SUPREMUM OF REPRESENTATION FUNCTIONS

G. Grekos
Université Jean Monnet, St-Etienne, France
grekosuniv-st-etienne.fr

L. Haddad
120 rue de Charonne, Paris, France
labib.haddadwanadoo.fr

C. Helou[1]
Pennsylvania State University, Media, Pennsylvania
cxh22psu.edu

J. Pihko
Kyläkunnantie 53, Helsinki, Finland
jukka.pihkohelsinki.fi

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Abstract

For a subset A of $\mathbb{N} = \{0, 1, 2, \dots\}$, the representation function of A is defined by $r_A(n) = |\{(a, b) \in A \times A : a + b = n\}|$, for $n \in \mathbb{N}$, where $|E|$ denotes the cardinality of a set E. Its supremum is the element $s(A) = \sup\{r_A(n) : n \in \mathbb{N}\}$ of $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$. Interested in the question "when is $s(A) = \infty$? ", we study some properties of the function $A \mapsto s(A)$, determine its range, and construct some subsets A of $\mathbb{N}$ for which $s(A)$ satisfies certain prescribed conditions.

1. Introduction

Let $A, B \subset \mathbb{N} = \{0, 1, 2, \dots\}$. The representation function for $A + B$ and its supremum are defined by

$$r_{A,B}(n) = |\{(a, b) \in A \times B : a + b = n\}| \ (\forall n \in \mathbb{N})$$

and

$$s(A, B) = \sup_{n \in \mathbb{N}} r_{A,B}(n) \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\},$$

where $|E|$ denotes the cardinality of a set E. In particular, for $A = B$, we write

$$r_A(n) = r_{A,A}(n) = |\{(a, b) \in A \times A : a + b = n\}| \ (\forall n \in \mathbb{N})$$

[1]Corresponding author.

and

$$s(A) = s(A, A) = \sup_{n \in \mathbb{N}} r_A(n) \in \overline{\mathbb{N}}.$$

The power series f_A associated with A, and its square g_A, which is the generating series of the sequence $(r_A(n))$, are

$$f_A(X) = \sum_{a \in A} X^a \quad , \quad g_A(X) = f_A(X)^2 = \sum_{n=0}^{\infty} r_A(n) X^n,$$

and $s(A)$ is simply the supremum of the coefficients of g_A. More generally,

$$g_{A,B}(X) = f_A(X) f_B(X) = \sum_{n=0}^{\infty} r_{A,B}(n) X^n.$$

Two celebrated conjectures of Erdős and Turán are ([1]):

(ET) If A is an asymptotic additive 2-basis of $\mathbb{N}$, then $s(A) = \infty$, i.e.,

$$(\exists n_0 \in \mathbb{N} : \; r_A(n) > 0, \; \forall n \geq n_0) \implies s(A) = \infty.$$

A more general one is

(GET) If $A = \{a_1 < a_2 < \cdots < a_n < \dots\} \subset \mathbb{N}$ is an infinite subset satisfying $a_n \leq cn^2$, for a constant $c > 0$ and all integers $n \geq 1$, then $s(A) = \infty$.

(GET) is more general than (ET), because if A is an asymptotic basis of $\mathbb{N}$, then there is a constant c such that $a_n \leq cn^2$ for all $n \geq 1$ ([4]).

This raises the more general question of determining the subsets A of $\mathbb{N}$ for which $s(A) = \infty$. This is a more restricted problem than the notoriously difficult and open one of determining all possible representation functions of bases for $\mathbb{N}$. It is to be noted that the difficulty in such problems seems to arise from the fact that $\mathbb{N}$ is just a semi-group for addition, since the analogue of the latter problem for the additive group of rational integers $\mathbb{Z}$ has been completely solved ([6]). In what follows, we first establish some fundamental properties of the function $A \mapsto s(A)$, for subsets A of $\mathbb{N}$, we then study its compatibility with a natural order relation on the set of strictly increasing sequences in $\mathbb{N}$, we establish that the range of the function $A \mapsto s(A)$ is the whole interval $[2, \infty]$ of $\overline{\mathbb{N}}$, and we construct a family of pairs of disjoint subsets A, B of $\mathbb{N}$ such that $s(A) = s(B) = 2$ and $s(A \cup B) = \infty$. We then introduce the notion of proximity of two subsets, viewed as strictly increasing sequences, of $\mathbb{N}$ and study its relation with the function $A \mapsto s(A)$; thus, for instance, if two subsets A, B of $\mathbb{N}$ are close, in the sense that their general n-th terms are at a bounded distance, then $s(A) = \infty$ if and only if $s(B) = \infty$. We also study the relations of the function $A \mapsto s(A)$ with the counting function $A(x) = |\{a \in A : a \leq x\}|$, where x is a real number, and the caliber $cal(A) = \liminf_{n\to\infty} \frac{a_n}{n^2}$

of a subset $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ of $\mathbb{N}$, thus showing, for instance, that $s(A) \geq \sup_{x\geq 0} \frac{A(x)^2}{2x+1}$, and that $s(A) \geq \frac{1}{2\ cal(A)}$. Some of these results are contained in previous papers ([2, 3]) considered from a different perspective, but they are included here to make the study of the function $A \mapsto s(A)$ as complete and self-contained as possible.

2. Some Properties

2.1. Notations and Definitions

Let $\mathcal{I}$ denote the set of infinite subsets $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ of $\mathbb{N}$. Such a subset A is often identified with the strictly increasing sequence $(a_n)_{n\geq 1}$ of its elements. A (partial) order relation $<<$ is defined on $\mathcal{I}$ by setting, for $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ and $B = \{b_1 < b_2 < \cdots < b_n < \dots\}$ in $\mathcal{I}$,

$$A << B \iff a_n \leq b_n,\ \forall n \in \mathbb{N}^*,$$

where $\mathbb{N}^* = \mathbb{N} \setminus \{0\} = \{1, 2, 3, \dots\}$.

For any subset A of $\mathbb{N}$ and any $t \in \mathbb{N}$, we set $t + A = \{t + a : a \in A\}$ (translation of A), and $t \cdot A = \{ta : a \in A\}$ (dilation of A).

Thus, if we denote by $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$ the set of squares in $\mathbb{N}^*$, the conjecture (GET) amounts to:

(GET) For any $A \in \mathcal{I}$,

$$(\exists c \in \mathbb{N}^* : A << c \cdot \mathbb{S}) \implies s(A) = \infty.$$

Remark 1. For any $A, B \in \mathcal{I}$, we clearly have

$$B \subset A \implies s(B) \leq s(A) \text{ and } A << B.$$

This leads to the natural question of whether $A << B$ implies that $s(B) \leq s(A)$. Moreover, it is known that $s(\mathbb{S}) = \infty$ (e.g., this follows from [5], Theorem 278), and (GET) says that $A << \mathbb{S}$ implies $s(A) = \infty$. So another question is whether the double condition $A << B$ and $s(B) = \infty$ implies that $s(A) = \infty$.

However, as shown in Theorem 7 below, the answer to both questions is negative, and the relation $A << B$ is compatible with any choice of values of $s(A)$ and $s(B)$. Furthermore, the range of the function $s(A)$ is the whole interval $[2, \infty]$ of $\overline{\mathbb{N}}$. But first, we need some technical results.

Remark 2. For any subsets A, B of $\mathbb{N}$, finite or infinite, we have:

(1) $s(A, B) = s(B, A) \leq \min(|A|, |B|)$.

Indeed, $r_{A,B}(n) = |\{(a, n - a) : a \in A,\ n - a \in B\}| \leq |\{(a, n - a) : a \in A\}| = |A|$, and by symmetry, $r_{A,B}(n) = r_{B,A}(n) \leq |B|$, for any $n \in \mathbb{N}$.

(2) If $A \cap B = \emptyset$, then

(i) $r_{A\cup B}(n) = r_A(n) + r_B(n) + 2r_{A,B}(n), \quad$ for all $n \in \mathbb{N}$,

(ii) $r_{A\cup B,B}(n) = r_{A,B}(n) + r_B(n), \quad$ for all $n \in \mathbb{N}$,

(iii) $\max(s(A), s(B), 2s(A,B)) \le s(A \cup B) \le s(A) + s(B) + 2s(A,B)$,

(iv) $s(A \cup B) \le s(A) + 2|B|$.

The proofs of these are as follows:

(i) Indeed, as A and B are disjoint, $f_{A\cup B} = f_A + f_B$, and therefore $g_{A\cup B} = f_{A\cup B}^2 = g_A + g_B + 2g_{A,B}$. By identification of the coefficients, the equation holds.

(ii) As in the proof of (i), $g_{A\cup B,B} = (f_A + f_B)f_B = g_{A,B} + g_B$.

(iii) It follows from (i) that $\max(r_A(n), r_B(n), 2r_{A,B}(n)) \le r_{A\cup B}(n) = r_A(n) + r_B(n) + 2r_{A,B}(n)$, for all $n \in \mathbb{N}$. Taking the supremum of the three terms yields the desired inequalities.

(iv) It follows from (i) and (ii) that $r_{A\cup B}(n) = r_A(n) + r_B(n) + 2r_{A,B}(n) = r_A(n) + r_{A\cup B,B}(n) + r_{A,B}(n) \le s(A) + s(A\cup B, B) + s(A,B)$, for all $n \in \mathbb{N}$, so that $s(A\cup B) \le s(A) + s(A\cup B, B) + s(A,B)$. And by (1) above, $s(A,B) \le |B|$ and $s(A \cup B, B) \le |B|$. Hence the inequality holds.

(3) In general, when A and B are not necessarily disjoint,

$$\begin{aligned} \max(s(A), s(B), s(A,B)) \le s(A \cup B) &\le s(A) + s(B \setminus A) + 2s(A, B \setminus A) \\ &\le s(A) + s(B) + 2s(A,B), \end{aligned}$$

$$s(A \cup B) \le s(A) + 2|B \setminus A| \le s(A) + 2|B|,$$

and by symmetry

$$s(A \cup B) \le s(B) + 2|A \setminus B| \le s(B) + 2|A|.$$

Indeed, letting $C = B \setminus A$, as $A \cup B = A \cup C$, with A and C disjoint and $C \subset B$, by (2), we have $s(A \cup B) = s(A \cup C) \le s(A) + s(C) + 2s(A,C) \le s(A) + s(B) + 2s(A,B)$, and $s(A\cup B) = s(A\cup C) \le s(A) + 2|C| \le s(A) + 2|B|$. This proves all inequalities except the first one, which follows from the fact that A and B are subsets of $A \cup B$.

(4) In particular, if B is finite, then $s(A) = \infty$ if and only if $s(A \cup B) = \infty$.

Indeed, by (3), we have $s(A) \le s(A \cup B) \le s(A) + 2|B|$.

(5) The last two inequalities in (3) are optimal, as seen from the following family of examples, where A and B are finite and disjoint, and satisfy $s(A \cup B) = s(A) + 2|B|$. Indeed, let $h, t \in \mathbb{N}$ such that $0 < 2h < t$, consider the integer intervals $U = [1, h]$ and $V = [2h+1, 2h+t]$, and set $A = U \cup V$ and $B = [h+1, 2h] \subset \mathbb{N}$. Then $|B| = h$, and $A \cup B = [1, 2h+t]$. Therefore $s(A \cup B) = 2h+t = t+2|B|$, and we claim that $s(A) = t$, thus implying the desired equality.

Proof of claim. First note that if $I = [0, m]$ and $J = [0, n]$ are intervals in $\mathbb{N}$, with $0 \leq m \leq n$, then $g_{I,J}(X) = (\sum_{i=0}^{m} X^i)(\sum_{j=0}^{n} X^j) = \sum_{k=0}^{m+n} r_{I,J}(k) X^k$, where

$$r_{I,J}(k) = \begin{cases} k+1 & \text{if } 0 \leq k \leq m \\ m+1 & \text{if } m \leq k \leq n \\ m+n-k+1 & \text{if } n \leq k \leq m+n. \end{cases}$$

So the monomials with largest coefficient in $g_{I,J}$ are $(m+1)X^k$ for $m \leq k \leq n$, and thus $s(I, J) = m + 1$.

Since $A = U \cup V$, with U and V disjoint, as in (2)(i), we have $g_A = g_U + g_V + 2g_{U,V}$, where $g_U(X) = X^2(\sum_{i=0}^{h-1} X^i)^2 = X^2 g_I(X)$, with $I = [0, h-1]$, and $g_V(X) = X^{4h+2}(\sum_{j=0}^{t-1} X^j)^2 = X^{4h+2} g_J(X)$, with $J = [0, t-1]$, and

$$2g_{U,V}(X) = 2X^{2h+2}(\sum_{i=0}^{h-1} X^i)(\sum_{j=0}^{t-1} X^j) = 2X^{2h+2} g_{I,J}(X).$$

So, applying what precedes with $m = h-1$ and $n = t-1$, we see that the only monomial with largest coefficient in g_U is hX^{h+1} (resp., in g_V, is tX^{4h+t+1}), and the monomials with largest coefficient in $2g_{U,V}$ are $2hX^k$ for $3h+1 \leq k \leq 2h+t+1$. Moreover, the degree of g_U is $2h$, while the least degree of a monomial in $g_V + 2g_{U,V}$ is $2h+2 > 2h$, so that g_U and $g_V + 2g_{U,V}$ have no common monomial. On the other hand, the sum of the common monomials in g_V and $2g_{U,V}$ is

$$\sum_{j=4h+2}^{2h+t+1} (j - 2h - 1)X^j + \sum_{j=2h+t+2}^{3h+t} (2h + 2t - j + 1)X^j,$$

in which the largest coefficient is t, as for g_V. We thus conclude that the largest coefficient in $g_A = g_U + g_V + 2g_{U,V}$ is t, i.e., $s(A) = t$. ⋄

Definition 3. A subset A of $\mathbb{N}$ (finite or infinite) is called *sparse* whenever the relation $a < b$ between two elements of A implies $2a < b$.

Notation 4. For two subsets X, Y of $\mathbb{N}$, we write $X < Y$ whenever for each $x \in X$ and each $y \in Y$ we have $x < y$. In this case, X is finite, possibly empty. When both $X, Y \neq \emptyset$, the relation $X < Y$ amounts to $\max(X) < \min(Y)$. When $X = \{x\}$ is a singleton, we simply write $x < Y$ instead of $\{x\} < Y$. Similarly, when $Y = \{y\}$, we write $X < y$ for $X < \{y\}$.

We similarly define $X \leq Y$, and $x \leq Y$ or $X \leq y$.

Lemma 5. *Let A, F be two subsets of $\mathbb{N}$, with A sparse, nonempty, and F finite, possibly empty, such that $2 \cdot F < A$. Then $s(F) \leq s(F \cup A) \leq \max(s(F), 2)$. If in addition $|F \cup A| \geq 2$, then $s(F \cup A) = \max(s(F), 2)$.*

Proof. Let $B = F \cup A$ and $T = \{(b, a) \in B \times A : b \leq a\}$, and define a function $\sigma : T \to \mathbb{N}$ by $\sigma(b, a) = b + a$. We first show that σ is injective. Indeed, for any $(b, a), (d, c) \in T$, if $(b, a) \neq (d, c)$, then either $a \lesseqgtr c$ or ($a = c$ and $b \neq d$). If $a < c$, then $c > 2a$ (since a, c lie in A which is sparse) and $d + c > 2a \geq b + a$. Similarly, if $a > c$, then $b + a > d + c$. If $a = c$ and $b \neq d$, then $b + a = b + c \neq d + c$. Thus, in all cases, $(b, a) \neq (d, c)$ implies $\sigma(b, a) \neq \sigma(d, c)$.

Now, for any $n \in \mathbb{N}$, if $n < A$, then $r_B(n) = r_F(n) \leq s(F)$. Otherwise, $2 \cdot F < a \leq n$ for some $a \in A$, so that $n \notin F + F$, and therefore $r_B(n) = |\{(x, y) \in (B \times A) \cup (A \times B) : x + y = n\}| \leq 2|\{(b, a) \in T : b + a = n\}| \leq 2$, since σ is injective. Thus $s(B) \leq \max(s(F), 2)$. Moreover, $s(F) \leq s(B)$, since $F \subset B$.

If, in addition, $|B| \geq 2$, then $s(B) \geq 2$, and therefore $s(B) = \max(s(F), 2)$. □

The following is the special case $F = \emptyset$ of Lemma 5.

Corollary 6. *If A is sparse, then $s(A) \leq 2$. If in addition $|A| \geq 2$, then $s(A) = 2$.*

Theorem 7. *For any A in $\mathcal{I}$ and any q in the interval $[2, \infty]$ of $\overline{\mathbb{N}}$, there exists B in $\mathcal{I}$ such that $A << B$ and $s(B) = q$.*

Proof. The proof is divided into two parts, according as $q \in [2, \infty) \subset \mathbb{N}$ or $q = \infty$.

i). Let q be an integer greater than or equal to 2. First note that there exists a sparse subset C of $\mathbb{N}$ such that $A << C$. Indeed, if $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$, define $C = \{c_1 < c_2 < \cdots < c_n < \dots\}$ by $c_1 = a_1$, and $c_{n+1} = \max(a_{n+1}, 2c_n + 1)$ for $n \geq 1$. So, replacing A by C, we may assume A sparse, and therefore $s(A) = 2$.

Let $h \in \mathbb{N}^*$ such that $h > a_q$, and $F = \{nh : 1 \leq n \leq q\}$. Also, let $p \in \mathbb{N}^*$ such that $a_p > 2qh$, and define $B = \{b_1 < b_2 < \cdots < b_n < \dots\}$ by

$$b_n = \begin{cases} nh & \text{if } 1 \leq n \leq q \\ a_{p+n} & \text{if } n > q. \end{cases}$$

Then $a_n \leq a_q < nh = b_n$ for $1 \leq n \leq q$, and $a_n < a_{p+n} = b_n$ for $n > q$, so that $A << B$. Also, $B = F \cup G$, where $G = \{a_{p+n} : n \geq q + 1\} \subset A$, so that G is sparse like A, and $2 \cdot F < a_p < G$. Therefore, by Lemma 5, $s(B) = \max(s(F), 2) = s(F) = q$, since F is an arithmetic progression of length q.

ii). If $q = \infty$, we define a sequence $(F_n)_{n \in \mathbb{N}^*}$ of subsets of $\mathbb{N}$ such that $F_1 < F_2 < \cdots < F_n < \dots$, and each F_n is an arithmetic progression of length n, say $F_n = \{kf_n : 1 \leq k \leq n\}$ with $f_n \in \mathbb{N}$, by setting $F_1 = \{a_1\}$, and inductively choosing an integer $f_{n+1} > \max(nf_n, a_{\frac{(n+1)(n+2)}{2}})$ and setting $F_{n+1} = \{kf_{n+1} : 1 \leq k \leq n+1\}$. We then let $B = \bigcup_{n=1}^{\infty} F_n$, so that $b_1 = a_1$, and for an index $m \geq 2$, if n is the unique integer such that $\frac{n(n+1)}{2} < m \leq \frac{(n+1)(n+2)}{2}$, and $1 \leq k := m - \frac{n(n+1)}{2} \leq n + 1$, then

$b_m = kf_{n+1} \geq f_{n+1} > a_{\frac{(n+1)(n+2)}{2}} \geq a_m$. Therefore $A << B$, and $s(B) \geq s(F_n) = n$ for all $n \in \mathbb{N}^*$, i.e., $s(B) = \infty$. □

Remark 8. For any subset A of $\mathbb{N}$ and any $t \in \mathbb{N}$, we have $s(t + A) = s(A)$, and if if $t \neq 0$, then we have $s(t \cdot A) = s(A)$.

The proofs present no difficulty, and are left to the reader.

Remark 9. In view of Remark 2 (3), if $s(A \cup B) = \infty$, then at least one of $s(A)$ or $s(B)$ or $s(A, B)$ is infinite. This naturally leads to the following question:

(Q1) If $s(A \cup B) = \infty$, does it follow that $s(A)$ or $s(B)$ is equal to ∞?

This question is also equivalent to the following one:

(Q2) Do the conditions $s(A) < \infty$ and $s(B) < \infty$ imply that $s(A, B) < \infty$?

In what follows (Theorem 12), we give examples of subsets A, B of $\mathbb{N}$ such that $s(A) = s(B) = 2$ and $s(A \cup B) = \infty$, thus showing that the answers to questions (Q1) and (Q2) are negative. To that end, we first introduce a useful technical tool in the next section.

3. Complementary Sets

Two finite subsets A, B of $\mathbb{N}$ are called *complementary* if there exists an integer $m \geq \max(A)$ such that $B = m - A = \{m - a : a \in A\}$; more specifically, A and B are then called m-complementary. In this case, $A = m - B$, and $|A| = |B|$. Moreover, we clearly have

$$f_B(X) = X^m f_A\left(\frac{1}{X}\right),$$

so that $s(B) = s(A)$. Similarly,

$$g_{A,B}(X) = X^m f_A(X) f_A\left(\frac{1}{X}\right),$$

and therefore $s(A, B) = |A| = |B|$.

Moreover, if (A, B) and (C, D) are two pairs of complementary subsets of $\mathbb{N}$, with $B = m - A$ and $D = n - C$, then

$$g_{B,D}(X) = X^{m+n} g_{A,C}\left(\frac{1}{X}\right),$$

so that $s(B, D) = s(A, C)$.

Whence the following result

Lemma 10. *For any pair (A, B) of finite complementary subsets A, B of $\mathbb{N}$, we have*

- $s(A) = s(B)$
- $s(A, B) = |A| = |B|$
- *If (C, D) is any other pair of complementary subsets of $\mathbb{N}$, then $s(A, C) = s(B, D)$.*

Remark 11. For a subset A of $\mathbb{N}$ and two integers $n, r \in \mathbb{N}$, with $r > 0$, the condition $r_A(n) \geq r$ is equivalent to the existence of two n-complementary subsets U and $V = n - U$ of A of common cardinality r. Indeed, $r_A(n) \geq r$ if and only if there exist r distinct pairs $(a_i, n - a_i) \in A \times A$ $(1 \leq i \leq r)$, i.e., there exists a subset $U = \{a_1, \ldots, a_r\}$ of r elements of A such that $n - U$ is a subset V of A.

Therefore $r_A(n)$ is the maximal common cardinality of n-complementary subsets of A. Thus $s(A)$ is the supremum of the common cardinalities $|U| = |V|$ of all pairs (U, V) of complementary subsets of A. In particular, $s(A) = \infty$ if and only if A has pairs of complementary subsets of arbitrarily large cardinalities.

4. An Example

Theorem 12. *There exist two infinite, disjoint subsets A and B of $\mathbb{N}$ such that $s(A) = s(B) = 2$ and $s(A \cup B) = \infty$.*

Proof. The proof is carried out in three stages.

i) Construction. We define inductively a sequence $(A_n)_{n\in\mathbb{N}}$ of finite **sparse** subsets of $\mathbb{N}$ and a sequence $(m_n)_{n\in\mathbb{N}}$ of integers, starting with $A_0 = \emptyset$ and $m_0 = 0$, and satisfying the following conditions for all $n \in \mathbb{N}$:

$$|A_n| = n \quad , \quad 2m_n < A_{n+1} \quad , \quad 2 \cdot (m_n + A_{n+1}) < m_{n+1}.$$

For $n \geq 1$, we clearly have $A_n < m_n$, and we let $B_n = m_n - A_n$ to get a pair (A_n, B_n) of m_n-complementary subsets of $\mathbb{N}$. We then set

$$A = \bigcup_{n=1}^{\infty} A_n \qquad , \qquad B = \bigcup_{n=1}^{\infty} B_n.$$

We undertake to show that $s(A) = s(B) = 2$, $s(A, B) = s(A \cup B) = \infty$, and A and B are disjoint.

ii) Steps in Proof.

(1) A is sparse, and therefore $s(A) = 2$, since, for $n \geq 1$, we have $2 \cdot A_n < m_n < A_{n+1}$, and the sets A_n are sparse.

(2) By the defining conditions, for $n \geq 1$, we have $2 \cdot (m_n - A_n) < 2m_n < m_{n+1} - A_{n+1}$, so that $2 \cdot B_n < B_{n+1}$. Therefore the sets B_n are pairwise disjoint.

(3) Similarly, for $n \geq 1$, that $m_n + 2 \cdot A_{n+1} - A_n - A_{n+1} < 2m_n + 2\max(A_{n+1}) < m_{n+1}$, and therefore $m_n - A_n + m_{n+1} - A_{n+1} < 2m_{n+1} - 2 \cdot A_{n+1}$, so that

$$B_n + B_{n+1} < 2 \cdot B_{n+1}.$$

(4) We claim (and prove below) that, for any $m, n, p, q \in \mathbb{N}^*$ such that $m \leq p,\ n \leq q$ and $(m, p) \neq (n, q)$, the sumsets $B_m + B_p$ and $B_n + B_q$ are disjoint.

(5) For $n \geq 1$, let $h_n(X) = f_{B_n}(X)$. As the sets B_n are pairwise disjoint, $f_B(X) = \sum_{n=1}^{\infty} h_n(X)$, and therefore the generating series of $(r_B(n))$ is

$$f_B(X)^2 = \sum_{n=1}^{\infty} h_n(X)^2 + 2 \sum_{0<n<p} h_n(X) h_p(X).$$

By the claim (4), no two polynomials in these sums have a common monomial.

(6) We have $s(B) = 2$, even though B need not be sparse.

Indeed, the pairs A_n, B_n are complementary, so, by 3.1, $s(B_n) = s(A_n) = 2$ for $n \geq 2$, and $s(B_n, B_p) = s(A_n, A_p) = 1$ for $0 < n < p$, since $s(A) = 2$. Hence, in view of (5), all the coefficients of f_B^2 are ≤ 2, with equality attained, i.e., $s(B) = 2$.

(7) The subsets A and B are disjoint.

Indeed, otherwise, for some $n, p \in \mathbb{N}^*$, there is an $x \in A_n \cap B_p$, i.e., there exist $x \in A_n$ and $y \in A_p$ such that $m_p = x + y$, which implies that $m_p \in A_n + A_p$. But this is impossible, since if $p < n$ then $2m_p \leq 2m_{n-1} < A_n$, and if $p \geq n$ then $A_n + A_p \leq 2\max(A_p) < m_p$.

(8) We have $s(A, B) = \infty$, since $s(A, B) \geq s(A_n, B_n) = |A_n| = n$ for all $n \geq 1$ (by 3.1). Note that, in view of 2.3, (2), iii), $s(A, B) = \infty$ is equivalent to $s(A \cup B) = \infty$.

iii) Proof of Claim (4). Let $m, n, p, q \in \mathbb{N}^*$ such that $m \leq p,\ n \leq q$ and $(m, p) \neq (n, q)$. We examine all essentially distinct cases: $p < q$, $\quad m < n = p = q$, $\quad m < n < p = q$. The remaining cases, where $q < p$ or $n < m$, follow similarly by exchange of p, q or of m, n. ◇

(9) If $p < q$, then, by (2), $B_m + B_p \leq 2\max(B_p) < \min(B_q) < B_n + B_q$, so that $B_m + B_p$ and $B_n + B_q$ are disjoint.

(10) If $m < n = p = q$, then, by (3), $B_m + B_p \leq \max(B_{p-1} + B_p) < 2\min(B_p) \leq B_n + B_q$, so that $B_m + B_p$ and $B_n + B_q$ are disjoint.

(11) If $m < n < p = q$, then $B_m + B_p$ and $B_n + B_q$ are also disjoint.

Indeed, otherwise there exist $x \in B_m$, $y \in B_n$ $u, v \in B_p$ such that $x+u = y+v$. As $m < n$, we have $x < y$ and therefore $v < u$, so that $m_p - u < m_p - v$ in the sparse set A_p. Hence $2(m_p - u) < m_p - v$, i.e. $m_p - u < u - v = y - x$. As $m_p - u \in A_p$, this implies that $y > y - x \geq \min(A_p)$. As $n < p$, it follows that $2 \cdot B_n < 2m_n < \min(A_p) < y$, which is impossible since $y \in B_n$.

□

Example 13. The construction above yields, as a special case starting with $m_n = 3^{\frac{n(n+3)}{2}-1}$ and $A_n = \{3^{\frac{(n-1)(n+2)}{2}+k-1} : 1 \leq k \leq n\}$, the pair

$$A = \{3^n : n \in \mathbb{N} \text{ and } n \neq \frac{k(k+3)}{2} - 1, \text{ for every } k \in \mathbb{N}^*\},$$

$$B = \{3^{\frac{n(n+1)}{2}-1}(3^n - 3^{k-1}) : k, n \in \mathbb{N}^*, \text{ with } 1 \leq k \leq n\}.$$

Next, we introduce a relation between infinite subsets of $\mathbb{N}$ which preserves the property of having unbounded corresponding representation functions.

5. Proximity

Definition 14. For $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ and $B = \{b_1 < b_2 < \cdots < b_n < \dots\}$ in $\mathcal{I}$, let

$$\delta(A, B) = \sup\{|a_n - b_n| : n \in \mathbb{N}^*\} \in \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}.$$

This defines a function $\delta : \mathcal{I} \times \mathcal{I} \longrightarrow \overline{\mathbb{N}}$. It is a pseudo-distance on $\mathcal{I}$, i.e., it has the properties of a distance, but it can be infinite:
i) $\delta(A, B) = 0$ if and only if $A = B$
ii) $\delta(A, B) = \delta(B, A)$
iii) $\delta(A, C) \leq \delta(A, B) + \delta(B, C)$, for any $A, B, C \in \mathcal{I}$.

Furthermore, we have:

- For any $A \in \mathcal{I}$, the *proximity* of A is, by definition, $\{B \in \mathcal{I} : \delta(A, B) < \infty\}$.
- If B is in the proximity of A, we say that A and B are *close.* More precisely, if $\delta(A, B) \leq d$, i.e., $|a_n - b_n| \leq d$ for $n \in \mathbb{N}^*$, with $d \in \mathbb{N}$, A and B are called *d-close.*

- The relation "A is close to B" is an equivalence relation on $\mathcal{I}$.
- The proximity of A is the union of all the open balls of finite radius centered at A.
- δ induces the discrete topology on $\mathcal{I}$, as the open ball $\{B \in \mathcal{I} : \delta(A,B) < 1\} = \{A\}$.

Lemma 15. *Let $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ and $B = \{b_1 < b_2 < \cdots < b_n < \dots\}$, in $\mathcal{I}$, be d-close, with $d \in \mathbb{N}$. Then for all $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that*

$$r_B(n) \geq \frac{r_A(m)}{4d+1}. \tag{1}$$

Proof. Let $m \in \mathbb{N}$ and $E(A,m) = \{(i,j) \in \mathbb{N}^* \times \mathbb{N}^* : a_i + a_j = m\}$. So $r_A(m) = |E(A,m)|$. If $r_A(m) = 0$, the property holds trivially. So we assume $r_A(m) > 0$, i.e., $E(A,m) \neq \emptyset$.

Let $\sigma : E(A,m) \longrightarrow \mathbb{N}$ be the map defined by $\sigma(i,j) = b_i + b_j$. For any $n \in \sigma(E(A,m))$, there exists $(i,j) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $a_i + a_j = m$ and $b_i + b_j = n$. Since $\delta(A,B) \leq d$, we have $|a_i - b_i| \leq d$ and $|a_j - b_j| \leq d$, so that $a_i + a_j - 2d \leq b_i + b_j \leq a_i + a_j + 2d$, i.e., $m - 2d \leq n \leq m + 2d$. Hence $\sigma(E(A,m)) \subset I = [m-2d, m+2d] \cap \mathbb{N}$.

Therefore $E(A,m) = \bigcup_{n \in I} \sigma^{-1}(n)$ is a finite union of pairwise disjoint sets $\sigma^{-1}(n) = \{(i,j) \in \mathbb{N}^* \times \mathbb{N}^* : a_i + a_j = m \text{ and } b_i + b_j = n\} \subset \{(i,j) \in \mathbb{N}^* \times \mathbb{N}^* : b_i + b_j = n\}$, satisfying $|\sigma^{-1}(n)| \leq r_B(n)$. Thus

$$\begin{aligned} r_A(m) = |E(A,m)| = \sum_{n \in I} |\sigma^{-1}(n)| \leq \sum_{n \in I} r_B(n) &\leq |I| \cdot \max\{r_B(n) : n \in I\} \\ &\leq (4d+1) r_B(n_0), \end{aligned}$$

where $n_0 \in I$ such that $r_B(n_0) = \max_{n \in I} r_B(n)$, and $|I| \leq 4d+1$.

Hence we have the existence of $n = n_0 \in \mathbb{N}$ such that $r_B(n) \geq \frac{r_A(m)}{4d+1}$. □

Corollary 16. *Let $A, B \in \mathcal{I}$ and $d \in \mathbb{N}$. If $\delta(A,B) \leq d$, then*

$$\frac{s(A)}{4d+1} \leq s(B) \leq (4d+1)s(A).$$

Proof. By Inequality (1), $r_A(m) \leq (4d+1)s(B)$ for all $m \in \mathbb{N}$. Thus $s(A) \leq (4d+1)s(B)$. Hence the first inequality. Exchanging A and B yields the second inequality. □

The following corollary follows immediately from Lemma 15 since A and B are d-close for some $d \in \mathbb{N}$.

Corollary 17. *Let $A, B \in \mathcal{I}$. If A and B are close, then $s(A) = \infty$ if and only if $s(B) = \infty$.*

Corollary 18. *Let $A \in \mathcal{I}$, and $\mathbb{S} = \{n^2 : n \in \mathbb{N}^*\}$. If there exists a constant $c \in \mathbb{N}^*$ such that A is close to $c \cdot \mathbb{S}$, then $s(A) = \infty$.*

Proof. By a classical result on the number of representations of a positive integer as a sum of two squares ([5], Theorem 278), this number is unbounded, i.e., $s(\mathbb{S}) = \infty$. Therefore, in view of 2.9, $s(c \cdot \mathbb{S}) = \infty$, and as A is close to $c \cdot \mathbb{S}$, by 5.4, we also have $s(A) = \infty$. □

Remark 19. *The result in Corollary 18 may be considered as a weak variant of the conjecture (GET).*

Corollary 20. *Let $A, B \in \mathcal{I}$ and $d \in \mathbb{N}$. If $\delta(A, B) \leq d$ and $s(A) + s(B) < \infty$, then*

$$|s(A) - s(B)| \leq 4d \cdot \min(s(A), s(B)).$$

Proof. Assume that $s(A) \leq s(B)$. Then, by Corollary 16, we have $s(B) \leq (4d + 1)s(A)$, i.e., $s(B) - s(A) \leq 4d \cdot s(A)$. Hence the result. □

Remark 21. The inequalities established in Corollaries 16 and 20 hold with $d = \delta(A, B)$, and they even hold trivially when $\delta(A, B) = \infty$. Hence

- for all $A, B \in \mathcal{I}$, $s(B) \leq (4\delta(A, B) + 1)s(A)$ and $s(A) \leq (4\delta(A, B) + 1)s(B)$.
- for all $A, B \in \mathcal{I}$, $s(A) + s(B) < \infty$ implies $|s(A) - s(B)| \leq 4 \min(s(A), s(B)) \cdot \delta(A, B)$.

6. Relations With the Counting Function and the Caliber

Definition 22. Let $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ be a subset of $\mathbb{N}$. For a real number $x \in \mathbb{R}$, setting $A[x] = \{a \in A : a \leq x\}$, the *counting function* of A is defined by $A(x) = |A[x]|$.

For $x \in \mathbb{R}$ and $n \in \mathbb{N}^*$, the condition $A(x) \geq n$ is equivalent to $a_n \leq x$, while the condition $A(x) = n$ is equivalent to $a_n \leq x < a_{n+1}$. In particular $A(a_n) = n$.

When A is infinite, we define its *caliber* by

$$cal(A) = \liminf_{n \to \infty} \frac{a_n}{n^2}.$$

Lemma 23. *For any subset A of $\mathbb{N}$ and any real number $x \geq 0$, we have*

$$\sum_{n \leq x} r_A(n) \leq A(x)^2 \leq \sum_{n \leq 2x} r_A(n),$$

and therefore

$$s(A) \geq \sup_{x \geq 0} \frac{A(x)^2}{2x + 1}.$$

Proof. Note that

$$\begin{aligned}\textstyle\sum_{n\le x} r_A(n) &= |\bigcup_{n\le x}\{(a,b)\in A\times A : a+b=n\}| = |\{(a,b)\in A\times A : a+b\le x\}|\\ &\le |A[x]\times A[x]| = A(x)^2.\end{aligned}$$

Similarly,

$$A(x)^2 = |A[x]\times A[x]| \le |\{(a,b)\in A\times A : a+b\le 2x\}| = \sum_{n\le 2x} r_A(n).$$

This proves the first double inequality. Moreover, we have

$$A(x)^2 \le \sum_{n\le 2x} r_A(n) \le \sum_{n\le 2x} s(A) \le (2x+1)s(A),$$

which yields the last inequality. □

Theorem 24. *For any infinite subset A of $\mathbb{N}$, we have*

$$s(A) \ge \frac{1}{2\ cal(A)}.$$

Thus, if $cal(A) = 0$, then $s(A) = \infty$.

Proof. Letting $A = \{a_1 < a_2 < \cdots < a_n < \dots\}$ and taking $x = a_n$ in the last inequality of Lemma 6.2, we get

$$s(A) \ge \sup_{n\ge 1} \frac{A(a_n)^2}{2a_n+1} \ge \limsup_{n\to\infty} \frac{n^2}{2a_n+1} = \frac{1}{2}\limsup_{n\to\infty}\frac{n^2}{a_n} = \frac{1}{2}\frac{1}{\liminf_{n\to\infty}\frac{a_n}{n^2}},$$

which yields the result. □

Remark 25. If there exist real constants $c > 0$ and $0 < t < 2$ such that

$$a_n \le cn^{2-t},$$

for large enough n, then $cal(A) = 0$, and therefore $s(A) = \infty$. This represents a weak variant of the conjecture (GET).

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