ON THE NUMBER OF POINTS IN A LATTICE POLYTOPE

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Abstract

In this article we will show that for every natural d and $n > 1$ there exists a natural number t such that for every d-dimensional simplicial complex $\mathcal{T}$ with vertices in $\mathbb{Z}^d$, the number of lattice points in the t^{th} dilate of $\mathcal{T}$ is exactly $\chi(\mathcal{T})$ modulo n, where $\chi(\mathcal{T})$ is the Euler characteristic of $\mathcal{T}$.

1 Introduction

This problem was given to one of the authors by Rom Pinchasi. He noticed that if we scale a segment with vertices in a lattice in two times, then the number of lattice points in the scaled segment will be odd. For polygons with vertices in a two-dimensional lattice, the same fact follows from Pick's formula except that this polygon must be scaled in four times. We will show that the following theorem holds:

Theorem 1. *For any natural numbers d and $n > 1$ there exists a natural number t such that if $\mathcal{T}$ is any simplicial complex in $\mathbb{R}^d$ with vertices in the integer lattice $\mathbb{Z}^d$ then the number of lattice points in the complex $t\mathcal{T}$ is equivalent to $\chi(\mathcal{T})$ modulo n.*

Here $\chi(\mathcal{T})$ is the Euler characteristic of the complex $\mathcal{T}$ and $t\mathcal{T}$ denotes the image of $\mathcal{T}$ under similarity with the center at the origin and ratio equal to t.

The proof is based on Stanley's theorem on the coefficients of Ehrhart polynomials [4]. Let us recall the definition of Ehrhart polynomial [2]. A polytope is called a

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lattice polytope if all the vertices lie on $\mathbb{Z}^d$. For any d-dimensional lattice polytope $\mathcal{P}$ in $\mathbb{R}^d$, there exists a polynomial

$$L(\mathcal{P},t) = a_d t^d + a_{d-1}t^{d-1} + \cdots + a_0, \tag{1}$$

such that the number of lattice points in the polytope $t\mathcal{P}$ is equal to $L(\mathcal{P},t)$. It is possible to prove that a_0 is the Euler characteristic of $\mathcal{P}$ (that is one for convex polytopes) and a_d is the volume of $\mathcal{P}$. Further important properties of Ehrhart polynomial and its connection with number theory, combinatorics and discrete geometry could be found in [1].

2 Proof

First we prove the following lemma. Here $[\cdot]$ is the floor function, that is, $[x]$ denotes the largest integer number not greater than x.

Lemma 2. *Let $\mathcal{P}$ be a convex polytope in $\mathbb{R}^d$ with vertices in the integer lattice $\mathbb{Z}^d$, p be any prime number and $l = \left[\log_p d\right]$. Then for any natural number $k > l$, the number of lattice points in the convex polytope $p^k\mathcal{P}$ is exactly one modulo p^{k-l}.*

Proof. From Stanley's nonnegativity theorem (more precisely Lemma 3.14 in [1]) it follows that in this case the number of lattice points in the convex polytope $t\mathcal{P}$ equals exactly:

$$\binom{t+d}{d} + h_1\binom{t+d-1}{d} + \cdots + h_{d-1}\binom{t+1}{d} + h_d\binom{t}{d}, \tag{2}$$

where h_1, h_2, $\ldots$, h_d are nonnegative integer numbers.

Suppose $t = p^k$ and $m \le d \le p^{l+1} - 1$. If α is the maximal power of p which divides m then $(m+p^k)/p^\alpha \equiv m/p^\alpha \pmod{p^{k-l}}$. Using this fact it is easy to show that $\binom{t+d}{d} \equiv 1 \pmod{p^{k-l}}$. Also from Kummer's theorem (see [3], exercise 5.36) it follows that for any $i = 1, 2, \ldots, d$ we have $\binom{t+d-i}{d} \equiv 0 \pmod{p^{k-l}}$. So as we can see, the number of lattice points equals exactly one modulo p^{k-l}. □

Remark 3. It is easy to see that the statement of Lemma 2 holds for dilation factor ap^k, $a \in \mathbb{N}$. For a proof it is sufficient to apply the Lemma to the polytope $a\mathcal{P}$.

Proof of Theorem 1. Consider the prime factorization of n: $n = p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\ldots p_s^{\alpha_s}$. Let $\beta_i = \alpha_i + \left[\log_{p_i} d\right]$. Define $t = p_1^{\beta_1}p_2^{\beta_2}p_3^{\beta_3}\ldots p_s^{\beta_s}$. Suppose Δ is a simplex. By Lemma 2 we have that the number of lattice points in $t\Delta$ equals 1 modulo $p_i^{\alpha_i}$ for any $i = 1, 2, \ldots, s$. From the Chinese remainder theorem, it follows that this number is equivalent to 1 modulo n.

We know that the Euler characteristic of every simplex (with its interior) equals 1 and the Euler characteristic is an additive function on simplicial complexes. Since the number of lattice points modulo n is also an additive function, we obtain that the number of lattice points is equivalent to exactly $\chi(\mathcal{T}) \pmod{n}$. □

Remark 4. As noted by the anonymous referee the statement of Theorem 1 is kind of obvious for $t = nd!$. It is well-known that for any d-dimensional lattice polytope, all the coefficients of the Ehrhart polynomial are rational numbers and all the denominators except for the constant term 1 are divisors of $d!$. In other words, the polynomial is of the form $L(\mathcal{P}, t) = 1 + t \cdot p(t)/d!$ where the polynomial $p(t)$ has integer coefficients. So if $t = n \cdot d!$ then $L(\mathcal{P}, nd!) = 1 + n \cdot p(n \cdot d!)$, which is 1 modulo n.

Let us show that the number t obtained in the proof of Theorem 1 is the minimal natural number which satisfies the condition of the Theorem.

Suppose t is not divisible by $p_i^{\beta_i}$ for some i. Let $d' = p_i^{\lceil \log_{p_i} d \rceil}$ and Δ be a d'-dimensional simplex with vertices $(0, 0, \dots, 0)$, $(1, 0, \dots, 0)$, $\dots$, $(0, 0, \dots, 1)$. Then the number of lattice points in the simplex $x\Delta$ is equal to $\binom{x+d'}{d'}$ (see [1], Section 2.3). It is easy to see that one can choose x such that $t \cdot x \equiv p_i^{\beta_i - 1} \pmod{p_i^{\beta_i}}$. Note that if $a \equiv b \pmod{p_i^{\beta_i}}$, then

$$\binom{a+k}{k} \equiv \binom{b+k}{k} \pmod{p_i^{\alpha_i}}, \text{ for all } k < p_i^{\beta_i - \alpha_i} = d'.$$

Since $\binom{p_i^{\beta_i-1}+d'-1}{d'-1} \equiv 1 \pmod{p_i^{\alpha_i}}$, we have

$$L(x\Delta, t) = \binom{xt + d'}{d'} \equiv \binom{p_i^{\beta_i - 1} + d'}{d'} =$$
$$= \binom{p_i^{\beta_i - 1} + d' - 1}{d' - 1} \cdot \frac{p_i^{\beta_i - 1} + d'}{d'} \equiv p_i^{\alpha_i - 1} + 1 \pmod{p_i^{\alpha_i}}. \quad (3)$$

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