#A05 INTEGERS 10 (2010), 31-64

# ON NORMAL NUMBERS AND POWERS OF ALGEBRAIC NUMBERS

**Hajime Kaneko**
*Department of Mathematics, Kyoto University, Oiwake-tyou, Kitashirakawa, Kyoto-shi, Kyoto, Japan*
kanekoha@math.kyoto-u.ac.jp

*Received: 7/14/09, Accepted: 11/17/09, Published: 3/5/10*

**Abstract**

Let $\alpha > 1$ be an algebraic number and $\xi > 0$. Denote the fractional parts of $\xi\alpha^n$ by $\{\xi\alpha^n\}$. In this paper, we estimate a lower bound for the number $\lambda_N(\alpha, \xi)$ of integers $n$ with $0 \le n < N$ and

$$\{\xi\alpha^n\} \ge \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\}.$$

Our results show, for example, the following: Let $\alpha$ be an algebraic integer with Mahler measure $M(\alpha)$ and $\xi > 0$ an algebraic number with $\xi \notin \mathbb{Q}(\alpha)$. Put $[\mathbb{Q}(\alpha, \xi) : \mathbb{Q}(\alpha)] = D$. Then there exists an absolute constant $c$ satisfying

$$\lambda_N(\alpha, \xi) \ge c\frac{(\log \alpha)^2}{(\log M(\alpha))^2(\log(6D))^{1/2}}\frac{(\log N)^{3/2}}{(\log\log N)^{1/2}}$$

for all large $N$.

## 1. Introduction

A *normal* number in an integer base $\alpha$ is a positive number for which all finite words with letters from the alphabet $\{0, 1, \ldots, \alpha - 1\}$ occur with the proper frequency. It is easily checked that a positive number $\xi$ is a normal number in base $\alpha$ if and only if the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1. Borel [6] proved that almost all positive $\xi$ are normal numbers in every integer base. Moreover, Koksma [16] showed that if any real number $\alpha > 1$ is given, then the sequence $\xi\alpha^n$ $(n = 0, 1, \ldots)$ is uniformly distributed modulo 1 for almost all positive $\xi$, which is a generalization of Borel's result. However, it is generally difficult to check a given geometric sequence is uniformly distributed modulo 1 or not. For instance, we even do not know whether the numbers $\sqrt{2}$, $\sqrt[3]{5}$ and $\pi$ are normal in base 10.

Borel [7] conjectured that each algebraic irrational number is normal in every integer base. However, we know no such number whose normality was proved. We now introduce some partial results.

Let $\alpha$ be a natural number greater than 1 and $\xi$ a positive algebraic irrational number. For simplicity, assume that $\xi < 1$. Write its $\alpha$-ary expansion as

$$\xi = \sum_{i=-\infty}^{-1} s_i(\xi)\alpha^i = .s_{-1}(\xi)s_{-2}(\xi)\cdots$$

with $s_i(\xi) \in \{0, 1, \ldots, \alpha - 1\}$. First, we measure the complexity of the infinite word $\boldsymbol{s} = s_{-1}(\xi)s_{-2}(\xi)\cdots$ by the number $p(N)$ of distinct blocks of length $N$ appearing in the word $\boldsymbol{s}$. If $\xi$ is normal in base $\alpha$, then $p(N) = \alpha^N$ for any positive $N$. Ferenczi and Mauduit [13] showed that

$$\lim_{N\to\infty}(p(N) - N) = \infty.$$

Adamczewski and Bugeaud [1] improved their result as follows:

$$\lim_{N\to\infty}\frac{p(N)}{N} = \infty.$$

Moreover, Bugeaud and Evertse [10] showed for any positive $\xi$ with $\eta < 1/11$ that

$$\limsup_{N\to\infty}\frac{p(N)}{N(\log N)^\eta} = \infty.$$

Bugeaud and Evertse [10] gave a lower bound of the number $\mathrm{ch}(N)$ of digit changes among the first $(N + 1)$ digits of the $\alpha$-ary expansion of $\xi$. Namely,

$$\mathrm{ch}(N) = \mathrm{Card}\{i \in \mathbb{N} | 1 \le i \le N, s_{-i}(\xi) \neq s_{-i-1}(\xi)\},$$

where Card denotes the cardinality. They showed for an algebraic irrational $\xi > 0$ of degree $D(\geq 2)$ that there exist an effectively computable absolute constant $c_1$ and an effectively computable constant $c_2(\alpha, \xi)$, depending only on $\alpha$ and $\xi$, satisfying

$$\mathrm{ch}(N) \geq c_1 \frac{(\log N)^{3/2}}{(\log 6D)^{1/2}(\log\log N)^{1/2}}$$

for any $N$ with $N \geq c_2(\alpha, \xi)$.

Next, we count the number $\lambda_N(\alpha, \xi)$ of nonzero digits among the first $N$ digits of the $\alpha$-ary expansion of $\xi$, where

$$\lambda_N(\alpha, \xi) = \mathrm{Card}\{i \in \mathbb{N} | 1 \le i \le N, s_{-i}(\xi) \neq 0\}. \tag{1}$$

Let $\xi$ be an algebraic irrational number of degree $D$ with $1 < \xi < 2$. In the case of $\alpha = 2$, Bailey, Borwein, Crandall, and Pomerance [4] showed that an arbitrary positive $\varepsilon$ is given, then

$$\lambda_N(\alpha, \xi) > (1-\varepsilon)(2A_D)^{-1/D} N^{1/D}$$

for all sufficiently large $N$, where $A_D (> 0)$ is the leading coefficient of the minimal polynomial of $\xi$. Moreover, in the same way as the proof of the inequality above, we can show for any natural number $\alpha \geq 2$ that there exists a positive constant $c_3(\alpha, \xi)$ depending only on $\alpha$ and $\xi$ satisfying

$$\lambda_N(\alpha, \xi) \geq c_3(\alpha, \xi) N^{1/D}$$

for every sufficiently large $N$.

In what follows, we consider the fractional parts of geometric progressions whose common ratios are algebraic numbers. Let $\alpha > 1$ be an algebraic number with minimal polynomial $a_d X^d + a_{d-1} X^{d-1} + \ldots + a_0 \in \mathbb{Z}[X]$, where $a_d > 0$ and $\gcd(a_d, a_{d-1}, \ldots, a_0) = 1$. Put

$$L_+(\alpha) = \sum_{a_i > 0} a_i, \ L_-(\alpha) = \sum_{a_i \leq 0} |a_i|. \tag{2}$$

Moreover, write the Mahler measure of $\alpha$ by

$$M(\alpha) = a_d \prod_{k=1}^{d} \max\{1, |\alpha_k|\},$$

where $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are the conjugates of $\alpha$. We now recall the definition of a Pisot and Salem number. A Pisot number is an algebraic integer greater than 1 whose conjugates different from itself have absolute values strictly less than 1. A Salem number is an algebraic integer greater than 1 which has at least one conjugate with modulus 1 and exactly one conjugate outside the unit circle. Take a positive number $\xi$. If $\alpha$ is a Pisot or Salem number, then assume $\xi \notin \mathbb{Q}(\alpha)$. Dubickas [11] showed for infinitely many $n \geq 1$ that

$$\{\xi\alpha^n\} \geq \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\},$$

where $\{\xi\alpha^n\}$ means the fractional part of $\xi\alpha^n$. In what follows we estimate the number of such $n$, namely, we give a lower bound of the number

$$\lambda_N(\alpha, \xi) = \mathrm{Card}\left\{n \in \mathbb{Z} \,\middle|\, 0 \leq n < N, \{\xi\alpha^n\} \geq \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\}\right\}. \tag{3}$$

(3) is generalization of (1). In fact, assume that $\alpha$ is a natural number greater than 1 and that $\xi$ is a positive number with $\xi < 1$. Then, for $n \geq 0$,

$$\{\xi\alpha^n\} \geq \min\left\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\right\} = \frac{1}{\alpha}$$

if and only if the $(n+1)$-th digit of $\alpha$-ary expansion of $\xi$ is nonzero.

Dubickas's result above implies

$$\lim_{N\to\infty} \lambda_N(\alpha, \xi) = \infty.$$

He verified this by showing that, for infinitely many $n \geq 0$,

$$s_{-n}(\xi) \neq 0,$$

where $s_{-n}(\xi)$ will be defined in Section 2. Moreover, in the same way as that of Theorem 3 of [11], we can show the following: Assume that $\alpha$ has at least one conjugate different from itself outside the unit circle. Then

$$\liminf_{N\to\infty} \frac{\lambda_N(\alpha,\xi)}{\log N} \geq \left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log(a_d\alpha)}\right)\right)^{-1}. \tag{4}$$

At the beginning of Section 5, we give another proof of (4). In this paper we improve this estimation in the case where $\alpha > 1$ and $\xi > 0$ are algebraic numbers with $\xi \notin \mathbb{Q}(\alpha)$ by using a version of the quantitative parametric subspace theorem of Bugeaud and Evertse [10]. First, we consider the case where $\alpha > 1$ is an algebraic integer.

**Theorem 1.** *Let $\alpha > 1$ be an algebraic integer with Mahler measure $M(\alpha)$. Let $\xi$ be a positive algebraic number with $\xi \notin \mathbb{Q}(\alpha)$. Put*

$$D = [\mathbb{Q}(\alpha, \xi) : \mathbb{Q}(\alpha)].$$

*Then there exists an effectively computable absolute constant $c > 0$ such that*

$$\lambda_N(\alpha, \xi) \geq c\frac{(\log\alpha)^2}{(\log M(\alpha))^2(\log(6D))^{1/2}}\frac{(\log N)^{3/2}}{(\log\log N)^{1/2}}$$

*for every sufficiently large $N$.*

Next we give a lower bound of $\lambda_N(\alpha, \xi)$ in the case where $\alpha > 1$ is not an algebraic integer.

**Theorem 2.** *Let $\alpha > 1$ be an algebraic number of degree $d$ with Mahler measure $M(\alpha)$. We denote the leading coefficient of the minimal polynomial of $\alpha$ by $a_d(\geq 1)$. Let $\xi$ be a positive algebraic number with $\xi \notin \mathbb{Q}(\alpha)$. Assume that $\alpha$ is not an algebraic integer. Then*

$$\liminf_{N\to\infty} \frac{\lambda_N(\alpha,\xi)}{\log N} \geq \left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log a_d}\right)\right)^{-1}.$$

Theorem 2 gives an improvement of (4) since

$$\left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log a_d}\right)\right)^{-1} > \left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log(a_d\alpha)}\right)\right)^{-1}.$$

We introduce a numerical example in the case of $\alpha = 4 + 1/\sqrt{2}$. The minimal polynomial of $\alpha$ is $2X^2 - 16X + 31$, so we have $a_d = 2$, $M(\alpha) = 31$, and

$$\min\{\frac{1}{L_+(\alpha)}, \frac{1}{L_-(\alpha)}\} = \min\{\frac{1}{33}, \frac{1}{16}\} = \frac{1}{33}.$$

Note that the conjugate of $\alpha$ is greater than 1. Thus by (4), for any positive $\xi$,

$$\liminf_{N\to\infty} \frac{\lambda_N(4+1/\sqrt{2},\xi)}{\log N} \geq \left(\log\left(\frac{\log(31)}{\log(31) - \log(8+\sqrt{2})}\right)\right)^{-1} = 0.944\ldots.$$

On the other hand, the second statement of Theorem 2 implies that if $\xi > 0$ is an algebraic number with $\xi \notin \mathbb{Q}(\sqrt{2})$, then

$$\liminf_{N\to\infty} \frac{\lambda_N(4+1/\sqrt{2},\xi)}{\log N} \geq \left(\log\left(\frac{\log(31)}{\log(31) - \log(2)}\right)\right)^{-1} = 4.43\ldots.$$

**Remark 3.** By using the same method for the proof of Theorem 2 and 1, Bugeaud [9] gave a lower bound for the number of digit changes in the $\beta$-expansion of algebraic numbers.

## 2. Preliminaries

Let $\alpha > 1$ be an algebraic number of degree $d$ and $\xi$ a positive number. Write the minimal polynomial of $\alpha$ by $P_\alpha(X) = a_d X^d + \cdots + a_0 \in \mathbb{Z}[X]$ $(a_d > 0)$. In this section, we study the sequence $(s_m(\xi))_{m=-\infty}^{\infty}$ defined by

$$s_m(\xi) = -\sum_{i=0}^{d} a_{d-i}\{\xi\alpha^{-m-i}\}.$$

Let $[x]$ be the integral part of a real number $x$. Since

$$0=\sum_{i=0}^{d}a_{d-i}\xi\alpha^{-m-i}=\sum_{i=0}^{d}a_{d-i}\Big([\xi\alpha^{-m-i}]+\{\xi\alpha^{-m-i}\}\Big),$$

we have

$$\begin{aligned} s_m(\xi) &= \sum_{i=0}^{d}a_{d-i}\Big([\xi\alpha^{-m-i}]-\xi\alpha^{-m-i}\Big) \\ &= \sum_{i=0}^{d}a_{d-i}[\xi\alpha^{-m-i}]. \end{aligned} \tag{5}$$

In particular, $s_m(\xi)$ is a rational integer. Thus we get the following:

**Lemma 4.** *Let $\xi$ be a positive number.*

(1) *If $s_m(\xi)\neq 0$ ,then*

$$\max_{-m-d\le n\le -m}\{\xi\alpha^n\}\ge \min\left\{\frac{1}{L_+(\alpha)},\frac{1}{L_-(\alpha)}\right\}.$$

(2) *$s_m(\xi)=0$ for all sufficiently large $m$.*

*Proof.* We first show the first statement. Since $s_m(\xi)$ is a nonzero integer, we have

$$1\le |s_m(\xi)|=\left|-\sum_{i=0}^{d}a_{d-i}\{\xi\alpha^{-m-i}\}\right|.$$

By using $0\le\{\xi\alpha^{-m-i}\}<1$, we obtain the first statement. The second statement follows from (5) and $[\xi\alpha^{-m}]=0$ for each sufficiently large $m$. □

**Proposition 5.** *Write the conjugates of $\alpha$ with moduli greater than 1 by $\alpha_1(=\alpha),\dots,\alpha_p$. Let $\xi$ be a positive number. Then*

(1) *For $2\le k\le p$,*

$$\sum_{i=-\infty}^{\infty}\alpha_k^i s_i(\xi)=0.$$

(2)

$$\sum_{i=-\infty}^{\infty} \alpha^i s_i(\xi) = -\frac{\xi}{\alpha}(P_\alpha^*)'\left(\frac{1}{\alpha}\right) \neq 0,$$

*where* $P_\alpha^*(X) = a_d + a_{d-1}X + \cdots + a_0 X^d$ *denotes the reciprocal polynomial of* $P_\alpha(X)$ *and* $(P_\alpha^*)'(X)$ *its derivative.*

**Remark 6.** By the second statement of Lemma 4, the series

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi)$$

converges for any $k$ with $1 \leq k \leq p$.

*Proof.* We first consider the case of $0 < \xi < 1$. Then, for any $m \leq 0$, $[\xi\alpha^m] = 0$, and so $s_{-m}(\xi) = 0$ by (5). Put

$$f(z) = \sum_{n=0}^{\infty}[\xi\alpha^n]z^n, \; g(z) = \sum_{n=0}^{\infty}\{\xi\alpha^n\}z^n.$$

Then we have

$$\begin{aligned}\left(\frac{\xi}{1-\alpha z} - g(z)\right) P_\alpha^*(z) &= f(z)P_\alpha^*(z) \\ &= \sum_{h=0}^{\infty} \sum_{\substack{i,j\geq 0 \\ i+j=h}} [\xi\alpha^i]a_{d-j}z^h \\ &= \sum_{h=0}^{\infty} \sum_{i=h-d}^{h} [\xi\alpha^i]a_{d-h+i}z^h = \sum_{h=0}^{\infty} s_{-h}(\xi)z^h.\end{aligned}$$

Consider the region of $z \in \mathbb{C}$ satisfying

$$\left(\frac{\xi}{1-\alpha z} - g(z)\right) P_\alpha^*(z) = \sum_{h=0}^{\infty} s_{-h}(\xi)z^h. \tag{6}$$

Since $0 \leq \{\xi\alpha^n\} < 1$ for any $n$, the left-hand side of (6) is a meromorphic function on $\{z : |z| < 1\}$. Moreover, because the sequence $s_{-m}(\xi)$ $(m = 0, 1, \ldots)$ is bounded, the right-hand side of (6) converges for $|z| < 1$. Hence (6) holds for

$|z| < 1$. In particular, since the left-hand side of (6) has a zero at $z = \alpha_k^{-1}$ with $2 \le k \le p$, we obtain

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi) = \sum_{i=0}^{\infty} \alpha_k^{-i} s_{-i}(\xi) = 0.$$

Let $\alpha_1 = \alpha, \ldots, \alpha_p, \alpha_{p+1}, \ldots, \alpha_d$ be the conjugates of $\alpha$. $P_\alpha^*(z)$ has a simple zero at $z = 1/\alpha$ since

$$P_\alpha^*(z) = z^d P_\alpha\left(\frac{1}{z}\right) = a_d(1-\alpha z)(1-\alpha_2 z)\cdots(1-\alpha_d z).$$

Note that $g(z)$ is holomorphic for $|z| < 1$. Hence

$$\begin{aligned}\sum_{i=-\infty}^{\infty} \alpha^i s_i(\xi) &= \sum_{i=0}^{\infty} \alpha^{-i} s_{-i}(\xi) \\ &= \lim_{z\to 1/\alpha} \frac{\xi P_\alpha^*(z)}{1-\alpha z} = -\frac{\xi}{\alpha}(P_\alpha^*)'\left(\frac{1}{\alpha}\right) \ne 0.\end{aligned}$$

Next, we check the case of $\xi \ge 1$. Take a positive integer $R$ satisfying $\xi\alpha^{-R} < 1$. Then we obtain

$$\sum_{i=-\infty}^{\infty} \alpha_k^i s_i(\xi) = \alpha_k^R \sum_{i=-\infty}^{\infty} \alpha_k^{i-R} s_{i-R}(\xi\alpha^{-R}) = 0$$

for $2 \le k \le p$, and

$$\sum_{i=-\infty}^{\infty} \alpha^i s_i(\xi) = \alpha^R \sum_{i=-\infty}^{\infty} \alpha^{i-R} s_{i-R}(\xi\alpha^{-R}) = -\frac{\xi}{\alpha}(P_\alpha^*)'\left(\frac{1}{\alpha}\right).$$

□

## 3. The Quantitative Subspace Theorem

First, we consider approximations of given algebraic numbers by algebraic numbers which lies in a fixed number field. We fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. In what follows, assume that all algebraic number fields are subfields of $\overline{\mathbb{Q}}$. Let us begin with some notation about the absolute values on $\mathbf{K}$, where $\mathbf{K}$ is a number field of degree $d$. Let $\mathcal{M}_{arc}(\mathbf{K})$ be the set of archimedean places of $\mathbf{K}$ and $\mathcal{M}_{non}(\mathbf{K})$ the set of non-archimedean places of $\mathbf{K}$, respectively. Moreover, put $\mathcal{M}(\mathbf{K}) = \mathcal{M}_{arc}(\mathbf{K}) \cup \mathcal{M}_{non}(\mathbf{K})$. We define the absolute values $|\cdot|_v$ and $||\cdot||_v$ associated with a place $v \in \mathbf{K}$. In the case of $\mathbf{K} = \mathbb{Q}$, we have

$$\mathcal{M}(\mathbb{Q}) = \{\infty\} \cup \{\text{primes}\}.$$

In the case of $v = \infty$, let $|\cdot|_\infty$ be the ordinary archimedean absolute value on $\mathbb{Q}$. If $v = p$ is a prime number, then denote by $|\cdot|_p$ the $p$-adic absolute value, normalized such that $|p|_p = p^{-1}$.

Next, we consider the case where $\mathbf{K}$ is an arbitrary number field. Suppose a place $v \in \mathcal{M}(\mathbf{K})$ lies above the place $p_v \in \mathcal{M}(\mathbb{Q})$. We choose the normalized absolute value $|\cdot|_v$ in such a way that the restriction of $|\cdot|_v$ to $\mathbb{Q}$ is $|\cdot|_{p_v}$. Let $\mathbf{K}_v$ (resp. $\mathbb{Q}_{p_v}$) be the completion of $(\mathbf{K}, |\cdot|_v)$ (resp. $(\mathbb{Q}, |\cdot|_{p_v})$). Put

$$d(v) = \frac{[\mathbf{K}_v : \mathbb{Q}_{p_v}]}{[\mathbf{K} : \mathbb{Q}]}.$$

and

$$||\cdot||_v = |\cdot|_v^{d(v)}.$$

Define the height of $x$ by

$$H(x) = \prod_{v \in \mathcal{M}(\mathbf{K})} \max\{1, ||x||_v\}.$$

By Lemma 3.10 of [20], we have

$$H(x)^{\deg x} = M(x) \tag{7}$$

Moreover, the product formula (for instance see [20], p. 74) implies for any nonzero $x \in \mathbf{K}$ that

$$H(x^{-1}) = H(x). \tag{8}$$

Now we introduce Theorem 2 of [17] in the case of $d = 1$, which we use to prove Theorem 2. Suppose every valuation of $\mathbf{K}$ to be extended to $\overline{\mathbb{Q}}$.

**Theorem 7 (Locher [17]).** *Let $0 < \varepsilon \leq 1$ and $\mathbf{F}/\mathbf{K}$ be an extension of number fields of degree $D$. Let $S$ be a finite set of places of $\mathbf{K}$ with cardinality $s$. Suppose that for each $v \in S$, a fixed element $\theta_v \in \mathbf{F}$ is given. Let $H$ be a real number with $H \geq H(\theta_v)$ for all $v \in S$. Consider the inequality*

$$\prod_{v \in S} \min\{1, ||\theta_v - \gamma||_v\} < H(\gamma)^{-2-\varepsilon} \tag{9}$$

*to be solved in elements $\gamma \in \mathbf{K}$. Then there are at most*

$$e^{7s+19}\varepsilon^{-s-4}\log(6D)\log\left(\varepsilon^{-1}\log(6D)\right)$$

*solutions $\gamma \in \mathbf{K}$ of (9) with*

$$H(\gamma) \geq \max\left\{H, 4^{4/\varepsilon}\right\}.$$

Next, we consider approximations of given algebraic numbers by algebraic numbers with arbitrary degree. Let us introduce the quantitative subspace theorem proved by Bugeaud and Evertse [10]. Let $\mathcal{L} = (L_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ be a tuple of linear forms with the following properties:

$$\begin{cases} L_{iv} \in \mathbf{K}[X, Y] \text{ for } v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2, \\ L_{1v} = X,\ L_{2v} = Y \text{ for all but finitely many } v \in \mathcal{M}(\mathbf{K}), \\ \det(L_{1v}, L_{2v}) = 1 \text{ for } v \in \mathcal{M}(\mathrm{K}), \\ \mathrm{Card}\left(\bigcup_{v\in\mathcal{M}(\mathbf{K})}\{L_{1v}, L_{2v}\}\right) \le r. \end{cases} \tag{10}$$

Put

$$\bigcup_{v\in\mathcal{M}(\mathbf{K})} \{L_{1v}, L_{2v}\} = \{L_1, \ldots, L_s\}$$

and

$$\mathcal{H} = \mathcal{H}(\mathcal{L}) = \prod_{v\in\mathcal{M}(\mathbf{K})} \max_{1\le i<j\le s} ||\det(L_i, L_j)||_v. \tag{11}$$

Moreover, let $\mathbf{c} = (c_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ be a tuple of reals with the following properties:

$$\begin{cases} c_{1v} = c_{2v} = 0 \text{ for all but finitely many } v \in \mathcal{M}(\mathbf{K}), \\ \sum_{v\in\mathcal{M}(\mathbf{K})} \sum_{i=1}^{2} c_{iv} = 0, \\ \sum_{v\in\mathcal{M}(\mathbf{K})} \max\{c_{1v}, c_{2v}\} \le 1. \end{cases} \tag{12}$$

Next, take any finite extension $\mathbf{E}$ of $\mathbf{K}$ and any place $w \in \mathcal{M}(\mathbf{E})$. Let $v \in \mathcal{M}(\mathbf{K})$ be the place lying below $w$. Write the completion of $(\mathbf{E}, |\cdot|_w)$ (resp. $(\mathbf{K}, |\cdot|_v)$) by $\mathbf{E}_w$ (resp. $\mathbf{K}_v$). For $i = 1, 2$, define the linear forms $L_{1w}, L_{2w}$ and the real numbers $c_{1w}, c_{2w}$ by

$$L_{iw} = L_{iv} \text{ and } c_{iw} = d(w|v)c_{iv}, \tag{13}$$

where

$$d(w|v) = \frac{[\mathbf{E}_w : \mathbf{K}_v]}{[\mathbf{E} : \mathbf{K}]}.$$

Note that

$$||x||_w = ||x||_v^{d(w|v)} \text{ for } x \in \mathbf{K} \tag{14}$$

and that

$$\sum_{\substack{w \in \mathcal{M}(\mathbf{E}) \\ w|v}} d(w|v) = 1 \text{ for } v \in \mathcal{M}(\mathbf{K}). \tag{15}$$

Take a positive number $Q$ and $\mathbf{x} = (x, y) \in \overline{\mathbb{Q}}^2$. We define the twisted height $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x})$. There exists a number field $\mathbf{E}$ including the field $\mathbf{K}(x, y)$. Then put

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) = \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}},$$

which is a finite product by the assumption of $\mathcal{L}$ and $\mathbf{c}$. We show that $H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x})$ does not depend on the choice of $\mathbf{E}$. Let $\mathbf{E}'$ be another number field including $\mathbf{K}(x, y)$. Take a number field $\mathbf{F}$ with $\mathbf{F} \supset \mathbf{E} \cup \mathbf{E}'$. By (13), (14), and (15)

$$\begin{aligned}
&\prod_{u \in \mathcal{M}(\mathbf{F})} \max_{1 \le i \le 2} ||L_{iu}(\mathbf{x})||_u Q^{-c_{iu}} \\
&\quad = \prod_{w \in \mathcal{M}(\mathbf{E})} \prod_{\substack{u \in \mathcal{M}(\mathbf{F}) \\ u|w}} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w^{d(u|w)} Q^{-d(u|w)c_{iw}} \\
&\quad = \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}}.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
\prod_{w' \in \mathcal{M}(\mathbf{E}')} \max_{1 \le i \le 2} ||L_{iw'}(\mathbf{x})||_{w'} Q^{-c_{iw'}} &= \prod_{u \in \mathcal{M}(\mathbf{F})} \max_{1 \le i \le 2} ||L_{iu}(\mathbf{x})||_u Q^{-c_{iu}} \\
&= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}}.
\end{aligned}$$

Now we consider the inequality

$$H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) \le Q^{-\delta}, \tag{16}$$

where $\mathbf{x} \in \overline{\mathbf{Q}}^2$ and $Q, \delta > 0$.

**Theorem 8 (Bugeaud and Evertse [10]).** *Let* $\mathcal{L} = (L_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ *be a tuple of linear forms satisfying* (10) *and* $\mathbf{c} = (c_{iv} : v \in \mathcal{M}(\mathbf{K}),\ i = 1, 2)$ *a tuple of reals fulfilling* (12). *Moreover, let* $0 < \delta \le 1$.

*Then there are proper linear subspaces* $T_1, \ldots, T_{t_1}$ *of* $\overline{\mathbb{Q}}^2$*, all defined over* $\mathbf{K}$*, with*

$$t_1 = t_1(r, \delta) = 2^{25}\delta^{-3}\log(2r)\log\left(\delta^{-1}\log(2r)\right) \tag{17}$$

*such that the following holds: for every real* $Q$ *with*

$$Q > \max\left(\mathcal{H}^{1/\binom{r}{2}}, 2^{2/\delta}\right) \tag{18}$$

*there is a subspace* $T_i \in \{T_1, \ldots, T_{t_1}\}$ *which contains all solutions* $\mathbf{x} \in \overline{\mathbf{Q}}^2$ *of (16).*

This is Proposition 4.1 of [10] in the case of $n = 2$.

## 4. Systems of Inequalities

In this section we apply Theorem 8 to certain systems of inequalities, which are generalization of Theorem 5.1 in [10]. Let $\mathbf{K} \subset \overline{\mathbb{Q}}$ be a number field of degree $d$. We define some notation about linear forms with algebraic coefficients. Take a linear form $L(X, Y) = \alpha X + \beta Y \in \overline{\mathbb{Q}}[X, Y]$ and put

$$\mathbf{K}(L) = \mathbf{K}(\alpha, \beta).$$

Define the inhomogeneous height $H^*(L)$ of $L$ by

$$H^*(L) = \prod_{v \in \mathcal{M}(\mathbf{K}(L))} \max\{1, ||\alpha||_v, ||\beta||_v\}.$$

Note that, for a number field $\mathbf{E}$ including $\mathbf{K}(L)$,

$$\begin{aligned}&\prod_{w \in \mathcal{M}(\mathbf{E})} \max\{1, ||\alpha||_v, ||\beta||_v\}\\ &= \prod_{v \in \mathcal{M}(\mathbf{K}(L))} \prod_{\substack{w \in \mathcal{M}(\mathbf{E})\\ w|v}} \max\{1, ||\alpha||_v, ||\beta||_v\}^{d(w|v)} = H^*(L)\end{aligned} \tag{19}$$

by (15). In what follows we put, for $w \in \mathbf{E}$,

$$||L||_w = \max\{||\alpha||_w, ||\beta||_w\}.$$

Moreover, if an automorphism $\sigma : \overline{\mathbb{Q}} \to \overline{\mathbb{Q}}$ is given, let

$$\sigma(L) = \sigma(\alpha)X + \sigma(\beta)Y.$$

Write the Archimedean place associated with the inclusion map $\mathbf{K} \hookrightarrow \mathbb{C}$ by $\infty$, namely,

$$||x||_\infty = |x|^{1/d} \text{ for } x \in \mathbf{K}. \tag{20}$$

Let $\varepsilon$ be a real with $0 < \varepsilon \leq 1/2$ and $S$ a finite subset of $\mathcal{M}(\mathbf{K})$ including all archimedean places of $\mathbf{K}$. Moreover, let $L_{iv}$ $(v \in S,\ i = 1, 2)$ be linear forms in $X, Y$ with coefficients in $\overline{\mathbb{Q}}$ such that

$$\begin{cases} \det(L_{1v}, L_{2v}) = 1 \text{ for } v \in S, \\ \text{Card}\left(\bigcup_{v \in S}\{L_{1v}, L_{2v}\}\right) \leq R, \\ [K(L_{iv}) : K] \leq D \text{ for } v \in S,\ i = 1, 2, \\ H^*(L_{iv}) \leq H \text{ for } v \in S,\ i = 1, 2, \end{cases} \tag{21}$$

and $e_{iv}$ $(v \in S,\ i = 1, 2)$ be reals satisfying

$$\sum_{v \in S} \sum_{i=1}^{2} e_{iv} = -\varepsilon. \tag{22}$$

Put

$$A = 1 + \sum_{v \in S} \max\{0, e_{1v}, e_{2v}\} \ (\geq 1).$$

Finally, let $\Psi$ be a function from $O_{\mathbf{K}}^2$ to $\mathbb{R}_{\geq 0}$, where $O_{\mathbf{K}}$ is the ring of integers of $\mathbf{K}$. Suppose every valuation $v$ of $\mathbf{K}$ to be extended to $\overline{\mathbb{Q}}$. Consider the system of inequalities

$$||L_{iv}(\mathbf{x})||_v \leq \Psi(\mathbf{x})^{e_{iv}} \ (v \in S, i = 1, 2), \tag{23}$$

where $\mathbf{x} \in O_{\mathbf{K}}^2$ with $\Psi(\mathbf{x}) \neq 0$.

**Proposition 9.** *The set of solutions $\mathbf{x} \in O_{\mathbf{K}}^2$ of (23) with*

$$\Psi(\mathbf{x}) > \max\{2H, 2^{4/\varepsilon}\} \tag{24}$$

*is contained in the union of at most*

$$2^{31} A^4 \varepsilon^{-3} \log(2RD) \log\left(\varepsilon^{-1} \log(2RD)\right)$$

*proper linear subspaces of $\mathbf{K}^2$.*

*Proof.* We can prove this proposition in the same way as Theorem 5.1 in [10]. Let $\mathbf{E}$ be a finite normal extension of $\mathbf{K}$, containing the coefficients of $L_{iv}$ as well as the conjugates over $\mathbf{K}$ of these coefficients, for $v \in S$, $i = 1, 2$. Let $\widetilde{S}$ denote the set of places of $\mathbf{E}$ lying above the places in $S$. Note that $\widetilde{S} \supset \mathcal{M}_{arc}(\mathbf{E})$. Take a place $w \in \mathcal{M}(\mathbf{E})$ above the place $v \in \mathcal{M}(\mathbf{K})$. For simplicity, put

$$d_w = d(w|v).$$

If $w \in \widetilde{S}$, then there exists an automorphism $\sigma_w$ of $\mathbf{E}$ satisfying

$$||x||_w = ||\sigma_w(x)||_v^{d_w} \text{ for } x \in \mathbf{E}.$$

For $i = 1, 2$, we define the linear forms $L_{iw}$ and the real numbers $e_{iw}$ by

$$L_{iw} = \begin{cases} \sigma_w^{-1}(L_{iv}) & (w \in \widetilde{S}), \\ X & (i = 1,\ w \notin \widetilde{S}) \\ Y & (i = 2,\ w \notin \widetilde{S}) \end{cases}$$

and

$$e_{iw} = \begin{cases} d_w e_{iv} & (w \in \widetilde{S}), \\ 0 & (w \notin \widetilde{S}), \end{cases}$$

respectively. Take an $\mathbf{x} \in O_{\mathbf{K}}^2$ with (23). If $w \notin \widetilde{S}$, then $w$ is non-archimedean, so

$$||L_{iw}(\mathbf{x})||_w \leq 1.$$

Moreover, since

$$||L_{iv}(\mathbf{x})||_v^{d_w} = ||\sigma_w\big(L_{iw}(\mathbf{x})\big)||_v^{d_w} = ||L_{iw}(\mathbf{x})||_w,$$

$\mathbf{x}$ satisfies the system of inequalities

$$||L_{iw}(\mathbf{x})||_w \leq \Psi(\mathbf{x})^{e_{iw}} \ (w \in \mathcal{M}(\mathbf{E}),\ i = 1, 2). \tag{25}$$

By using (15) and (22) we get

$$\begin{aligned} \sum_{w \in \mathcal{M}(\mathbf{E})} \sum_{i=1}^{2} e_{iw} &= \sum_{v \in S} \sum_{\substack{v \in \mathcal{M}(\mathbf{E}) \\ w|v}} \sum_{i=1}^{2} d_w e_{iv} \\ &= \sum_{v \in S} \sum_{i=1}^{2} e_{iv} = -\varepsilon. \end{aligned} \tag{26}$$

By the definition of $L_{iw}$ with $1 \le i \le 2$ and $w \in \mathcal{M}(\mathbf{E})$

$$\mathrm{Card}\left(\bigcup_{w\in\mathcal{M}(\mathbf{E})}\{L_{1w},L_{2w}\}\right) \le 2+DR.$$

Let $\mathcal{L} = (L_{iw} : w \in \mathcal{M}(\mathbf{E}),\ i = 1,2)$. Define the tuple of reals $\mathbf{c} = (c_{iw} : w \in \mathcal{M}(\mathbf{E}),\ i = 1,2)$ by

$$c_{iw} = A^{-1}\left(e_{iw} - \frac{1}{2}\sum_{j=1}^{2} e_{jw}\right).$$

We apply Theorem 8 with $\mathcal{L}$, $\mathbf{c}$, $r = 2 + DR(\ge 4)$, and

$$\delta = \frac{\varepsilon}{2A}.$$

It is easy to check the condition (10). We verify the condition (12). The first statement is clear by the definition of $c_{iw}$ and $e_{iw}$. The second statement follows from $c_{1w} + c_{2w} = 0$ for each $w \in \mathcal{M}(\mathbf{E})$. Moreover, by using (22) and (26), we obtain

$$\begin{aligned}
A\sum_{w\in\mathcal{M}(\mathbf{E})}\max\{c_{1w},c_{2w}\} &= \sum_{w\in\mathcal{M}(\mathbf{E})}\max\{e_{1w},e_{2w}\} - \frac{1}{2}\sum_{w\in\mathcal{M}(\mathbf{E})}\sum_{j=1}^{2}e_{jw}\\
&= \sum_{v\in S}\sum_{\substack{w\in\mathcal{M}(\mathbf{E})\\ w|v}}\max\{d_we_{1v},d_we_{2v}\} + \frac{\varepsilon}{2}\\
&\le \frac{\varepsilon}{2} + \sum_{v\in S}\max\{0,e_{1v},e_{2v}\}\sum_{\substack{w\in\mathcal{M}(\mathbf{E})\\ w|v}}d_w\\
&\le 1 + \sum_{v\in S}\max\{0,e_{1v},e_{2v}\} = A.
\end{aligned}$$

Therefore we proved the last inequality of (12).

Let $\mathbf{x} \in O_{\mathbf{K}}^2$ be a solution of (23) with (24). Then $\mathbf{x}$ also fulfills (25). Put

$$Q = \Psi(\mathbf{x})^A.$$

Finally, we show that such an $\mathbf{x}$ satisfies (16) and (18). By (25) and the definition of $c_{iw}$,

$$\begin{aligned}
||L_{iw}(\mathbf{x})||_wQ^{-c_{iw}} &= ||L_{iw}(\mathbf{x})||_w\Psi(\mathbf{x})^{-e_{iw}}\Psi(\mathbf{x})^{(e_{1w}+e_{2w})/2}\\
&\le \Psi(\mathbf{x})^{(e_{1w}+e_{2w})/2}
\end{aligned}$$

for $w \in \mathcal{M}(\mathbf{E})$, $i = 1, 2$. By taking product over $w \in \mathcal{M}(\mathbf{E})$ and using (26), we get

$$\begin{aligned} H_{Q,\mathcal{L},\mathbf{c}}(\mathbf{x}) &= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i \le 2} ||L_{iw}(\mathbf{x})||_w Q^{-c_{iw}} \\ &\le \prod_{w \in \mathcal{M}(\mathbf{E})} \Psi(\mathbf{x})^{(e_{1w}+e_{2w})/2} \\ &= \Psi(\mathbf{x})^{-\varepsilon/2} = Q^{-\delta}. \end{aligned}$$

Thus (16) is verified. Put

$$\bigcup_{w \in \mathcal{M}(\mathbf{E})} \{L_{1w}, L_{2w}\} = \{L_1, \dots, L_s\},$$

where $s \le r$. We check $H^*(L_{iw}) \le H$ for $w \in \mathcal{M}(\mathbf{E})$ and $i = 1, 2$. We may assume that $w \in \widetilde{S}$. There exists an automorphism $\sigma_w$ of $\mathbf{E}$ such that $L_{iw} = \sigma_w^{-1}(L_{iv})$, where $v \in S$ is the place below $w$. By (19) and (21)

$$\begin{aligned} H^*(L_{iw}) &= \prod_{u \in \mathcal{M}(\mathbf{E})} \max\{1, ||\sigma_w^{-1}(L_{iv})||_u\} \\ &= \prod_{u \in \mathcal{M}(\mathbf{E})} \max\{1, ||L_{iv}||_u\} = H^*(L_{iv}) \le H. \end{aligned}$$

Let $\widetilde{D} = [\mathbf{E} : \mathbb{Q}]$ and $1 \le i < j \le s$. If $w$ is an Archimedean place, then

$$\begin{aligned} ||\det(L_i, L_j)||_w &\le 2^{[\mathbf{E}_w:\mathbb{R}]/\widetilde{D}} ||L_i||_w ||L_j||_w \\ &\le 2^{[\mathbf{E}_w:\mathbb{R}]/\widetilde{D}} \prod_{l=1}^{s} \max\{1, ||L_l||_w\}. \end{aligned}$$

Similarly, if $w$ is non-Archimedean, then by the ultrametric inequality

$$||\det(L_i, L_j)||_w \le ||L_i||_w ||L_j||_w \le \prod_{l=1}^{s} \max\{1, ||L_l||_w\}.$$

Since $\sum_{w \in \mathcal{M}_{arc}(\mathbf{E})} [\mathbf{E}_w : \mathbb{R}] = \widetilde{D}$, we conclude that

$$\begin{aligned} \mathcal{H}(\mathcal{L}) &= \prod_{w \in \mathcal{M}(\mathbf{E})} \max_{1 \le i < j \le s} ||\det(L_i, L_j)||_w \\ &\le \prod_{w \in \mathcal{M}_{arc}(\mathbf{E})} 2^{[\mathbf{E}_w:\mathbb{R}]/\widetilde{D}} \prod_{l=1}^{s} H^*(L_l) \le 2H^r, \end{aligned}$$

hence

$$\begin{aligned}\max\left\{\mathcal{H}(\mathcal{L})^{1/\binom{r}{2}}, 2^{2/\delta}\right\} &\leq \max\left\{2^{1/\binom{r}{2}} H^{r/\binom{r}{2}}, 2^{4A/\varepsilon}\right\} \\ &\leq \max\left\{2H, 2^{4/\varepsilon}\right\}^A < \Psi(\mathbf{x})^A = Q.\end{aligned}$$

Let $t_1 = t_1(r, \delta)$ be defined as (17). Theorem 8 implies the following: there are proper subspaces $T_1, \ldots, T_{t_1}$ of $\overline{\mathbb{Q}}$ all defined over $\mathbf{E}$ such that any solution $\mathbf{x} \in O_{\mathbf{K}}^2$ of (23) with (24) satisfies

$$\mathbf{x} \in \bigcup_{i=1}^{t_1} (T_i \setminus \mathbf{K}^2).$$

Therefore, for the proof of the proposition it suffices to check

$$t_1 \leq 2^{31} A^4 \varepsilon^{-3} \log(2RD) \log\left(\varepsilon^{-1} \log(2RD)\right) \tag{27}$$

Since $DR \geq 2$, we have

$$\log(2r) \leq 2\log(2DR).$$

Moreover, by $0 < \varepsilon \leq 1/2$

$$\begin{aligned}\log\left(\delta^{-1}\log(2r)\right) &\leq \log\left(4A\varepsilon^{-1}\log(2DR)\right) \\ &\leq 4A\log\left(\varepsilon^{-1}\log(2DR)\right).\end{aligned}$$

Thus (27) follows. □

## 5. Proof of Main Results

We give another proof of the inequality (4). Without loss of generality, we may assume that $1/\alpha \leq \xi < 1$. In fact, there is an integer $R$ with $1/\alpha \leq \xi\alpha^R < 1$. Then since $\xi\alpha^n = (\xi\alpha^R)\alpha^{n-R}$, we have

$$|\lambda_N(\alpha, \xi) - \lambda_N(\alpha, \xi\alpha^R)| \leq |R|.$$

In particular, for any $n \leq 0$, $[\xi\alpha^n] = 0$, and so $s_{-n}(\xi) = 0$ by (5).

Recall that $s_{-n}(\xi) \neq 0$ for infinitely many positive $n$, which we introduced in Section 1. Define the increasing sequence of positive integers $(n_j)_{j=1}^{\infty}$ by $s_{-n}(\xi) \neq 0$ if and only if $n = n_j$ for some $j \geq 1$. By the first statement of Lemma 4, it suffices to show that

$$\liminf_{j\to\infty} \frac{j}{\log n_j} \geq \left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha) - \log(a_d\alpha)}\right)\right)^{-1}.$$

Write the conjugates of $\alpha$ by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Without loss of generality, we may assume that

$$|\alpha_k| > 1 \ (1 \le k \le p),$$

where $p$ is the number of the conjugates of $\alpha$ whose absolute values are greater than 1. In what follows, $C_1(\alpha), C_2(\alpha), \ldots$ denote positive constants depending only on $\alpha$. We first check

$$C_1(\alpha)\alpha^{n_j} \le \left|\sum_{i=0}^{n_j} \alpha^i s_{i-n_j}(\xi)\right| \le C_2(\alpha)\alpha^{n_j} \tag{28}$$

for any sufficiently large $j \ge 1$. By using $s_m(\xi) = 0$ for any $m > 0$ and the second statement of Proposition 5, we get

$$\begin{aligned}
\sum_{i=0}^{n_j} \alpha^i s_{i-n_j}(\xi) &= \sum_{i=0}^{\infty} \alpha^i s_{i-n_j}(\xi) \\
&= \alpha^{n_j} \sum_{i=-\infty}^{\infty} \alpha^{i-n_j} s_{i-n_j}(\xi) - \sum_{i=-\infty}^{-1} \alpha^i s_{i-n_j}(\xi) \\
&= -\xi\alpha^{-1+n_j}(P_\alpha^*)'\left(\frac{1}{\alpha}\right) - \sum_{i=-\infty}^{-1} \alpha^i s_{i-n_j}(\xi),
\end{aligned}$$

where $(P_\alpha^*)'$ is defined in Proposition 5. Thus

$$\begin{aligned}
\left|\sum_{i=0}^{n_j} \alpha^i s_{i-n_j}(\xi) + \xi\alpha^{-1+n_j}(P_\alpha^*)'\left(\frac{1}{\alpha}\right)\right| &\le \max\{L_+(\alpha), L_-(\alpha)\} \sum_{i=-\infty}^{-1} \alpha^i \\
&\le C_3(\alpha).
\end{aligned}$$

By considering $(P_\alpha^*)'(1/\alpha) \ne 0$, we obtain (28). Recall that, for any nonempty subset $I$ of $\{1, 2, \ldots, d\}$, the number

$$a_d \prod_{k\in I} \alpha_k$$

is an algebraic integer (for example, see pages 71 and 72 of [20]). So (28) implies that

$$1 \le \left| a_d^{n_j} \prod_{k=1}^{d} \left( \sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi) \right) \right| \tag{29}$$

since the right-hand side of this inequality is the absolute value of a nonzero rational integer. By the first statement of Proposition 5, for $2 \le k \le p$,

$$\left|\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi)\right| = \left|-\sum_{i=1+n_j}^{\infty} \alpha_k^i s_{i-n_j}(\xi) - \sum_{i=-\infty}^{-1} \alpha_k^i s_{i-n_j}(\xi)\right|.$$

Because $s_{i-n_j}(\xi) = 0$ for each $i$ with $i \ge 1 + n_j$, we have

$$\begin{aligned} \left|\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi)\right| &\le \max\{L_+(\alpha), L_-(\alpha)\} \sum_{i=-\infty}^{n_j - n_{1+j}} |\alpha_k^i| \\ &\le C_4(\alpha)|\alpha_k|^{n_j - n_{1+j}}. \end{aligned} \tag{30}$$

Similarly, if $p + 1 \le k \le d$, then

$$\left|\sum_{i=0}^{n_j} \alpha_k^i s_{i-n_j}(\xi)\right| \le C_5(\alpha) n_j. \tag{31}$$

Take an arbitrary positive $\varepsilon$. By combining (28), (29), (30), and (31), we conclude for sufficiently large $j$ that

$$\begin{aligned} 1 &\le C_6(\alpha) a_d^{n_j} \alpha^{n_j} \left(\prod_{k=2}^{p} |\alpha_k|^{n_j - n_{1+j}}\right) \left(\prod_{k=1+p}^{d} n_j\right) \\ &\le |\alpha_2 \cdots \alpha_p|^{-n_{1+j}} \left((1+\varepsilon)M(\alpha)\right)^{n_j}. \end{aligned}$$

Hence, for $j \ge j_0$,

$$\frac{n_{1+j}}{n_j} \le \frac{\log((1+\varepsilon)M(\alpha))}{\log M(\alpha) - \log(a_d \alpha)} =: F_1(\varepsilon)$$

and

$$n_j \le n_{j_0} F_1(\varepsilon)^{j - j_0}.$$

Therefore we conclude that

$$\liminf_{j \to \infty} \frac{j}{\log n_j} \ge \frac{1}{\log F_1(\varepsilon)}.$$

Since $\varepsilon$ is an arbitrary positive number, (4) is proved.

*Proof of Theorem 2.* Theorem 3 of [11] shows for infinitely many $n \geq 0$ that $s_{-n}(\xi) \neq 0$. There exists the unique increasing sequence of positive integers $(n_j)_{j=1}^{\infty}$ such that $s_{-n}(\xi) \neq 0$ if and only if $n = n_j$ for some $j \geq 1$. Put

$$\xi' = \sum_{i=-\infty}^{-1} \alpha^i s_i(\xi) \text{ and } \xi_j = \sum_{i=-n_j}^{-1} \alpha^i s_i(\xi).$$

We may assume $\xi \in [1/\alpha, 1)$. Then $s_n(\xi) = 0$ for any $n \geq 0$. By Proposition 5, $\xi' \notin \mathbb{Q}(\alpha)$. Thus we get $\xi' \neq \xi_j$ for any $j \geq 1$. Recall that $\infty$ is the archimedean place defined by (20). In what follows, let $C_1(\alpha)$, $C_2(\alpha), \ldots$ be positive constants depending only on $\alpha$. Then

$$\begin{cases} 0 < ||\xi' - \xi_j||_\infty \leq C_1(\alpha)\alpha^{-n_{1+j}/d}, \\ ||\xi_j||_\infty \leq C_1(\alpha). \end{cases} \tag{32}$$

Take an arbitrary positive number $\varepsilon$. Apply Theorem 7 with

$$\mathbf{K} = \mathbb{Q}(\alpha),\ S = \mathcal{M}_{arc}(\mathbf{K}) \cup \left\{ v \in \mathcal{M}_{non}(\mathbf{K}) \middle| ||\alpha||_v < 1 \right\},$$

and

$$\theta_v = \begin{cases} 1/\xi' & (\text{if } v = \infty), \\ 0 & (\text{otherwise}). \end{cases}$$

Consider solutions $\gamma$ of (9) satisfying $\gamma = 1/\xi_j$ for some $j$. Let us take any $j_0 \geq 0$. By (32) there exists at most finitely many $j \geq 1$ with $\xi_j = \xi_{j_0}$. Thus by Theorem 7 there exist at most finitely many $j$ such that $\gamma = 1/\xi_j$ fulfills (9). Namely, for all sufficiently large $j$,

$$\prod_{v \in S} \min \left\{ 1, \left\| \theta_v - \frac{1}{\xi_j} \right\|_v \right\} \geq\ H\left(\frac{1}{\xi_j}\right)^{-2-\varepsilon} = H(\xi_j)^{-2-\varepsilon}. \tag{33}$$

We have

$$\begin{aligned} &\prod_{v \in S} \min \left\{ 1, \left\| \theta_v - \frac{1}{\xi_j} \right\|_v \right\} \\ &\qquad = \left\| \frac{1}{\xi'} - \frac{1}{\xi_j} \right\|_\infty \prod_{v \in S \setminus \{\infty\}} \min \left\{ 1, \left\| \frac{1}{\xi_j} \right\|_v \right\} \\ &\qquad = \left\| \frac{1}{\xi'} - \frac{1}{\xi_j} \right\|_\infty \max\{1, ||\xi_j||_\infty\} \left( \prod_{v \in S} \max\{1, ||\xi_j||_v\} \right)^{-1} \\ &\qquad \leq C_1(\alpha)^2 \alpha^{-n_{1+j}/d} \left( \prod_{v \in S} \max\{1, ||\xi_j||_v\} \right)^{-1}. \end{aligned}$$

Note that if $v \in \mathcal{M}(\mathbf{K})\backslash S$, then $||\xi_j||_v \leq 1$ by the ultrametric inequality. Hence

$$\prod_{v\in S} \min\left\{1, \left|\left|\theta_v - \frac{1}{\xi_j}\right|\right|_v\right\}$$

$$\leq C_1(\alpha)^2 \alpha^{-n_{1+j}/d} H(\xi_j)^{-1} \prod_{v\in\mathcal{M}(\mathbf{K})\backslash S} \max\{1, ||\xi_j||\}$$

$$\leq C_1(\alpha)^2 \alpha^{-n_{1+j}/d} H(\xi_j)^{-1}.$$

By combining the inequality above and (33), we obtain, for any sufficiently large $j$,

$$\alpha^{n_{1+j}} \leq C_1(\alpha)^{2d} H(\xi_j)^{(1+\varepsilon)d}.$$

Write the conjugates of $\alpha$ by $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. Let $p$ (resp. $q$) be the number of the conjugates of $\alpha$ whose absolute values are greater (resp. smaller) than 1. Without loss of generality we may assume $|\alpha_k| > 1$ if $1 \leq k \leq p$, $|\alpha_k| < 1$ if $p+1 \leq k \leq p+q$, and $|\alpha_k| = 1$ otherwise. By the ultrametric inequality

$$\prod_{v\in\mathcal{M}_{non}(\mathbf{K})} \max\{1, ||\xi_j||_v\} \quad \leq \quad \prod_{v\in\mathcal{M}_{non}(\mathbf{K})} \max\{1, ||\alpha^{-1}||_v\}^{n_j}.$$

Since $s_n(\xi) \leq \max\{L_+(\alpha), L_-(\alpha)\}$ for every integer $n$

$$\prod_{k=p+1}^{d} \left|\sum_{i=-n_j}^{-1} \alpha_k^i s_i(\xi)\right|^{1/d} \leq \prod_{k=p+1}^{p+q} \left|C_2(\alpha)\alpha_k^{-n_j}\right|^{1/d} \prod_{k=p+q+1}^{d} (C_2(\alpha)n_j)^{1/d}$$

$$= C_2(\alpha)^{(d-p)/d} n_j^{(d-p-q)/d} \prod_{k=p+1}^{p+q} |\alpha_k|^{n_j/d}.$$

By using the first statement of Proposition 5 and $s_n(\xi) = 0$ for any $n \geq 0$,

$$\prod_{k=1}^{p} \left|\sum_{i=-n_j}^{-1} \alpha_k^i s_i(\xi)\right|^{1/d} = \left|\sum_{i=-n_j}^{-1} \alpha^i s_i(\xi)\right|^{1/d} \prod_{k=2}^{p} \left|\sum_{i=-\infty}^{-n_{1+j}} \alpha_k^i s_i(\xi)\right|^{1/d}$$

$$\leq C_3(\alpha)^{1/d} \prod_{k=2}^{p} \left|C_3(\alpha)\alpha_k^{-n_{1+j}}\right|^{1/d}$$

$$= C_3(\alpha)^{p/d} \prod_{k=2}^{p} \left|\alpha_k^{-n_{1+j}}\right|^{1/d}.$$

Hence we obtain

$$\begin{aligned}\prod_{v\in\mathcal{M}_{arc}(\mathbf{K})}\max\{1,||\xi_j||_v\} &= \prod_{k=1}^{d}\left|\sum_{i=-n_j}^{-1}\alpha_k^i s_i(\xi)\right|^{1/d} \\ &\leq C_4(\alpha)n_j^{(d-p-q)/d}\prod_{v\in\mathcal{M}_{arc}(\mathbf{K})}\max\{1,||\alpha^{-1}||_v\}^{n_j} \\ &\quad\times\prod_{k=2}^{p}\left|\alpha_k^{-n_{1+j}}\right|^{1/d}\end{aligned}$$

and so

$$H(\xi_j)\leq C_4(\alpha)n_j^{(d-p-q)/d}H(\alpha^{-1})^{n_j}\prod_{k=2}^{p}\left|\alpha_k^{-n_{1+j}}\right|^{1/d}.$$

Finally, we conclude for sufficiently large $j$ that

$$\begin{aligned}\alpha^{n_{1+j}} &\leq C_5(\alpha)n_j^{(1+\varepsilon)(d-p-q)}H(\alpha^{-1})^{(1+\varepsilon)dn_j}\prod_{k=2}^{p}\left|\alpha_k^{-n_{1+j}}\right|^{(1+\varepsilon)} \\ &\leq H(\alpha^{-1})^{(1+2\varepsilon)dn_j}\prod_{k=2}^{p}\left|\alpha_k^{-n_{1+j}}\right|^{(1+\varepsilon)} \\ &= M(\alpha)^{(1+2\varepsilon)n_j}\prod_{k=2}^{p}\left|\alpha_k^{-n_{1+j}}\right|^{(1+\varepsilon)},\end{aligned}$$

where for the last equality we use (7). Taking logarithms of both sides of the inequality above, we get

$$\frac{n_{1+j}}{n_j}\leq\frac{(1+2\varepsilon)\log M(\alpha)}{\log\alpha+(1+\varepsilon)\log|\alpha_2\cdots\alpha_p|}=:F_2(\varepsilon), \tag{34}$$

consequently

$$\liminf_{j\to\infty}\frac{j}{\log n_j}\geq\frac{1}{\log F_2(\varepsilon)}.$$

Therefore by the first statement of Lemma 4

$$\liminf_{N\to\infty}\frac{\lambda_N(\alpha,\xi)}{\log N}\geq\frac{1}{\log F_2(\varepsilon)}.$$

Since $\varepsilon$ is an arbitrary positive number, we proved the theorem. In fact,

$$\lim_{\varepsilon\to+0}\frac{1}{\log F_2(\varepsilon)}=\left(\log\left(\frac{\log M(\alpha)}{\log M(\alpha)-\log a_d}\right)\right)^{-1}.$$

□

*Proof of Theorem 1.* We may assume $\xi\in[1/\alpha,1)$, and so $s_n(\xi)=0$ for any $n\geq 0$. Put

$$\xi'=\sum_{i=-\infty}^{-1}\alpha^i s_i(\xi). \tag{35}$$

By the second statement of Proposition 5, we have $\xi'\notin\mathbb{Q}(\alpha)$. Let $p$ be the number of the conjugates of $\alpha$ whose absolute values are greater than 1. Write the conjugates of $\alpha$ by $\alpha_1=\alpha,\alpha_2,\dots,\alpha_d$, where $d$ is the degree of $\alpha$. Without loss of generality, we may assume that $|\alpha_k|>1$ for $k=1,2,\dots,p$.

First we show the following:

**Lemma 10.** *There is a sequence of integers* $\mathbf{y}=(y_n)_{n=1}^{\infty}$ *satisfying the following:*

1. $y_n=0$ *or* $y_n=s_{-n}(\xi)$;
2. $\sum_{i=1}^{\infty}y_i\alpha^{-i}=\xi'$;
3. $\sum_{i=1}^{\infty}y_i\alpha_k^{-i}=0$ *for any* $k$ *with* $2\leq k\leq p$;
4. *Put*

$$\{n\geq 1|y_n\neq 0\}=:\{n_1<n_2<\cdots\}$$

*and*

$$\xi_j=\sum_{i=1}^{n_j}y_i\alpha^{-i}.$$

*Then, for any* $h$ *and* $l$ *with* $h<l$, $\xi_h\neq\xi_l$.

*Proof.* We construct the bounded sequences of integers $\mathbf{y}_m=(y(m,n))_{n=1}^{\infty}$ ($m=1,2,\dots$) by induction on $m$ fulfilling the following:

1. For any $n\geq 1$,

$$y(m,n)=0\text{ or }y(m,n)=s_{-n}(\xi); \tag{36}$$

2.

$$\sum_{i=1}^{\infty} \alpha^{-i} y(m,i) = \xi'; \tag{37}$$

3. For any $k$ with $2 \le k \le p$,

$$\sum_{i=1}^{\infty} \alpha_k^{-i} y(m,i) = 0. \tag{38}$$

In particular, we have, for any $m, n \ge 1$,

$$|y(m,n)| \le |s_{-n}(\xi)| \le \max\{L_+(\alpha), L_-(\alpha)\}.$$

Define $\mathbf{y}_1 = (y(1,n))_{n=1}^{\infty}$ by

$$y(1,n) = s_{-n}(\xi) \ (n \ge 1).$$

For $m = 1$, (36) and (37) hold. Moreover, (38) follows from the first statement of Proposition 5.

Next, assume that we have a sequence of integers $\mathbf{y}_m$ with (36), (37), and (38) for $m \ge 1$. Let

$$\begin{aligned} \Xi_m &= \{n \ge 1 | y(m,n) \ne 0\} \\ &=: \{n(m,1) < n(m,2) < \cdots\} \end{aligned}$$

and

$$\xi(m,j) = \sum_{i=1}^{n(m,j)} \alpha^{-i} y(m,i).$$

By (37) and $\xi' \notin \mathbb{Q}(\alpha)$, $\Xi_m$ is an infinite set. If $\xi(m,h) \ne \xi(m,l)$ for any $h \ne l$, then then $\mathbf{y} = \mathbf{y}_m$ satisfies the last condition of Lemma 10. Moreover, first, second, and third conditions of Lemma 10 follow immediately from (36), (37), and (38). Otherwise, we define $\mathbf{y}_{m+1}$ by using $\mathbf{y}_m$. There exists an $h \ge 1$ such that $\xi(m,h) = \xi(m,l)$ for some $l > h$. For such an $h$, write the minimal value by $h_m$. Put

$$\Lambda_m = \{l > h_m | \xi(m,l) = \xi(m,h_m)\}.$$

Then $\Lambda_m$ is a finite set. In fact, if $\Lambda_m$ is an infinite set, then

$$\xi' = \lim_{\substack{n \to \infty \\ n \in \Lambda_m}} \xi(m,n) = \xi(m,h_m) \in \mathbb{Q}(\alpha),$$

which contradicts to $\xi \notin \mathbb{Q}(\alpha)$. So let

$$l_m = \max \Lambda_m.$$

We define $\mathbf{y}_{m+1} = (y(m+1,n))_{n=1}^{\infty}$ by

$$y(m+1,n) = \begin{cases} 0 & (\text{if } 1 + n(m,h_m) \le n \le n(m,l_m)), \\ y(m,n) & (\text{otherwise}). \end{cases}$$

Note that

$$\xi(m+1,j) = \begin{cases} \xi(m,j) & (\text{if } j \le h_m) \\ \xi(m, j + l_m - h_m) & (\text{if } j > h_m). \end{cases} \tag{39}$$

Now we verify that $\mathbf{y}_{m+1}$ fulfills (36), (37), and (38). (36) is obvious by the definition of $\mathbf{y}_{m+1}$. By the inductive hypothesis and

$$0 = \xi(m,l_m) - \xi(m,h_m) = \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha^{-i} y(m,i), \tag{40}$$

we get

$$\sum_{i=1}^{\infty} \alpha^{-i} y(m+1,i) = \sum_{i=1}^{\infty} \alpha^{-i} y(m,i) - \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha^{-i} y(m,i) = \xi'.$$

By taking the conjugate of (40), we deduce for any $k$ with $2 \le k \le p$ that

$$0 = \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha_k^{-i} y(m,i).$$

Thus

$$\begin{aligned} \sum_{i=1}^{\infty} \alpha_k^{-i} y(m+1,i) &= \sum_{i=1}^{\infty} \alpha_k^{-i} y(m,i) - \sum_{i=1+n(m,h_m)}^{n(m,l_m)} \alpha_k^{-i} y(m,i) \\ &= \sum_{i=1}^{\infty} \alpha_k^{-i} y(m,i) = 0. \end{aligned}$$

For the proof of Lemma 10 we may assume that, for any $m \ge 1$, $\mathbf{y} = \mathbf{y}_m$ is defined and does not satisfy the conditions of Lemma 10. We verify that $h_{m+1} > h_m$ for each $m \ge 1$. It suffices to check for $1 \le h < l$ with $h \le h_m$ that

$$\xi(m+1,l) \ne \xi(m+1,h).$$

In the case of $h < h_m$, this follows from (39) and the definition of $h_m$. So consider the case of $h = h_m$. Since $l + l_m - h_m > l_m$ we get

$$\xi(m+1,l) = \xi(m, l + l_m - h_m) \neq \xi(m, h_m) = \xi(m+1, h_m)$$

by the definition of $l_m$. Hence the sequence $h_m$ $(m = 1, 2, \ldots)$ is strictly increasing. In particular, $h_m \geq m$.

Let $n \geq 1$. Take an integer $m$ with $m \geq n$. Note that

$$n \leq m \leq h_m \leq n(m, h_m).$$

So, by the definition of $\mathbf{y}_{m+1}$, we have $y(m+1,n) = y(m,n)$. Thus

$$y(m,n) = y(n,n) \text{ for any } m \geq n. \tag{41}$$

We define the sequence $\mathbf{y} = (y_n)_{n=1}^{\infty}$ by

$$y_n = y(n,n).$$

In what follows we check the conditions of Lemma 10. The first condition is clear. Let $m \geq 1$ be any integer. Then by (41)

$$\begin{aligned} \left| \xi' - \sum_{i=1}^{\infty} \alpha^{-i} y_i \right| &= \left| \sum_{i=1}^{\infty} \alpha^{-i} (y(m,i) - y(i,i)) \right| \\ &= \left| \sum_{i=m+1}^{\infty} \alpha^{-i} (y(m,i) - y(i,i)) \right| \\ &\leq 2 \max\{L_+(\alpha), L_-(\alpha)\} \frac{1}{(\alpha-1)\alpha^m}. \end{aligned}$$

Similarly, for $2 \leq k \leq p$,

$$\begin{aligned} \left| \sum_{i=1}^{\infty} \alpha_k^{-i} y_i \right| &= \left| \sum_{i=1}^{\infty} \alpha_k^{-i} (y(m,i) - y(i,i)) \right| \\ &= \left| \sum_{i=m+1}^{\infty} \alpha_k^{-i} (y(m,i) - y(i,i)) \right| \\ &\leq 2 \max\{L_+(\alpha), L_-(\alpha)\} \frac{1}{(\alpha_k-1)\alpha_k^m}, \end{aligned}$$

where for the first equality we use (38). Since $m$ is arbitrary, we obtain the second and third conditions.

Finally, assume that $\xi_h = \xi_l$ for some $h < l$. Take an integer $m$ with $m > l$. Then by (41)

$$\xi(m, h) = \xi_h = \xi_l = \xi(m, l).$$

By the definition of $h_m$, we get $h_m \leq h < m$, which contradicts to $h_m \geq m$. Therefore, the last condition follows. □

For $N \geq 1$ put

$$\tau_N = \mathrm{Card}\{n \in \mathbb{Z} | 1 \leq n \leq N, y_n \neq 0\}.$$

By the first condition of Lemma 10

$$\tau_N \leq \mathrm{Card}\{n \in \mathbb{Z} | 1 \leq n \leq N, s_{-n}(\xi) \neq 0\}. \tag{42}$$

In what follows, we verify for all sufficiently large $N$ that

$$\tau_N \geq c \frac{(\log \alpha)^2}{(\log M(\alpha))^2 (\log(6D))^{1/2}} \frac{(\log N)^{3/2}}{(\log \log N)^{1/2}}. \tag{43}$$

Theorem 1 follows from (42), (43), and the first statement of Lemma 4.

Put $\mathbf{K} = \mathbb{Q}(\alpha)$. Let $\infty$ be the Archimedean place in $\mathbf{K}$ which is defined by (20). In what follows, let $C_1(\alpha, \xi)$, $C_2(\alpha, \xi), \ldots$ be positive constants depending only on $\alpha$ and $\xi$. Put

$$C_1(\alpha, \xi) = \max\{L_+(\alpha), L_-(\alpha)\}^{1/d} \max_{1 \leq k \leq p} \left( \frac{1}{1 - |\alpha_k^{-1}|} \right)^{1/d}.$$

Then we have

$$\begin{aligned} 0 < ||\xi' - \xi_j||_\infty &= \left| \sum_{i=n_{1+j}}^{\infty} y_i \alpha^{-i} \right|^{1/d} \\ &\leq C_1(\alpha, \xi) ||\alpha||_\infty^{-n_{1+j}}. \end{aligned} \tag{44}$$

Let $\varepsilon$ be an arbitrary positive number with $\varepsilon \leq 1/2$ and $F_2(\varepsilon)$ be defined by (34). In the same way as the proof of Theorem 2, we can verify that

$$\frac{n_{1+j}}{n_j} \leq F_2(\varepsilon)$$

for sufficiently large $j$. In particular, since

$$\lim_{\varepsilon \to +0} F_2(\varepsilon) = 1,$$

we get, for $j \geq C_2(\alpha, \xi)$,

$$n_{1+j} \leq 2n_j. \tag{45}$$

We count the numbers of $j$ fulfilling

$$n_{1+j} \geq (1+2\varepsilon)n_j. \tag{46}$$

Assume (46) and

$$n_j \geq C_3(\alpha, \xi)\varepsilon^{-9/8}. \tag{47}$$

We determine $C_3(\alpha, \xi)$ later. Let

$$S = \mathcal{M}_{arc}(\mathbf{K}) \cup \left\{ v \in \mathcal{M}_{non}(\mathbf{K}) \middle| ||\alpha||_v < 1 \right\}.$$

Define the linear forms $L_{i,v}$ ($v \in S$, $i = 1, 2$) by

$$\begin{aligned} L_{1v} &= \begin{cases} X - \xi' Y & \text{for } v = \infty, \\ X & \text{for } v \in S \backslash \{\infty\}, \end{cases} \\ L_{2v} &= Y \text{ for } v \in S. \end{aligned}$$

Then (21) is satisfied with $R = 3$, $D = [\mathbb{Q}(\alpha, \xi) : \mathbb{Q}(\alpha)]$, and $H = H(\xi)$. Consider the system of inequalities (23) with

$$\begin{aligned} e_{1v} &= \begin{cases} -(5\varepsilon)/4 & \text{for } v = \infty, \\ \varepsilon/(4d') & \text{for } v \in \mathcal{M}_{arc}(\mathbf{K}) \backslash \{\infty\}, \\ 0 & \text{for } v \in \mathcal{M}_{non}(\mathbf{K}) \setminus S, \end{cases} \\ e_{2v} &= (\log ||\alpha||_v)/(\log ||\alpha||_\infty), \text{ for } v \in S \\ \Psi(x, y) &= ||y||_\infty, \end{aligned}$$

where $d' = \mathrm{Card}(\mathcal{M}_{arc}(\mathbf{K}) \backslash \{\infty\})$. Then (22) follows from the product formula. Apply Proposition 9 with

$$\mathbf{x}_j = \left( \sum_{i=1}^{n_j} y_i \alpha^{-i+n_j}, \alpha^{n_j} \right) \in O_{\mathbf{K}}^2.$$

If $C_3(\alpha, \xi)$ is sufficiently large, then (24) follows from (47). In fact,

$$\log \Psi(\mathbf{x}_j) = n_j \log ||\alpha||_\infty > \max \left\{ \log(2H), \frac{4}{\varepsilon} \log 2 \right\}.$$

We check that $\mathbf{x}_j$ satisfies the system of inequalities (23). If $i = 2$, then

$$\begin{aligned} ||L_{2v}(\mathbf{x}_j)||_v &= ||\alpha^{n_j}||_v = ||\alpha^{n_j}||_\infty^{(\log ||\alpha||_v)/(\log ||\alpha||_\infty)} \\ &= \Psi(\mathbf{x}_j)^{e_{2v}}. \end{aligned}$$

In the case of $v \in S \setminus \mathcal{M}_{non}(\mathbf{K})$, by the ultrametric inequality

$$||L_{1v}(\mathbf{x}_j)||_v = \left\| \sum_{i=1}^{n_j} y_i \alpha^{-i+n_j} \right\|_v \leq 1 = \Psi(\mathbf{x}_j)^{e_{1v}}.$$

Now we show that

$$n_j^{1/d} C_1(\alpha, \xi) \leq ||\alpha^{n_j}||_\infty^{\varepsilon/(4d')} = \Psi(\mathbf{x}_j)^{\varepsilon/(4d')}. \tag{48}$$

Note that (48) is equivalent to

$$\frac{4d'}{\log ||\alpha||_\infty} \left( \frac{1}{d} + \frac{\log C_1(\alpha, \xi)}{\log n_j} \right) \leq \frac{\varepsilon n_j}{\log n_j}.$$

In what follows, constants implied by the Vinogradov symbols $\ll$, $\gg$ are absolute. If $n_j \geq C_3(\alpha, \xi)\varepsilon^{-9/8}$, then

$$\frac{\varepsilon n_j}{\log n_j} \gg \varepsilon n_j^{8/9} \geq C_3(\alpha, \xi)^{8/9}.$$

Thus, if $C_3(\alpha, \xi)$ is sufficiently large, then (48) follows. By (44), (46), and (48), we get

$$\begin{aligned} ||L_{1\infty}(\mathbf{x}_j)||_\infty &= ||\alpha^{n_j}||_\infty ||\xi_j - \xi'||_\infty \\ &\leq C_1(\alpha, \xi) ||\alpha||_\infty^{n_j - n_{1+j}} \leq C_1(\alpha, \xi) ||\alpha^{n_j}||_\infty^{-2\varepsilon} \\ &\leq ||\alpha^{n_j}||_\infty^{-(5\varepsilon)/4} = \Psi(\mathbf{x}_j)^{-(5\varepsilon)/4}. \end{aligned}$$

Let $v \in \mathcal{M}_{arc}(\mathbf{K}) \backslash \{\infty\}$. Then there exists an embedding $\sigma : \mathbf{K} \hookrightarrow \mathbb{C}$ such that

$$||x||_v = ||\sigma(x)||_\infty$$

for any $x \in \mathbf{K}$. Let $\sigma(\alpha) = \alpha_k$, where $2 \leq k \leq d$. If $2 \leq k \leq p$, then by (48) and the third condition of Lemma 10

$$\begin{aligned} ||L_{1v}(\mathbf{x}_j)||_v &= \left\| \sum_{i=1}^{n_j} y_i \alpha_k^{-i+n_j} \right\|_\infty = \left\| \alpha_k^{n_j} \right\|_\infty \left\| \sum_{i=n_{1+j}}^{\infty} y_i \alpha_k^{-i} \right\|_\infty \\ &\leq C_1(\alpha, \xi) \left\| \alpha_k^{n_j - n_{1+j}} \right\|_\infty \\ &\leq C_1(\alpha, \xi) \leq \Psi(\mathbf{x}_j)^{\varepsilon/(4d')} = \Psi(\mathbf{x}_j)^{e_{1v}}. \end{aligned}$$

In the case of $k \geq p+1$, by using $|\alpha_k| \leq 1$ and (48), we obtain

$$\begin{aligned} ||L_{1v}(\mathbf{x}_j)||_v &= \left\| \sum_{i=1}^{n_j} y_i \alpha_k^{-i+n_j} \right\|_\infty \\ &\leq |n_j \max\{L_+(\alpha), L_-(\alpha)\}|^{1/d} \leq n_j^{1/d} C_1(\alpha, \xi) \\ &\leq \Psi(\mathbf{x}_j)^{\varepsilon/(4d')} = \Psi(\mathbf{x}_j)^{e_{1v}}. \end{aligned}$$

Since

$$\begin{aligned} A &= 1 + \sum_{v \in S} \max\{0, e_{1v}, e_{2v}\} \\ &\leq 1 + \frac{\varepsilon}{4} + \sum_{k=1}^{p} \frac{\log ||\alpha_k||_\infty}{\log ||\alpha||_\infty} \ll \frac{\log M(\alpha)}{\log \alpha}, \end{aligned}$$

Proposition 9 indicates that the vectors $\mathbf{x}_j$ satisfying (46) and (47) lie in

$$\ll \left( \frac{\log M(\alpha)}{\log \alpha} \right)^4 \varepsilon^{-3} \log(6D) \log(\varepsilon^{-1} \log(6D))$$

one-dimensional linear subspaces of $\mathbf{K}^2$. By the last condition of Lemma 10, if $j \neq l$, then $\mathbf{x}_j$ and $\mathbf{x}_l$ are linearly independent over $\mathbf{K}$. Thus we obtain

$$\begin{aligned} &\mathrm{Card}\{j \geq 0 | n_j \geq C_3(\alpha, \xi)\varepsilon^{-9/8}, n_{1+j} \geq (1+2\varepsilon)n_j\} \\ &\qquad \ll \left( \frac{\log M(\alpha)}{\log \alpha} \right)^4 \varepsilon^{-3} \log(6D) \log(\varepsilon^{-1} \log(6D)). \end{aligned} \tag{49}$$

Let $j_1$ be the smallest $j$ such that $n_j \geq C_2(\alpha, \xi)$ and $J$ an integer with

$$J > \max\{n_{j_1}^3, 2^{12} C_3(\alpha, \xi)^{12}\}. \tag{50}$$

Moreover, let $j_2$ be the largest integer with $n_{j_2} \leq 2C_3(\alpha, \xi) J^{5/12}$. Then since

$$n_{j_1} \leq J^{1/3} \leq 2C_3(\alpha, \xi) J^{5/12},$$

we get

$$n_{j_2} \geq n_{j_1} \geq C_2(\alpha, \xi). \tag{51}$$

So by (45)

$$n_{j_2} \geq \frac{n_{1+j_2}}{2} \geq C_3(\alpha, \xi) J^{5/12}.$$

For a positive integer $u(\geq 2)$, put

$$\varepsilon_1 = \frac{(\log(6D))^{1/3}(\log M(\alpha))^{4/3}}{(\log \alpha)^{4/3}} \left(\frac{\log J}{J}\right)^{1/3}, \varepsilon_u = \frac{1}{2}.$$

Note that if $C_3(\alpha, \xi)$ is sufficiently large, then $\log(\varepsilon_1^{-1}) \geq \log(6D)$. Next, let $\varepsilon_2, \ldots, \varepsilon_{u-1}$ be any reals satisfying

$$\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{u-1} < \varepsilon_u.$$

Then we have

$$n_{j_2} \geq C_3(\alpha, \xi)\varepsilon_h^{-9/8} \tag{52}$$

for $h = 1, \ldots, u$. In fact,

$$n_{j_2} C_3(\alpha, \xi)^{-1} \varepsilon_h^{9/8} \geq J^{5/12} \varepsilon_1^{9/8} \geq J^{5/12} \cdot J^{-3/8} \geq 1.$$

Let $\mathcal{S}_0 = \{j_2, 1 + j_2, \ldots, J\}$ and, for $h = 1, \ldots, u$, let $\mathcal{S}_h$ denote the set of positive integers $j$ such that $j_2 \leq j < J$ and $n_{1+j} \geq (1 + 2\varepsilon_h)n_j$. Moreover, write the cardinality of $\mathcal{S}_h$ by $T_h$ for $h = 1, \ldots, u$. Then $\mathcal{S}_0 \supset \mathcal{S}_1 \supset \cdots \supset \mathcal{S}_u$. If $j \in \mathcal{S}_0$, then by (51) and (52) we have

$$\frac{n_{1+j}}{n_j} \leq 2$$

and

$$n_j \geq C_3(\alpha, \xi)\varepsilon_h^{-9/8} \text{ for } h = 1, \ldots, u.$$

Thus we get

$$\begin{aligned}
\frac{n_J}{n_{j_2}} &= \frac{n_J}{n_{-1+J}} \frac{n_{-1+J}}{n_{-2+J}} \cdots \frac{n_{1+j_2}}{n_{j_2}} \\
&\leq \left(\prod_{j \in \mathcal{S}_u} \frac{n_{1+j}}{n_j}\right) \prod_{h=0}^{u-1} \left(\prod_{j \in \mathcal{S}_h \setminus \mathcal{S}_{1+h}} \frac{n_{1+j}}{n_j}\right) \\
&\leq 2^{T_u}(1 + 2\varepsilon_1)^J \prod_{h=1}^{u-1} (1 + 2\varepsilon_{h+1})^{T_h - T_{1+h}}.
\end{aligned}$$

Taking logarithms, we obtain

$$\begin{aligned}\log\left(\frac{n_J}{n_{j_2}}\right) &\leq T_u \log 2 + 2\varepsilon_1 J + \sum_{h=1}^{u-1} 2\varepsilon_{1+h}(T_h - T_{1+h}) \\ &\leq T_u \log 2 + 2\varepsilon_1 J + 2\varepsilon_2 T_1 + 2\sum_{h=2}^{u-1}(\varepsilon_{1+h} - \varepsilon_h)T_h - T_u.\end{aligned}$$

(49) implies

$$T_h \ll \left(\frac{\log M(\alpha)}{\log\alpha}\right)^4 \log(6D)\varepsilon_h^{-3}\log(\varepsilon_h^{-1}\log(6D))$$

for $h = 1, \ldots, u$, and so

$$\begin{aligned}\log\left(\frac{n_J}{n_{j_2}}\right) &\ll \varepsilon_1 J + \left(\frac{\log M(\alpha)}{\log\alpha}\right)^4 \log(6D) \\ &\quad \times\Bigg(\log(\log(6D)) + \varepsilon_2\varepsilon_1^{-3}\log(\varepsilon_1^{-1}\log(6D)) \\ &\quad + \sum_{h=2}^{u-1}(\varepsilon_{1+h} - \varepsilon_h)\varepsilon_h^{-3}\log(\varepsilon_h^{-1}\log(6D))\Bigg).\end{aligned}$$

If $u$ tends to infinity and if $\max_{1\leq h\leq u-1}(\varepsilon_{1+h} - \varepsilon_h)$ tends to zero, then the sum converges to a Riemann integral, so

$$\begin{aligned}&\lim_{u\to\infty}\sum_{h=2}^{u-1}(\varepsilon_{1+h} - \varepsilon_h)\varepsilon_h^{-3}\log(\varepsilon_h^{-1}\log(6D)) \\ &\quad = \int_{\varepsilon_1}^{1/2} x^{-3}\log(x^{-1}\log(6D))dx \\ &\quad \ll \varepsilon_1^{-2}\log(\varepsilon_1^{-1}) + \log(\log(6D))\varepsilon_1^{-2} \ll \varepsilon_1^{-2}\log(\varepsilon_1^{-1}).\end{aligned}$$

Thus,

$$\begin{aligned}\log\left(\frac{n_J}{n_{j_2}}\right) &\ll \varepsilon_1 J + \left(\frac{\log M(\alpha)}{\log\alpha}\right)^4 \log(6D)\varepsilon_1^{-2}\log(\varepsilon_1^{-1}) \\ &\ll \left(\frac{\log M(\alpha)}{\log\alpha}\right)^{4/3} (\log(6D))^{1/3} J^{2/3}(\log J)^{1/3},\end{aligned}$$

where for the second inequality we use $\log(\varepsilon_1^{-1}) \leq \log J$. By using (50) and the definition of $j_2$ we obtain

$$\frac{n_{j_2}}{n_J^{1/2}} \leq \frac{n_{j_2}}{J^{1/2}} \leq 2C_3(\alpha,\xi)J^{-1/12} \leq 1,$$

and so

$$\log n_J \ll \left(\frac{\log M(\alpha)}{\log \alpha}\right)^{4/3} (\log(6D))^{1/3} J^{2/3} (\log J)^{1/3} =: G(J).$$

Hence

$$J \gg \frac{(\log \alpha)^2}{(\log M(\alpha))^2 (\log(6D))^{1/2}} \frac{(\log n_J)^{3/2}}{(\log\log n_J)^{1/2}}.$$

In fact, since the function $x^{3/2}(\log x)^{-1/2}$ is monotone increasing for $x > e$,

$$\frac{(\log n_J)^{3/2}}{(\log\log n_J)^{1/2}} \ll \frac{(\log G(J))^{3/2}}{(\log\log G(J))^{1/2}} \ll \left(\frac{\log M(\alpha)}{\log \alpha}\right)^2 (\log(6D))^{1/2} J.$$

Therefore, we proved the theorem. □

**Acknowledgements** I am very grateful to Prof. Masayoshi Hata for careful reading and for improving the language of this paper. I would like to thank Prof. Shigeki Akiyama for useful suggestions. I am sincerely grateful to Prof. Yann Bugeaud for giving me fruitful information about the reference paper [9, 10, 19]. This work is supported by the JSPS fellowship.

## References

[1] B. Adamczewski and Y. Bugeaud, On the complexity of algebraic numbers. I. Expansions in integer bases, Ann. of Math. **165** (2007), 547-565.

[2] B. Adamczewski and Y. Bugeaud, On the independence of expansions of algebraic numbers in an integer base, Bull. Lond. Math. Soc. **39** (2007), 283-289.

[3] S. Akiyama and Y. Tanigawa, Salem numbers and uniform distribution modulo 1, Publ. Math. Debrecen **64** (2004), 329-341.

[4] D. H. Bailey, J. M. Borwein, R. E. Crandall and C. Pomerance, On the binary expansions of algebraic numbers, J. Théor. Nombres Bordeaux **16** (2004), 487-518.

[5] M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse, and J. P. Schreiber, Pisot and Salem numbers, Birkhäuser Verlag, Basel, 1992.

[6] É. Borel, Les probabilités dénombrables et leurs applications arithmétiques, Rend. circ. Mat. Palermo **27** (1909), 247-271.

[7] É. Borel, Sur les chiffres décimaux de $\sqrt{2}$ et divers problèmes de probabilités en chaîne, C. R. Acad. Sci. Paris **230** (1950), 591-593.

[8] Y. Bugeaud, On the $b$-ary expansion of an algebraic number, Rend. Semin. Mat. Univ. Padova **118** (2007), 217-233.

[9] Y. Bugeaud, On the $\beta$-expansion of an algebraic number in algebraic base $\beta$, Integers **9** (2009), 215-226.

[10] Y. Bugeaud and J.-H. Evertse, On two notions of complexity of algebraic numbers, Acta Arith. **133** (2008), 221-250.

[11] A. Dubickas, Arithmetical properties of powers of algebraic numbers, Bull. London Math. Soc. **38** (2006), 70-80.

[12] A. Dubickas, On the distance from a rational power to the nearest integer, J. Number Theory. **117** (2006), 222-239.

[13] S. Ferenczi and C. Mauduit, Transcendence of numbers with a low complexity expansion, J. Number Theory **67** (1997), 146-161.

[14] H. Kaneko, Distribution of geometric sequences modulo 1, Result. Math. **52** (2008), 91-109.

[15] H. Kaneko, Limit points of fractional parts of geometric sequences, Unif. Distrib. Theory **5** (2009), 1-37.

[16] J. F. Koksma, Ein mengen-theoretischer Satz über Gleichverteilung modulo eins, Compositio Math. **2** (1935), 250-258.

[17] H. Locher, On the number of good approximations of algebraic numbers by algebraic numbers of bounded degree, Acta Arith. **89** (1999), 97-122.

[18] D. Ridout, Rational approximations to algebraic numbers, Mathematika **4** (1957), 125-131.

[19] T. Rivoal, On the bits counting function of real numbers, J. Aust. Math. Soc. **85** (2008), 95-111.

[20] M. Waldschmidt, Diophantine approximation on linear algebraic groups. Transcendence properties of the exponential function in several variables. Grundlehren der Mathematischen Wissenschaften 326. Springer-Verlag, Berlin, 2000.