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FINDING ALMOST SQUARES V

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Abstract
An almost square of type 2 is an integer n that can be factored in two different ways as $n = a_1 b_1 = a_2 b_2$ with a_1, a_2, b_1, $b_2 \approx \sqrt{n}$. In this paper, we continue the study of almost squares of type 2 in short intervals and improve the 1/2 upper bound. We also draw connections with almost squares of type 1.

1. Introduction and Main Results

An almost square (of type 1) is an integer n that can be factored as $n = ab$ with a, b close to $\sqrt{n}$. For example, $9999 = 99 \times 101$. We call an integer n an almost square of type 2 if it has two different such representations, $n = a_1 b_1 = a_2 b_2$ where a_1, b_1, a_2, b_2 are close to $\sqrt{n}$. For example $99990000 = 9999 \times 10000 = 9900 \times 10100$. Of course, this depends on what we mean by close. More precisely, for $0 \leq \theta \leq 1/2$ and $C > 0$,

Definition 1 An integer n is a (θ, C)-almost square of type 1 if $n = ab$ for some integers $a < b$ in the interval $[n^{1/2} - Cn^{\theta}, n^{1/2} + Cn^{\theta}]$.

Definition 2 An integer n is a (θ, C)-almost square of type 2 if $n = a_1 b_1 = a_2 b_2$ for some integers $a_1 < a_2 \leq b_2 < b_1$ in the interval $[n^{1/2} - Cn^{\theta}, n^{1/2} + Cn^{\theta}]$.

In a series of papers [1], [2], [3], [4], the author was interested in finding almost squares of either types in short intervals. In particular, given $0 \leq \theta \leq \frac{1}{2}$, we want to find "admissible" exponent $\phi_i \geq 0$ (as small as possible) such that, for some constants $C_{\theta,i}, D_{\theta,i} > 0$, the interval $[x - D_{\theta,i} x^{\phi_i}, x + D_{\theta,i} x^{\phi_i}]$ contains a $(\theta, C_{\theta,i})$-almost square of type i $(i = 1, 2)$ for all sufficiently large x. These lead to the following

Definition 3 We let $f(\theta) := \inf \phi_1$ and $g(\theta) := \inf \phi_2$, where the infima are taken over all the "admissible" ϕ_i $(i = 1, 2)$ respectively.

Clearly f and g are non-increasing functions of θ. It was conjectured (and partially verified) that

Conjecture 4 For $0 \le \theta \le \frac{1}{2}$,

$$f(\theta) = \begin{cases} \frac{1}{2}, & \text{if } 0 \le \theta < \frac{1}{4}, \\ \frac{1}{2} - \theta, & \text{if } \frac{1}{4} \le \theta \le \frac{1}{2}; \end{cases}$$

and

$$g(\theta) = \begin{cases} \text{does not exist}, & \text{if } 0 \le \theta < \frac{1}{4}, \\ 1 - 2\theta, & \text{if } \frac{1}{4} \le \theta \le \frac{1}{2}. \end{cases}$$

In [3], it was proved that

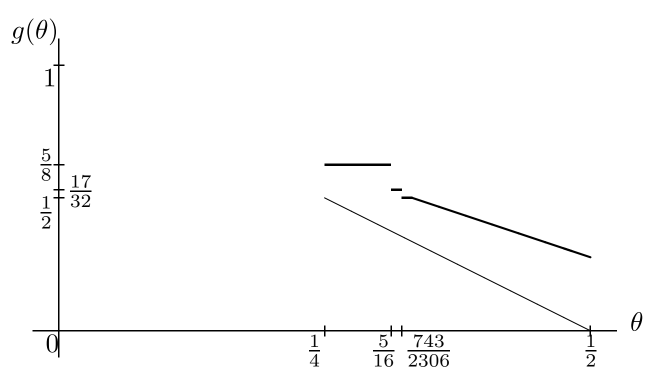
Theorem 5 *For* $\frac{1}{4} \le \theta \le \frac{1}{2}$,

$$g(\theta) \le \begin{cases} \frac{5}{8}, & \text{if } \frac{1}{4} \le \theta \le \frac{5}{16}, \\ \frac{17}{32}, & \text{if } \frac{5}{16} \le \theta \le \frac{743}{2306}, \\ \frac{1}{2}, & \text{if } \frac{743}{2306} < \theta \le \frac{1}{2}. \end{cases}$$

The purpose of this paper is to improve the $\frac{1}{2}$ upper bound for $g(\theta)$ in certain range of θ, namely

Theorem 6 *For* $\frac{1}{3} \le \theta \le \frac{1}{2}$, *we have* $g(\theta) \le 1 - \frac{3\theta}{2}$.

Combining the above two theorems, we have the following picture:



The thin downward sloping line is the conjectural lower bound $1 - 2\theta$ while the thick line segments above are the upper bounds from Theorems 5 and 6.

Furthermore, there are some connections between almost squares of type 1 and almost squares of type 2.

Theorem 7 *If Conjecture 4 is true for $f(\theta)$ when $\frac{1}{4} \le \theta \le \frac{1}{2}$, then*

$$g(\theta) \le \frac{3}{2} - 3\theta$$

for $\frac{1}{3} \le \theta \le \frac{1}{2}$.

Notation. Both $f(x) = O(g(x))$ and $f(x) \ll g(x)$ mean that $|f(x)| \le Cg(x)$ for some constant $C > 0$.

2. Unconditional Result: Theorem 6

Proof. We shall use the fact: for any real number $x \ge 1$, there exists a perfect square a^2 such that $a^2 \le x < (a+1)^2$. Hence $|x - a^2| \ll \sqrt{x}$.

Given $x \ge 1$ sufficiently large. The almost square of type 2 close to x we have in mind has the form

$$n = (G^2 - 1)(H^2 - h^2) = a_1 b_1 = a_2 b_2,$$

where $\{a_1, b_1\} = \{(G-1)(H-h), (G+1)(H+h)\}$ and $\{a_2, b_2\} = \{(G-1)(H+h), (G+1)(H-h)\}$.

Let $0 < \lambda < \frac{1}{4}$. We choose $G = [x^{1/4-\lambda}]$.

First, we approximate $\frac{x}{G^2-1}$ by H^2 where $H = [\sqrt{\frac{x}{G^2-1}} + 1]$. Then $0 < H^2 - \frac{x}{G^2-1} \ll H$. One can check that

$$GH = G\Big[\sqrt{\frac{x}{G^2-1}} + 1\Big] = x^{1/2}\Big(1 + O\Big(\frac{1}{G^2}\Big)\Big) + O(G) = x^{1/2} + O(x^{2\lambda}) + O(x^{1/4-\lambda}).$$

Next, we approximate $H^2 - \frac{x}{G^2-1}$ by h^2 for some $0 < h \ll H^{1/2} \ll x^{1/8+\lambda/2}$. We can get within a distance $H^2 - \frac{x}{G^2-1} - h^2 \ll H^{1/2} \ll x^{1/8+\lambda/2}$. Therefore

$$|x - (G^2-1)(H^2-h^2)| \le \Big|\frac{x}{G^2-1} - (H^2-h^2)\Big| G^2 \ll G^2 x^{1/8+\lambda/2} \ll x^{5/8-3\lambda/2}.$$

The number $n = (G^2-1)(H^2-h^2) = (G-1)(G+1)(H-h)(H+h)$. Notice that

$$\begin{aligned}(G-1)(H-h) &= GH - H - Gh + h \\ &= x^{1/2} + O(x^{2\lambda}) + O(x^{1/4-\lambda}) \\ &\quad + O(x^{1/4+\lambda}) + O(x^{3/8-\lambda/2}) + O(x^{1/8+\lambda/2}) \\ &= x^{1/2} + O(x^{1/4+\lambda}) + O(x^{3/8-\lambda/2}).\end{aligned}$$

One can check that the same asymptotic holds for $(G-1)(H+h)$, $(G+1)(H-h)$ and $(G+1)(H+h)$. When $\lambda \geq \frac{1}{12}$, the first error term dominates. Thus, we just have found a $(\frac{1}{4}+\lambda, C)$-almost square of type 2 within a distance $O(x^{5/8-3\lambda/2})$ from x for some $C > 0$.

Set $\theta = \frac{1}{4} + \lambda$. The condition $\frac{1}{12} \leq \lambda < \frac{1}{4}$ becomes $\frac{1}{3} \leq \theta < \frac{1}{2}$. Meanwhile $\frac{5}{8} - \frac{3\lambda}{2} = 1 - \frac{3\theta}{2}$. Therefore, for any $\frac{1}{3} \leq \theta < \frac{1}{2}$, there exists a (θ, C)-almost square of type 2 within a distance $O(x^{1-3\theta/2})$ from x. So $g(\theta) \leq 1 - \frac{3\theta}{2}$ when $\frac{1}{3} \leq \theta < \frac{1}{2}$. When $\lambda = \frac{1}{4}$ (i.e. $\theta = \frac{1}{2}$), one simply uses $G = 2$ and the above argument works in the same way to give $g(\frac{1}{2}) \leq \frac{1}{4}$. □

3. Connection to Almost Squares of Type 1: Theorem 7

Proof. In the previous section, we used an elementary method to approximate $\frac{x}{G^2-1}$ by $(H-h)(H+h)$, a $(\frac{1}{4}, C)$-almost square of type 1, since $h \ll H^{1/2} \ll (\frac{x}{G^2-1})^{1/4}$. So one should expect to do better using (ϕ, C)-almost square of type 1 for some $\frac{1}{4} \leq \phi \leq \frac{1}{2}$.

Again we choose $G = [x^{1/4-\lambda}]$ and let $H = \sqrt{\frac{x}{G^2-1}}$. By Conjecture 4 on $f(\theta)$, we can find a (ϕ, C)-almost square of type 1, say ab, such that $H - CH^{2\phi} \leq a < b \leq H + CH^{2\phi}$ and

$$\Big|\frac{x}{G^2-1} - ab\Big| \ll \Big(\frac{x}{G^2-1}\Big)^{1/2-\phi+\epsilon} \ll x^{1/4-\phi/2+\lambda-2\lambda\phi+\epsilon}$$

for x sufficiently large. Hence

$$|x - (G-1)(G+1)ab| \ll x^{3/4-\lambda-\phi/2-2\lambda\phi+\epsilon}. \tag{1}$$

Similar to the previous section, one has

$$\begin{aligned}(G-1)a = (G-1)(H + O(H^{2\phi})) &= GH - H + O(GH^{2\phi}) \\ &= x^{1/2} + O(x^{1/4+\lambda}) + O(x^{1/4-\lambda+\phi/2+2\lambda\phi}).\end{aligned}$$

The same is true for $(G-1)b$, $(G+1)a$ and $(G+1)b$. One can check that $\frac{1}{4}+\lambda \geq \frac{1}{4}-\lambda+\frac{\phi}{2}+2\lambda\phi$ if and only if $\lambda \geq \frac{\phi}{4-4\phi}$.

Let $\theta = \frac{1}{4}+\lambda$ and $\lambda = \frac{\phi}{4-4\phi}$. Then the exponent in (1) satisfies $\frac{3}{4}-\lambda-\frac{\phi}{2}-2\lambda\phi = 1-\theta(1+2\phi)$. Therefore, for any $\frac{1}{4}+\frac{\phi}{4-4\phi} \leq \theta \leq \frac{1}{2}$, there exists a (θ, C')-almost square of type 2 within a distance of $O(x^{1-\theta(1+2\phi)+\epsilon})$ from x for some $C' > 0$.

Given $\frac{1}{3} \leq \theta \leq \frac{1}{2}$, the bigger the ϕ, the better the above result. Since $\frac{\phi}{4-4\phi}$ is an increasing function of ϕ, the biggest ϕ we can use is when $\frac{1}{4}+\frac{\phi}{4-4\phi} = \theta$. This gives $\phi = 1-\frac{1}{4\theta} \leq \frac{1}{2}$ as $\theta \leq \frac{1}{2}$. Using this value of ϕ, we have a (θ, C')-almost square of type 2 within a distance of $O(x^{3/2-3\theta+\epsilon})$ from x. This proves Theorem 7 as ϵ can be arbitrarily small. □

Remark. The exponent $\frac{3}{2}-3\theta \to 0$ as $\theta \to \frac{1}{2}$. However $\frac{3}{2}-3\theta$ is always greater than the conjectural value $1-2\theta$ for $g(\theta)$ which is no surprise as part of the almost square has the special form G^2-1. It would be interesting to see how one could incorporate the extra degree of freedom, namely G^2-g^2 for some g, for further improvements.

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