ON THE TENNIS BALL PROBLEM

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Abstract

We generalize the original tennis ball problem to the case in which the number of balls received and the number of balls discarded can vary from move to move, and derive formulas solving the problem in that case. We then use these formulas to obtain new and simpler proofs of the relation between the solution to the original tennis ball problem and the Catalan numbers, as well as a considerably simpler proof for the generalization in which a fixed number s of balls are received in each move, and one ball is discarded.

1. Introduction and Notation

The *tennis ball problem* was introduced, in a somewhat different guise, by T. Tymoczko and J. Henle in their textbook on Logic [6]. It can be described as follows. Consider a game or process of n moves. In the first move you are given two tennis balls labeled 1 and 2. You throw out one of them. In the second move you are given balls labeled 3 and 4. You now have three tennis balls; the one you retained from the first move and the two new ones. You throw out one of these three balls. And so forth. In the n-th move you receive balls labeled $2n-1$ and $2n$, you add them to the $n-1$ tennis balls you still have to get a set of $n+1$ balls, and you throw out one ball from this set. The discarded balls are arranged in an increasing sequence by their labels. The tennis ball problem consists in determining the total number of such sequences. It should be emphasized that the order in which the balls are discarded is disregarded. Thus if $n = 3$ one possible set of moves is to discard ball

1 in the first move. In the second move one gets balls 3,4; one can discard number 4. In the third move one gets balls 5,6; one can discard ball 2 (still in one's possession). This produces the sequence $(1, 2, 4)$. This is the same sequence produced if we discard ball 2 in the first move, ball 1 in the second move, and ball 4 in the third move. In [3], R.P. Grimaldi and J. Moser prove that the number of distinct increasing sequences after n moves equals C_{n+1}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Since the publication of the article of Grimaldi and Moser [3], a number of papers have appeared dealing with generalizations of the problem and related problems. Among them we want to mention the article by D. Merlini, R. Sprugnoli, and M.C. Verri that considers the s-ball tennis problem, in which in each move s balls are added, one is discarded (and also a particular case of the s, t problem; s balls are added in each move and t are discarded), the article by C.L. Mallows and L. Shapiro [4] which asks for the sum over all possible sequences in the (original) tennis ball problem, and several as yet unpublished articles by J. Fallon, S. Gao, and H. Niederhausen [1], [2].

In this paper we consider at first a more general form of the tennis ball problem. We assume given positive integers $s_1, \ldots, s_n, t_1, \ldots, t_n$ such that $t_j < s_j$ for $j = 1, \ldots, n$ and consider an n-moves tennis ball game with $\bar{s} = s_1 + \cdots + s_n$ tennis balls labeled from 1 to $\bar{s}$. In the j-th move of this game, you are handed the s_j balls labeled from $s_1 + \cdots + s_{j-1} + 1$ to $s_1 + \cdots + s_j$ (with $s_1 + \cdots s_{j-1} = 0$ if $j = 1$), and you throw out t_j of the balls from among all the balls you have. As in the original game, the labels of the discarded balls get arranged into an increasing sequence (regardless of the order in which they were discarded). We will call such a sequence an $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-set and the problem consists in determining the total number, to be denoted by $N(s_1, \ldots, s_n; t_1, \ldots, t_n)$, of $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-sets. We find a formula for $N(s_1, \ldots, s_n; t_1, \ldots, t_n)$ and then use it to produce simplified proofs of the known results; for the s, t problem (in the terminology of [5]) if $t = 1$, as well as a second short proof for the original $2, 1$ problem. We abbreviate $N(s_1, \ldots, s_n; t_1, \ldots, t_n)$ by $N(s, t)$ in case $s_1 = \cdots = s_n = s$, $t_1 = \cdots = t_n = t$, $1 \le t < s$.

In addition to the notation already introduced, the following notation will be used throughout: We set $\bar{t} = t_1 + \cdots + t_n$. $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}^0$ the set of non-negative integers, $\mathbb{Z}^+$ the set of positive integers. If $(k_1, \ldots, k_n), (\ell_1, \ldots, \ell_n) \in (\mathbb{Z}^0)^n$, we define

$$(k_1, \ldots, k_n) \succeq (\ell_1, \ldots, \ell_n)$$

to mean

$$k_1 + \cdots k_j \ge \ell_1 + \cdots \ell_j \quad \text{for } j = 1, \ldots, n-1; \; k_1 + \cdots k_n = \ell_1 + \cdots \ell_n.$$

We will need to consider the following sets of n-tuples.

$$\begin{aligned} I_n &= \{(\gamma_1,\ldots,\gamma_n)\in(\mathbb{Z}^+)^n\ :\gamma_{j-1}+t_j\le\gamma_j\le s_1+\cdots+s_j\ (1\le j\le n)\},\\ &\quad (\text{where } \gamma_{j-1}=0 \text{ if } j=1).\\ M_n &= \{(m_1,\ldots,m_n)\ :\ m_j\ge t_j-1, m_1+\cdots+m_j\le s_1+\cdots+s_j-j,\\ &\qquad\qquad 1\le j\le n\}.\\ K_n &= \{(k_1,\ldots,k_n)\in(\mathbb{Z}^0)^n\ :\ (k_1,\ldots,k_n)\succeq(t_1,\ldots,t_n)\}. \end{aligned}$$

By C_n we denote the n-th Catalan number: $C_n=\dfrac{1}{n+1}\left(\begin{array}{c}2n\\ n\end{array}\right)$. We recall that the Catalan numbers satisfy the recurrence relation $C_0=1$, and

$$C_{n+1}=\sum_{j=0}^{n}C_jC_{n-j} \tag{1}$$

for $n=0,1,2,\ldots$.

2. Results

Theorem 1. *An ascending sequence*

$$1\le i_1<i_2<\cdots<i_{\bar t}\le\bar s \tag{2}$$

is an $(s_1,\ldots,s_n;t_1,\ldots,t_n)$-set if and only if

$$(i_{t_1},i_{t_1+t_2},\ldots,i_{t_1+t_2+\cdots+t_n})\in I_n$$

Furthermore, given an n-tuple $\gamma=(\gamma_1,\ldots,\gamma_n)\in I_n$, then the number of $(s_1,\ldots,s_n;t_1,\ldots,t_n)$-sets $(i_1,\ldots,i_{\bar t})$ with $(i_{t_1},i_{t_1+t_2},\ldots,i_{t_1+t_2+\cdots+t_n})=\gamma$ is given by

$$N^{(\gamma)}(s_1,\ldots,s_n;t_1,\ldots,t_n)=\prod_{j=1}^{n}\left(\begin{array}{c}\gamma_j-\gamma_{j-1}-1\\ t_j-1\end{array}\right) \tag{3}$$

Proof. Assume first $(i_1,\ldots,i_{\bar t})$ is an $(s_1,\ldots,s_n;t_1,\ldots,t_n)$-set. After the j-th move, a total of $t_1+\cdots+t_j$ balls with labels in the range $1,\ldots,s_1+\cdots+s_j$ have been thrown out. The highest labeled ball, the ball now in position $i_{t_1+\cdots+t_j}$, must have a label $\le s_1+\cdots+s_j$. As more balls get discarded in succeeding moves, this ball can only be shifted to the right, so that the ball in position $i_{t_1+\cdots+t_j}$ at the end of the game has label less than or equal to the one in position $i_{t_1+\cdots+t_j}$ after the j-th move. It follows that $i_{t_1+\cdots+t_j}\le s_1+\cdots+s_j$. Moreover, since there are t_j steps between $i_{t_1+\cdots+t_{j-1}}$ and $i_{t_1+\cdots+t_j}$; that is,

$$i_{t_1+\cdots+t_{j-1}}<i_{t_1+\cdots+t_{j-1}+1}<\cdots<i_{t_1+\cdots+t_j},$$

we see that $i_{t_1+\cdots+t_{j-1}}+t_j \le i_{t_1+\cdots+t_j}$ for $j = 1, \ldots, n$. This proves that $(i_{t_1}, \ldots, i_{t_n}) \in I_n$. Conversely, assume that $(i_1, \ldots, i_{\bar{t}})$ satisfies (2) and $(i_{t_1}, \ldots, i_{t_n}) \in I_n$. One way of playing a game that results in that sequence is as follows. Since $i_{t_1} \le s_1$, we can throw out balls labeled from 1 to i_{t_1} in the first move, since $i_{t_1+t_2} \le s_1 + s_2$, we can throw out the balls labeled from $t_1 + 1$ to $t_1 + t_2$ in the second move, and so forth. It follows that $(i_1, \ldots, i_{\bar{t}})$ is an $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-set.

Finally, assume $\gamma = (\gamma_1, \ldots, \gamma_n) \in I_n$. To get an $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-set, for $j = 1, \ldots, n$ with $(i_{t_1}, \ldots, i_{t_n}) = \gamma$ we are free to choose for the $t_j - 1$ labels in the range $[i_{t_1+\cdots+t_{j-1}+1}, i_{t_1+\cdots+t_j-1}] = [\gamma_{j-1}+1, \gamma_j-1]$ any strictly increasing sequence of length $t_j - 1$. Since there are

$$\binom{\gamma_j - \gamma_{j-1} - 1}{t_j - 1}$$

such choices, Formula (3) follows. □

Our first formula for the solution to the generalized tennis ball problem follows easily from this result.

Theorem 2. *The following holds:*

$$N(s_1, \ldots, s_n; t_1, \ldots, t_n) = \sum_{(m_1,\ldots,m_n)\in M_n} \left(\prod_{j=1}^{n} \binom{m_j}{t_j - 1} \right).$$

Proof. It is easy to see that there is a bijection between I_n and M_n given by

$$(\gamma_1, \ldots, \gamma_n) \mapsto (m_1, \ldots, m_n)$$

where $m_j = \gamma_j - \gamma_{j-1} - 1$ for $j = 1, \ldots, n$. If $\gamma = (\gamma_1, \ldots, \gamma_n)$ and $(m_1, \ldots, m_n)$ are so related then Theorem 1 proves that

$$N^{(\gamma)}(s_1, \ldots, s_n; t_1, \ldots, t_n) = \prod_{j=1}^{n} \binom{m_j}{t_j - 1},$$

thus

$$\begin{aligned} N(s_1, \ldots, s_n; t_1, \ldots, t_n) &= \sum_{\gamma\in I_n} N^{(\gamma)}(s_1, \ldots, s_n; t_1, \ldots, t_n) \\ &= \sum_{(m_1,\ldots,m_n)\in M_n} \prod_{j=1}^{n} \binom{m_j}{t_j - 1}. \end{aligned}$$

□

As a corollary, we get our first proof of the formula for $N(2, 1)$.

Theorem 3. *We have*

$$N(2,1) = C_{n+1} = \frac{1}{n+2}\binom{2(n+1)}{n+1}.$$

Proof. We are assuming $s_1 = \cdots = s_n = 2$, $t_1 = \cdots = t_n = 1$. By Theorem 2 and the one-to-one correspondence between M_n and I_n, we get

$$N(2,1) = |M_n| = |I_n|.$$

Let $\sigma_n = |I_n|$, $n = 1, 2, \ldots$. We verify directly that $\sigma_1 = 2 = C_2$, $\sigma_2 = 5 = C_3$. We set $\sigma_0 = 1 = C_1$. We assume from now on $n \geq 2$. To count the elements of I_n we can use the following strategy. A sequence $(\gamma_1, \ldots, \gamma_n) \in I_n$ (in the case we consider) if and only if $1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_n$ and $\gamma_j \leq 2j$ for $j = 1, \ldots, n$. Let I_n^j be the set of all sequences in I_n such that $\gamma_j = 2j$ but $\gamma_\ell < 2\ell$ for $\ell = 1, \ldots, j-1$ $(1 \leq j \leq n)$. Let us denote by L_n the set of all sequences in I_n that are not in I_n^j for $j = 1, \ldots, n$. If $\gamma = (\gamma_1, \ldots, \gamma_n) \in L_n$ then $\gamma_1 = 1$ and it is easy to see that $\tilde{\gamma} = (\gamma_2 - 1, \ldots, \gamma_n - 1) \in I_{n-1}$. In fact, $\tilde{\gamma}$ is still strictly increasing and since the condition of being in L_n implies that $\gamma_j \leq 2j - 1$ for $j = 1, \ldots, n$, we see that the $\ell - th$ term of $\tilde{\gamma}$ satisfies

$$\tilde{\gamma}_\ell = \gamma_{\ell+1} - 1 \leq 2(\ell+1) - 2 = 2\ell$$

for $\ell = 1, \ldots, n-1$, so that the assertion follows. Conversely, given $\gamma = (\tilde{\gamma}_1, \ldots, \tilde{\gamma}_{n-1}) \in I_{n-1}$, one sees at once that $\gamma = (1, \tilde{\gamma}_1{+}1, \ldots, \tilde{\gamma}_{n-1}{+}1) \in L_n$. It follows that $|L_n| = |I_{n-1}| = \sigma_{n-1}$. Assume now $2 \leq j \leq n$. Then, as is immediately verified, a sequence $(\gamma_1, \ldots, \gamma_n) \in I_n^j$ satisfies that $(\gamma_1, \ldots, \gamma_{j-1}) \in L_{j-1}$, $(\gamma_{j+1} - 2j, \cdots, \gamma_n - 2j) \in I_{n-j}$, and conversely, given a sequence $(\gamma_1, \ldots, \gamma_{j-1}) \in L_{j-1}$, $\delta = (\delta_1, \ldots, \delta_{n-j}) \in I_{n-j}$, then $(\gamma_1, \ldots, \gamma_{j-1}, 2j, \delta_1 + 2j, \ldots, \delta_{n-j} + 2j) \in I_n^j$. Thus

$$|I_n^j| = |L_{j-1}| \cdot |I_{n-j}| = \sigma_{j-2}\sigma_{n-j}.$$

If $(\gamma_1, \ldots, \gamma_n) \in I_n^1$, then $\gamma_1 = 2$ and one verifies that the map $(\gamma_1, \ldots, \gamma_n) \mapsto (\gamma_2 - 2, \ldots, \gamma_n - 2)$ is a bijection from I_n^1 onto I_{n-1}; thus $|I_n^1| = |I_{n-1}| = \sigma - n - 1$. Since I_n is clearly the disjoint union of L_n and $I_n^1, \ldots, I_n^n$, we get that

$$\sigma_n = 2\sigma_{n-1} + \sum_{j=2}^{n} \sigma_{j-2}\sigma_{n-j}. \tag{4}$$

Introducing $\tau_0 = 1$, $\tau_n = \sigma_{n-1}$ for $n = 1, 2, \ldots$, we can write (4) in the form

$$\tau_{n+1} = \tau_0\tau_n + \sum_{j=2}^{n} \tau_{j-1}\tau_{n-j+1} + \tau_n\tau_0 = \sum_{j=0}^{n} \tau_j\tau_{n-j}.$$

This is the recursion relation (1) for the Catalan numbers; since $\tau_0 = 1 = C_0$, and $\tau_1 = \sigma_0 = 1 = C_1$, we conclude that $\tau_n = C_n$, hence $\sigma_n = C_{n+1}$, proving the theorem. □

Next we get another formula for the general case.

Theorem 4. *A sequence $(i_1, i_2, \ldots, i_{\bar{t}})$ satisfying (2) is an $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-set if and only if $(k_1, \ldots, k_n) \in K_n$, where*

$$k_j = \left| \mathbb{Z}[s_1 + \cdots + s_{j-1} + 1, s_1 + \cdots + s_j] \cap \{i_1, i_2, \ldots, i_t\} \right|, \tag{5}$$

for $j = 1, \ldots, n$, where $\mathbb{Z}[a, b]$ denotes the set of integers m such that $a \le m \le b$.

Proof. Suppose first that $(i_1, i_2, \ldots, i_t)$ is an $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-set. The rules of the game imply that $t_1 + \cdots + t_j$ balls are thrown out from the set labeled from 1 to $s_1 + \cdots + s_j$. That means that at the end of the game the total number of balls thrown out from the interval $[1, s_1 + \cdots + s_j]$ will be at least $t_1 + \cdots t_j$; proving $k_1 + \cdots + k_j \ge t_1 + \cdots + t_j$ for $j = 1, \ldots, n$. However, the case n is special; all balls have labels in the interval $[1, \bar{s}]$, thus $k_1 + \cdots + k_n$ must coincide with the total number of balls thrown out; i.e., must equal t, completing the proof that $(k_1, \ldots, k_n) \in K_n$. Conversely assume that $(i_1, i_2, \ldots, i_t)$ is an ascending sequence and that $k = (k_1, \ldots, k_n)$ defined by (5) is in K_n. For $j = 1, \ldots, n$, there are $k_1 + \cdots + k_j$ balls in the interval $[1, s_1 + \cdots + s_j]$; since $k_1 + \cdots + k_j \ge t_1 + \cdots + t_j$, the balls labeled from 1 to $i_{t_1 + \cdots + t_j}$ must be among these. It follows that there is a game in which in the j-th round we throw out the balls labeled from $i_{t_1 + \cdots + t_{j-1}} + 1$ to $i_{t_1 + \cdots + t_j}$; in other words, $(i_1, i_2, \ldots, i_t)$ is an $(s_1, \ldots, s_n; t_1, \ldots, t_n)$-set. ☐

Theorem 5. *The following holds:*

$$N(s_1, \ldots, s_n; t_1, \ldots, t_n) = \sum_{(k_1, \ldots, k_n) \in K_n} \binom{s_1}{k_1} \cdots \binom{s_n}{k_n}.$$

Proof. This follows at once from Theorem 4 ☐

As an easy consequence, we obtain another proof of the formula for the classical tennis ball problem, the case in which $s_1 = \cdots = s_n = 2$ and $t_1 = \cdots t_n = 1$.

Corollary 6. *We have*

$$N_n(2, 1) = \frac{1}{n+2} \binom{2(n+1)}{n+1}.$$

Proof. We write

$$\sigma_n = \sum_{(k_1, \ldots, k_n) \in K_n} \binom{2}{k_1} \cdots \binom{2}{k_n},$$

where now K_n consists of all n-tuples $(k_1, \ldots, k_n)$ such that $(k_1, \ldots, k_n) \succeq (1, \ldots, 1)$. By Theorem 5, $N_n(2, 1) = \sigma_n$. We now follow a strategy similar to that used in the proof of Theorem 3. Let us denote by K_n^j the set of all elements $(k_1, \ldots, k_n) \in K_n$

such that j is the first integer ≥ 1 such that $\sum_{\nu=1}^{j} k_\nu = j$. Clearly K_n is the disjoint union of the sets K_n^j, $j = 1, \ldots, K_n^n$ so that

$$\sigma_n = \sum_{j=1}^{n} \sum_{(k_1,\ldots,k_n)\in K_n^j} \begin{pmatrix} 2 \\ k_1 \end{pmatrix} \cdots \begin{pmatrix} 2 \\ k_n \end{pmatrix}.$$

Assume $2 \leq j \leq n$ and let $k = (k_1, \ldots, k_n) \in K_n^j$. Then $(k_{j+1}, \ldots, k_n) \in K_{n-j}$. Concerning $(k_1, \ldots, k_j)$, because $j \geq 2$, we must have $k_1 = 2$. Moreover, $k_j = 0$, otherwise we would have had $k \in K_n^i$ for some $i < j$. It is then easy to see that $(k_2, \ldots, k_{j-1}) \in K_{j-2}$. This defines a bijective mapping $K_n^j \to K_{j-2} \times K_{n-j}$, where we identify $K_0 \times K_{n-2} = K_{n-2} = K_{n-2} \times K_0$. The set K_1 consists of all elements k with $k_1 = 1$; dropping that initial 1 what remains is an element of K_{n-1}. It follows that

$$\begin{aligned} \sigma_n \;&=\; \sum_{(k_2,\ldots,k_n)\in K_{n-1}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdots \begin{pmatrix} 2 \\ k_n \end{pmatrix} + \\ &\quad \sum_{j=2}^{n} \sum_{\substack{(k_2,\ldots,k_{j-1}) \in K_{j-2} \\ (k_{j+1},\ldots,k_n) \in K_n^j}} \begin{pmatrix} 2 \\ k_2 \end{pmatrix} \cdots \begin{pmatrix} 2 \\ k_{j-1} \end{pmatrix} \begin{pmatrix} 2 \\ k_{j+1} \end{pmatrix} \cdots \begin{pmatrix} 2 \\ k_n \end{pmatrix} \\ &=\; 2\sigma_{n-1} + \sum_{j=2}^{n} \sigma_{j-2}\sigma_{n-j}, \end{aligned}$$

where $\sigma_0 = 1$ and $\sigma_1 = 2$. The numbers σ_n satisfy thus the relation (4) of Theorem 3, with the same initial conditions. As in that theorem, it follows that

$$\sigma_n = C_{n+1} = \frac{1}{n+2} \begin{pmatrix} 2(n+1) \\ n+1 \end{pmatrix}.$$

□

We want to generalize the result of the corollary to the case $s_1 = \cdots = s_n = s$, were $s \geq 2$; $t_1 = \cdots = t_n = 1$. For this we need two preparatory lemmas.

Lemma 7. *Let* $X_n = \{(x_0, \ldots, x_n) \in \left(Z^0\right)^{n+1} : \sum_{j=0}^{n} x_j = n\}$ *and let* $s \geq 2$. *Then*

$$\sum_{(x_0,\ldots,x_n)\in X_n} \begin{pmatrix} s \\ x_0 \end{pmatrix} \cdots \begin{pmatrix} s \\ x_n \end{pmatrix} = \begin{pmatrix} (n+1)s \\ n \end{pmatrix}, \qquad n = 0, 1, 2, \ldots$$

Proof. We have

$$\begin{aligned}(1+y)^{(n+1)s} &= \underbrace{(1+y)^s(1+y)^s\cdots(1+y)^s}_{n+1} = \prod_{j=0}^{n}\sum_{x_j=0}^{s}\binom{s}{x_j}y^{x_j}\\ &= \sum_{i=0}^{(n+1)s}\left(\sum_{x_0+\cdots+x_n=i}\prod_{j=0}^{n}\binom{s}{x_j}\right)y^i.\end{aligned}$$

On the other hand,

$$(1+y)^{(n+1)s} = \sum_{i=0}^{(n+1)s}\binom{(n+1)s}{i}y^i.$$

The result follows equating the coefficient for y^n in the two expressions for $(1+y)^{(n+1)s}$. □

Note. When $x_i > s$ we have $\binom{s}{x_i} = 0$.

Lemma 8. *Let X_n be as in Lemma 7 and let $(x_0,\ldots,x_n)\in X_n$. There exists then a unique index k such that*

$$(x_k,\ldots,x_{k+n}) \succeq (\underbrace{1,1,\ldots,1}_{n},0), \tag{6}$$

where the indices are computed modulo $n+1$.

Proof. We begin proving uniqueness. Assume two such indices exist. Without loss of generality, we may assume that one of these indices is 1, the other one $k>1$; i.e., we may assume that

$$(x_0,\ldots,x_n) \succeq (1,1,\ldots,1,0), \tag{7}$$

$$(x_k,\ldots,x_{n+k}) \succeq (1,1,\ldots,1,0). \tag{8}$$

In particular, by (7),

$$x_1+\cdots+x_{k-1}\geq k.$$

Subtracting from $x_0+\cdots x_n = n$ we get

$$x_k+\cdots+x_{n+1}\leq n-k,$$

a contradiction since, by (8), we should have $x_k+\cdots+x_{n+1}\geq n-k+1$. This proves uniqueness.

We prove existence by induction on n. The result being obvious for $n=1$, assume it has been proved for some $n\geq 1$. Let $(x_0,\ldots,x_{n+1})\in X_{n+1}$. There has to exist at least one index j with $x_j = 0$. Since not all components can be 0, without loss of generality we may assume that $x_n > 0$, $x_{n+1} = 0$. Now set $y_j = x_j$ for $j = 0, \dots, n-1$, $y_n = x_n - 1$. Then $(y_0, \dots, y_n) \in X_n$ and by the induction hypothesis there is k such that $(y_k, \dots, y_{n+k}) \succeq (1, \dots, 1, 0)$. Let r be the index such that $k + r = 0$ modulo $n + 1$; it is then easy to see that

$$(\underbrace{1, \dots, 1}_{n+1}, 0) \preceq (y_k, \dots, y_{k+s} + 1, 0, y_{k+s+1}, \dots, y_{k+n}) = (x_k, \dots, x_{k+n+1}).$$

The lemma is proved. □

We are ready to prove the promised generalization of Corollary 6.

Theorem 9. *Assume $s_1 = s_2 = \dots = s_n = s \geq 2$, $t_1 = \dots = t_n = 1$. Then*

$$N_n(s, 1) = \frac{1}{n+1} \binom{(n+1)s}{n}.$$

Proof. By Theorem 5,

$$N_n(s, 1) = \sum_{(x_0, \dots, x_n) - 1 \in K_n} \binom{s}{x_0} \cdots \binom{s}{x_{n-1}}.$$

Let $x = (x_0, \dots, x_n) \in X_n$. Suppose that there are two indices $0 \leq l < m \leq n$ such that $(x_l, \dots, x_{l+n}) = (x_m, \dots, x_{m+n})$. Without loss of generality, we may assume $l = 0$. By Lemma 8, there is an index k such that $(x_k, \dots, x_{k+n}) \succeq (1, 1, \dots, 1, 0)$. Then $(x_{k+m}, \dots, x_{k+m+n}) = (x_k, \dots, x_{k+n}) \succeq (1, 1, \dots, 1, 0)$, and $k + m \not\equiv k \pmod{n+1}$. This contradicts the uniqueness of such an index k stated in Lemma 8. Therefore, there is an $(n+1)$ to one correspondence between the set X_n and the set K_n; if $x = (x_0, \dots, x_n) \in X_n$ and if k is the unique index such that $(x_k, \dots, x_{k+n}) \succeq (1, 1, \dots, 1, 0)$, then $(x_k, \dots, x_{k+n}) \in K_n$. We assign to $x \in K_n$ the $n + 1$ distinct vectors that can be obtained from x by a cyclic shift. Since we must have $x_{k+n} = 0$, hence $\binom{s}{x_{k+n}} = 1$, we have

$$\begin{aligned} \sum_{(x_0, \dots, x_{n-1}) \in K_n} \binom{s}{x_0} \cdots \binom{s}{x_{n-1}} &= \frac{1}{n+1} \sum_{(x_0, \dots, x_n) \in X_n} \binom{s}{x_0} \cdots \binom{s}{x_n} \\ &= \frac{1}{n+1} \binom{(n+1)s}{n}, \end{aligned}$$

the last equality being a consequence of Lemma 7. □

Note. If $s = 2$ then

$$\frac{1}{n+1} \binom{(n+1)s}{n} = \frac{[2(n+2)]!}{(n+1)(n+2)!n!} = \frac{1}{n+2} \binom{2(n+1)}{n+1}$$

so that we recover the formula of Corollary 6.

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