu \leq m_\gamma - 1.$$

Since $\rho_0 > 1$, there is a $\varepsilon > 0$, such that $\rho_0 = 1 + \varepsilon$. Consequently there also exists a $k^*(\varepsilon) > k^{'}(\varepsilon)$, such that for all $k \geq k^*(\varepsilon)$ the term ρ_0^k grows faster than $\binom{k}{\mu} \left|\rho_\gamma^{k-\mu}\right|$ for any μ, γ with $0 \leq \mu \leq m_\gamma - 1$ and $1 \leq \gamma \leq s$. So we derive from Equation (4), for $d(i,j) \neq 0$, the limit

$$\lim_{k\to\infty} \frac{B_i^{(kp+j)}}{\rho_0^k} = d(i,j), \tag{5}$$

as n is an integer and $b_{\gamma\mu}$ represents a constant for all γ, μ with $1 \leq \gamma \leq s, 0 \leq \mu \leq m_\gamma$.

Now, we use the recursion relations from Section 3.4 to get

$$\begin{aligned} B_i^{(s+n)} &= k_{s+n} B_i^{(s)} + B_i^{(s-1)} \\ B_i^{(s+n+1)} &= k_{s+n+1} B_i^{(s+1)} + B_i^{(s)}. \end{aligned}$$

Clearly, if $B_i^{(s)} \gg \rho_0^k$ then $B_i^{(s+j)} \gg \rho_0^k$ for all $j \in \mathbb{N}$, $j \geq n \cdot (n-1)$. Since $d(i,j)$ doesn't depend on k and n is finite, it follows that, if $d(i,j) \neq 0$ for some j, then $d(i,j) \neq 0$ for all j, $0 \leq j < p$.

Due to periodicity, the matrices M^k will be part of the expansion and using the eigenvalues of M we know that for the trace of M^k the relation

$$B_0^{(kp-n+1)} + \ldots + B_{n-1}^{(kp)} + B_n^{(kp-n)} = \rho_0^k + \ldots + \rho_n^k \tag{6}$$

holds. Due to periodicity, Equation (6) is also valid for arbitrarily large k and as n is finite, there is at least one summand $B_0^{(kp-n+1)}, \ldots, B_{n-1}^{(kp)}, B_n^{(kp-n)}$, for whose related $d(i,j)$ we have $d(i,j) \neq 0$ by Equation (4). Hence for this i we obtain $d(i,j) \neq 0$ for all $j, 0 \leq j < p$, due to the recursion relations.

Now, we consider the relation for x_i in Equation (2). As $y = T^s x \in B(k_{s+1}) := \{x \in B^n : \frac{1}{k_{s+1}+1} < x_n \leq \frac{1}{k_{s+1}}\}$ we get

$$\frac{1}{k_{s+1}+1} \leq \frac{B_i^{(s-n+1)} + \ldots + B_i^{(s)} + B_i^{(s-n)}}{B_0^{(s-n+1)} + \ldots + B_0^{(s)} + B_0^{(s-n)}} \leq k_{s+1} + 1, \quad i = 1, \ldots, n.$$

Due to periodicity, we can define $k^* := \max(k_1, \ldots, k_p) + 1$ to obtain

$$\frac{1}{k^*} \leq \frac{B_i^{(s-n+1)} + \ldots + B_i^{(s)} + B_i^{(s-n)}}{B_0^{(s-n+1)} + \ldots + B_0^{(s)} + B_0^{(s-n)}} \leq k^*, \quad i = 1, \ldots, n. \tag{7}$$

As a result, Equation (7) is also bounded for arbitrarily large s, but as $d(i,j) \neq 0$ is valid for $0 \leq j < p$ and a fixed $i \in \{0,\ldots,n\}$, the boundedness implies $d(i,j) \neq 0$ for all $0 \leq i \leq n$, $0 \leq j < p$.

Again, due to periodicity, with M being part of the expansion, M^k is also part of the expansion and the equation

$$M^k \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \lambda^k \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

shows that for a positive eigenvalue λ of the matrix M the relation

$$B_0^{(kp-n+1)} + x_1 B_0^{(kp-n+2)} + \ldots + x_{n-1} B_0^{(kp)} + x_n B_0^{(kp-n)} = \lambda^k \tag{8}$$

holds.

But, as Equation (8) is, due to periodicity, valid for arbitrarily large k and $d(0,j) \neq 0$ for all $0 \leq j < p$, we get $\lambda = \rho_0$. Thus, by Equation (2) we derive for $T^p x = x$ and for all $1 \leq i \leq n$ the relation

$$x_i = \rho_0^{-k} \left(B_i^{(kp-n+1)} + x_1 B_i^{(kp-n+2)} + \ldots + x_{n-1} B_i^{(kp)} + x_n B_i^{(kp-n)} \right). \tag{9}$$

Due to periodicity, Equation (9) holds for arbitrarily large k; hence, since $d(i,j)$ exists for all $0 \leq i \leq n$, $0 \leq j < p$, every single limit in the relation

$$x_i = \lim_{k\to\infty} \rho_0^{-k} B_i^{(kp-n+1)} + x_1 \lim_{k\to\infty} \rho_0^{-k} B_i^{(kp-n+2)} + \ldots + x_n \lim_{k\to\infty} \rho_0^{-k} B_i^{(kp-n)}$$

exists for all i, $1 \leq i \leq n$. Due to periodicity, we can write $\beta^{2kp+j} = M^k \beta^{kp+j}$ and thus see that for the entries of the matrices the relations

$$B_i^{2kp+j} = B_i^{(kp-n+1)} B_0^{(kp+j)} + \ldots + B_i^{(kp)} B_{n-1}^{(kp+j)} + B_i^{(kp-n)} B_n^{(kp+j)}$$

hold for $1 \leq i \leq n$ and $0 \leq j \leq p-1$. As $\rho_0^k > |\rho_1^k| \geq \ldots \geq |\rho_n^k|$ follows from $\rho_0 > |\rho_1| \geq \ldots \geq |\rho_n|$, we have

$$d(i,j) = d(0,j) \lim_{k\to\infty} \rho_0^{-k} B_i^{(kp-n+1)} + \ldots + d(n,j) \lim_{k\to\infty} \rho_0^{-k} B_i^{(kp-n)}. \tag{10}$$

Hence, we set

$$x_i = \frac{d(i,j)}{d(0,j)}, \quad 1 \leq i \leq n \tag{11}$$

in Equation (9) and obtain Equation (10) as a result. By Equation (1) we know that the determinant of M^k is not zero. Thus, Equation (9) represents a system of

n equations in n variables and x_i is uniquely determined by Equation (11). Since this is valid for all j, $0 \le j < p$, and $d(i,j)$ does not depend on k, we finally get

$$\lim_{s\to\infty}\left(\frac{B_1^{(s)}}{B_0^{(s)}},\dots,\frac{B_n^{(s)}}{B_0^{(s)}}\right) = x.$$

□

Lemma 12. *Let M be a $(n+1)\times(n+1)-$matrix defined by*

$$M := \begin{pmatrix} 0 & \dots & k & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix},$$

where $k \in \mathbb{N}$ is positive. Then there is a $p(n) \in \mathbb{N}$, such that M^p is a positive matrix for all $p \ge p(n)$.

Proof. Notice that $M = E' + K$, where

$$E' := \begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & 0 \end{pmatrix}, \quad K := \begin{pmatrix} 0 & \dots & k & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix}.$$

Furthermore $(E')^{n+1} = \not\Vdash$ represents the unit matrix and K^2 the zero matrix. Then clearly the relation

$$K(E')^{n-1} = \begin{pmatrix} k & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

holds.

Now, we set $p = n^2+1$ and consider M^p. Then we get $M^p = K^* + A$, where K^* is given by

$$K^* = \begin{pmatrix} k^{n-1} & \cdots & k & 1 & k^n \\ k^n & \ddots & & k & 1 \\ 1 & \ddots & & & k \\ \vdots & \ddots & & & \vdots \\ k^{n-2} & \cdots & 1 & k^n & k^{n-1} \end{pmatrix}$$

and $A \geq 0$. The shape of K^* results from

$$\begin{aligned}
K^* &= (E')^{n^2+1} + \sum_{j=0}^{n-1}\sum_{i=0}^{n}(E')^i[K(E')^{n-1}]^j K(E')^{n^2-jn-i} \\
&= (E')^{n^2+1} + \sum_{i=0}^{n}(E')^i K(E')^{n^2-i} \\
&\quad + \sum_{i=0}^{n}(E')^i K(E')^{n-1}K(E')^{n^2-n-i} \\
&\quad + \cdots \\
&\quad + \sum_{i=0}^{n}(E')^i \underbrace{K(E')^{n-1}\ldots K(E')^{n-1}}_{[K(E')^{n-1}]^{n-1}} K(E')^{n-i}.
\end{aligned}$$

Hence, for all $p \geq n^2+1$ the matrix M^p has only posivite entries. Therefore, the matrix β^p of a periodic MSA is a positive matrix for $k \geq 1$. □

3.5.2. Example of a Periodic MSA

Unfortunately the MSA of $x := (\sqrt[3]{4}, \sqrt[3]{2})$ does not become periodic within the first 40 steps of expansion. Although we are unaware of whether or not periodicity eventually occurs in this expansion, there certainly are periodic expansions, some even of period length 1.

Example. We consider the tuple $x := (x_1, x_2) = (\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2})$ and apply Selmer's multiplicative algorithm. Note that $x_2 = x_1^2$. Then we obtain

$$Tx = \left(\frac{3-\sqrt{5}}{\sqrt{5}-1}, \frac{-4+2\sqrt{5}}{\sqrt{5}-1}\right)$$

and by multiplying each fraction by $\frac{\sqrt{5}+1}{\sqrt{5}+1}$ we obtain

$$Tx = \left(\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2}\right).$$

Hence the MSA for $x := (x_1, x_2) = (\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2})$ becomes periodic with a period of length 1.

3.5.3. Some General Results on the Periodic MSA

Since we proved in Theorem 11 that the MSA is uniformly weakly convergent we can show that the coordinates of the limit points are rational functions of the largest eigenvalue of the periodicity matrix. Due to the weakly convergence of a periodic MSA, we can apply a theorem on approximation properties of MCFs which can be traced back to Perron [9]. It states, in terms of the MSA, that for all $\varepsilon > 0$ and $g > g(\varepsilon)$ we have $|B_0^{(pg)} x_i - B_i^{(pg)}| < |\rho_1(1+\varepsilon)|^g, \quad i = 1, \ldots, n$, where p is the period length and ρ_1 is, as in Theorem 11, the second largest eigenvalue of M. The general form of these theorems for multidimensional continued fractions can be found in [11, p. 115-119, 157-162].

Theorem 13. *Assume that the algorithm of $x = (x_1, \ldots, x_n)$ eventually becomes periodic with period length p. Then $x_1, \ldots, x_n$ are rational functions in ρ_0, where ρ_0 denotes the largest eigenvalue of the characteristic polynomial of the periodicity matrix $\beta^{(p)}$. Therefore $x_1, \ldots, x_n$ belong to a number field of degree at most $n+1$.*

Proof. Clearly we can assume that the expansion is purely periodic with period length p. Due to Theorem 11, we know that

$$M^k \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \rho_0^k \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and thus obtain $B_0^{(kp-n+1)} + x_1 B_0^{(kp-n+2)} + \ldots + x_{n-1} B_0^{(kp)} + x_n B_0^{(kp-n)} = \rho_0^k$. Therefore we can calculate $x_1, \ldots, x_n$ as rational functions in ρ_0 from the equation

$$M \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \rho_0 \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.$$

□

4. Outlook

Due to Jeffrey C. Lagarias [4] it is already known that the subtractive versions of Jacobi-Perron's, Brun's, and Selmer's algorithms are weakly convergent for periodic expansions. Now, due to Theorem 11 of this paper, we know that the same is true for the "odd" multiplicative Selmer algorithm (MSA). Hence, we can apply a theorem linking the convergence speed of a periodic expansion to the eigenvalues of the periodicity matrix [11, p. 157-162], which yields the approximation result mentioned in Section 3.5.3. But, what about convergence in the non-periodic case?

Lagarias [5] was able to show that every multidimensional continued fraction algorithm of dimension $n \geq 2$ must include approximations which are not best approximations. However, this does not exclude the possibility that multidimensional continued fractions can yield successive best approximations. But, it seems much more difficult to find results on approximation properties in the general case than in the purely periodic one. After all, the unanswered question of periodicity is what makes multidimensional continued fractions so fascinating.

Interestingly, Leon Bernstein [1] showed periodicity for certain groups of tuples using the Jacobi-Perron algorithm. And originally, Jacobi's intentions were to find an algorithm that yields periodic expansions for every cubic or higher-dimensional irrational number, just as the simple continued fraction algorithm does for quadratic irrationals. With the technique of singularization [10], emerging fields of application [13, 3] and the help of computers, there should be a way to get hold of the periodicity of the multidimensional continued fractions.

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