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THE SHORTEST GAME OF CHINESE CHECKERS AND RELATED PROBLEMS

George I. Bell
Tech-X Corporation, 5621 Arapahoe Ave Suite A, Boulder, CO 80303, USA
gibell@comcast.net

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Abstract

In 1979, David Fabian found a complete game of two-person Chinese Checkers in 30 moves (15 by each player). This solution requires that the two players cooperate to generate a win as quickly as possible for one of them. We show, using computational search techniques, that no shorter game is possible. We also consider a solitaire version of Chinese Checkers where one player attempts to move her pieces across the board in as few moves as possible. In 1971, Octave Levenspiel found a solution in 27 moves; we demonstrate that no shorter solution exists. To show optimality, we employ a variant of A* search, as well as bidirectional search.

1. Introduction

The game of *Halma* was invented in the 1880's by George H. Monks [10, 13]. This game is played on a rather large 16×16 board and is still popular in parts of Europe. In 1892, a significant variation appeared in Germany played on a triangular grid, originally called *Stern-Halma* [13]. When this game was marketed in the United States it was given the more exotic-sounding name **Chinese Checkers**, although it did not originate in China and is not a variant of Checkers. Chinese Checkers remains a popular children's game in the United States.

Chinese Checkers is normally played using marbles on a 121-hole, star-shaped board. However, the two-player version of the game can equivalently be played on a square 9×9 board, with move directions as given in Figure 1a[1]. Although the board symmetry is harder to see, it allows us to consider Halma and Chinese Checkers as games played on the same board shape.

We will refer to a board location as a **cell**, with coordinates given in Figure 1a. We will also find it useful to use Cartesian Coordinates to refer to a cell, with the origin at the center of the board[2]. Each player begins with a certain number of

[1]This is possible because the two players move between two opposing "points" of the star. The 3 or 6-player versions of Chinese Checkers cannot be played on a square 9×9 board.

[2]The Halma board has no central hole, we just use the cell identified in Figure 1b as the origin.

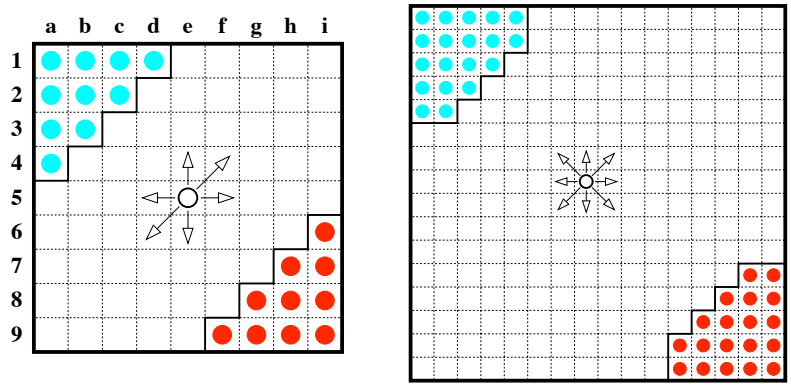

Figure 1: A Chinese Checkers board (left) and Halma board (right) with men in their starting positions. The central man shows the directions of allowed movement.

identical game pieces, called *men*[3]. The set of men owned by one player will be referred to as her *army*. The standard Chinese Checkers army has 10 men, and the standard Halma army has 19 men.

A *move* in either game is a step move or a jump move, as defined by:

1. A *step*, where a man is moved one cell in the direction indicated by the arrows in Figure 1.
2. A *chain* of one or more jumps by the same man. A *jump* is where a man hops over another man (of either army) into an empty cell. Jumps are allowed in any of the directions indicated by the arrows in Figure 1. The jumped piece is *not removed* from the board – there are no captures. Jumps are never compulsory – a player may choose to stop a chain whenever she pleases.

The area from where a player's army begins (outlined in Figure 1) will be called that player's *base*. Players alternate moves; the first player to fully occupy the opposing base is the winner. A complicating factor is how to deal with a player who refuses to vacate their own base, preventing the other player from winning. Various additional rules have been proposed to prevent this [13], but these will not concern us here.

In this square-board geometry the movement rules for Chinese Checkers and Halma are identical except for the directions of possible movement. Halma allows moves in all eight directions from a cell, along rows, columns, and both diagonals – we will refer to these movement rules as *8-move* rules. In Chinese Checkers, one parallel direction of diagonal movement is not allowed; these will be referred to as *6-move* rules. Although it may appear artificial to remove only one direction of diagonal jump, this rule variation is completely natural on the triangular grid on which Chinese Checkers is normally played.

[3]To compensate for this male-oriented terminology, all players will be female.

A final variation is to allow steps and jumps only along columns and rows; this variation will be called **4-move** rules. Figure 2a shows a standard game of Checkers (English Draughts) viewed with the board rotated 45°. We can remove the white squares (they are not used), and play Checkers on the 32-cell board in Figure 2b under 4-move rules. Although Checkers has more complex jumping rules (captures), this shows how the basic jumps and steps of the game can be viewed as moves restricted to columns and rows – that is, 4-move rules.

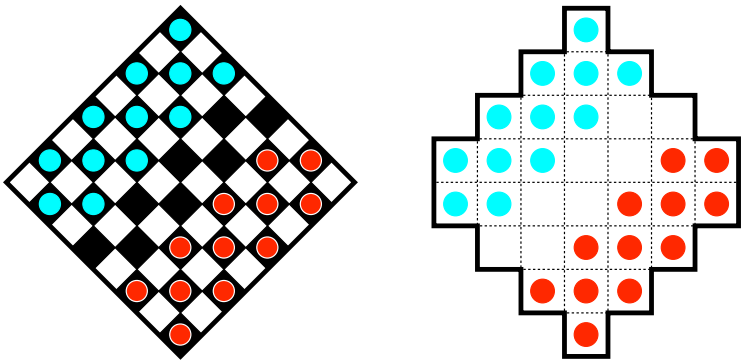

Figure 2: (a) A standard game of checkers. (b) The same game under 4-move rules.

Halma game play naturally divides into three distinct phases (using George Monks' original terminology [10]):

1. the *gambit*, where the armies advance toward one another but do not interact,
2. the *melee*, where the two armies interact and eventually pass through one another,
3. the *packing*, where the armies separate and attempt to fill the opposing bases as quickly as possible.

This paper does not address game play directly, but considers two puzzles based on Chinese Checkers and Halma. The first puzzle concerns the shortest possible game. In 1979, David Fabian, working by hand, found a complete game of Chinese Checkers in 30 moves [6, p. 309] (15 moves by each player). This is remarkably short considering that each of the 20 men (both armies) must make at least one move. Such a solution requires that both sides cooperate so that one of them wins as quickly as possible, and thus has little to do with a competitive game. We will show that no game can be shorter than 30 moves.

A second puzzle we consider is a solitaire version of the game, where the goal is to advance an army across the board in as few moves as possible (with no opponent's pieces in the way). We will refer to these puzzles as "army transfer problems." Such problems were a favorite of Martin Gardner – in three of his Scientific American columns he discusses army transfer problems on three different boards (first with

regard to a Checkers board (Figure 2) under 4-move rules [5], then in the context of Halma on a 9×9 board under 8-move rules [4], and finally in the context of Chinese Checkers under 6-move rules [6]). We will find the shortest possible solution to these three army transfer problems, and consider various generalizations.

2. The Shortest Possible Game

The game is considered ended when one player fully occupies her opponent's base, even if the other player does the same on her next move. Draws can't occur except in artificial situations where a player refuses to leave her base, or the players move pieces back and forth so that the game goes on forever. On some boards one player can be stalemated with no possible moves. This is not possible in standard Chinese Checkers or Halma (there are not enough men in one army to trap the opposing army in a corner), but can occur among cooperating players for other variants. We will not consider such stalemated games as candidates for the shortest possible game.

The length of a completed game is defined as the total number of moves taken by both players. We assume that the *blue* player, starting from the upper left in Figure 1, always moves first. The *red* player moves second; if she wins first on her 15th move, it is a 30-move game. If the blue player wins first on her 15th move, it is a 29-move game.

2.1 Types of Men

	a	**b**	**c**	**d**	**e**	**f**	**g**	**h**	**i**
1	0	1	0	1	0	1	0	1	0
2	2	3	2	3	2	3	2	3	2
3	0	1	0	1	0	1	0	1	0
4	2	3	2	3	2	3	2	3	2
5	0	1	0	1	0	1	0	1	0
6	2	3	2	3	2	3	2	3	2
7	0	1	0	1	0	1	0	1	0
8	2	3	2	3	2	3	2	3	2
9	0	1	0	1	0	1	0	1	0

Figure 3: Type labeling of a Chinese Checker board.

Clearly jumps are more effective than steps to move an army quickly across the board. However, a certain number of steps are generally needed. A skilled player selects a careful mixture of jumps and steps to advance her army across the board.

There is a fundamental difference between steps and jumps. In Figure 3, we checker the board with a pattern of four type labels 0–3. We label the men in an army by their type, and observe that *only step moves can change a man's type.* On

a Chinese Checker board, both bases have the same number of men of each type. Therefore, the winning player will have the same types of men at the end of the game that she started with. Interestingly, this is not true of Halma, where the type 0 and type 3 men must effectively change places during the game. The counts of the number of types of men for each game are given in Table 1.

	Chinese Checkers		Halma	
Type	# starting	# finishing	# starting	# finishing
0	3	3	6	3
1	3	3	5	5
2	3	3	5	5
3	1	1	3	6
total men	10	10	19	19

Table 1: The number of men of each type for the games of Chinese Checkers and Halma (from the blue player's perspective).

This suggests that it may be possible to play an entire game of Chinese Checkers without making any step moves, and we will see that it is possible. For Halma, however, the same argument shows that *at least three step moves are required* to win the game. The difference is due to the fact that the board side length in Halma is even (20), while for Chinese Checkers it is odd (9). In general, we will refer to these as *even* or *odd* boards, and this type of disparity will be seen on any even board. On an odd board, the number of starting and finishing types will be the same, as long as the starting army is symmetric about the diagonal line $x = -y$.

In Chinese Checkers, to advance a type 0 or 3 man (move it closer to the opponent's base) requires a jump over a type 1 or 2 man, or a step, which converts it to a type 1 or 2 man. Similarly, advancing a type 1 or 2 man requires a jump over or conversion to a type 0 or 3 man. In Halma, a man of a certain type can advance by jumping or converting into any of the other three types (because it is now possible to move diagonally between the bases).

2.2 Ladders

A *ladder* is any configuration of men that allows for quick transfer of men between the bases by means of long chain jump moves. Figure 4 shows several ladders on a Chinese Checkers board. These ladders are shown using the same color men to visually separate them, but in general they can be composed of either army.

The red (top) ladder in Figure 4 is composed of men of types 0 and 3, and can transport men of the opposite types 1 and 2. We refer to such a ladder as a "type 0&3 ladder." The blue (middle) ladder has these roles reversed, it is a type 1&2 ladder. The green (bottom) ladder is also a type 1&2 ladder, but can only transport

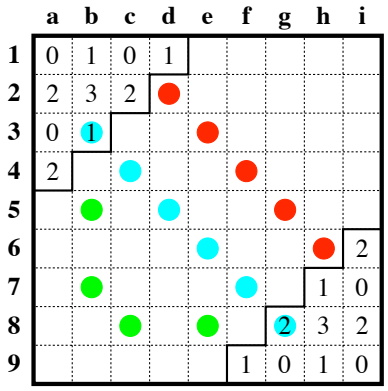

Figure 4: Three ladders on a Chinese Checkers board.

type 3 men. Because it can transport only one type of man, the green ladder is less useful. Any ladder can transport at most two of the four types of men, for the fastest transport of an army (with men of all types), *at least two ladders of different types will be needed.*

The game would be relatively easy if the ladders were in place at the beginning. In reality, ladders are not only used, they must be built and disassembled. Plus, in a competitive game ladders will seldom be separate or complete as in Figure 4, and one player can strategically "block" a ladder by stopping a man in the middle. But our focus now is on the shortest possible game, where the players must cooperate to build the most efficient ladders, and they need not be broken down at all if they are composed of the losing player's men.

2.3 A Lower Bound on Game Length

Given two cells with Cartesian coordinates $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$, what is the minimum number of step moves needed to move a man between the two? To answer this question, we define the norm of a cell with Cartesian coordinates (x, y) as

$$||(x,y)|| = \begin{cases} ||(x,y)||_\infty & = \max\{|x|,|y|\}, & \text{Halma or 8-move,} \\ ||(x,y)||_1 & = |x|+|y|, & \text{4-move,} \\ ||(x,y)||_\triangle & = \frac{1}{2}(|x|+|y|+|x-y|), & \text{Chinese Checker or 6-move.} \end{cases} \tag{1}$$

The norm $||(x,y)||_\triangle$ is a combination of the first two norms, as shown by the alternate formula

$$||(x,y)||_\triangle = \begin{cases} \max(|x|,|y|) & \text{if } \operatorname{sgn}(x) = \operatorname{sgn}(y), \\ |x|+|y| & \text{otherwise.} \end{cases} \tag{2}$$

The formula (2) was given in 1976 in the context of image processing to calculate distances on a hexagonal grid [8].

The distance between two board locations $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ is then defined as

$$d(\mathbf{a}, \mathbf{b}) = ||\mathbf{a} - \mathbf{b}|| = ||(a_x - b_x, a_y - b_y)||, \tag{3}$$

with norm appropriate for the game as defined in (1). Note that $d(\mathbf{a}, \mathbf{b})$ is the minimum number of steps needed to move a man from $\mathbf{a}$ to $\mathbf{b}$.

Given two armies B and R, we define the distance between them as:

$$d(B, R) = \min\{d(\mathbf{b}, \mathbf{r}) : \mathbf{b} \in B \text{ and } \mathbf{r} \in R\}. \tag{4}$$

If we let B and R be the initial positions of two armies at the beginning of a game, for Chinese Checkers we can compute that $d(B, R) = 10$. Interestingly, for Halma we also have $d(B, R) = 10$. Thus, despite the large difference in board size, the initial distance between the two armies is the same for Chinese Checkers and Halma.

Theorem 1. *If B and R are the initial positions of two equal-sized armies ($s = |B| = |R|$), no game can be shorter than h moves, where*

$$h = \max\{0, d(B, R) - 2\} + 2s - 1. \tag{5}$$

Proof: After one player moves, $d(B, R)$ can decrease by at most 1. As soon as $d(B, R) \leq 2$, the next move can cross between the two armies. The best that can happen is that $d(B, R)$ decreases to 2, the next player then wins by placing a man from her army into her opponent's base on *each* of her subsequent s moves. The number of moves is then given by (5). □

For a standard game of Chinese Checkers, the bound given by Theorem 1 is 27 moves, and for Halma 45 moves.

2.4 A 30 Move Game of Chinese Checkers

David Fabian found a 30 move Chinese Checkers game in 1979 and sent it to Martin Gardner [6, p. 309]. I contacted David Fabian about how he found his solution, he said he used "logic and patience" to find it by hand. We can come up with a set of properties that a short solution is likely to have. We can't prove that the shortest solution must have these properties, but they can guide us on our search for them, both by hand and using a computer.

From Section 2.1, if the winning player in Chinese Checkers makes one step, she will be obliged at some point to make another. Therefore, it seems reasonable that in the shortest possible game, the winning player only jumps. The losing player has no constraints on the types of finishing men, and is likely to make some steps.

If the game is to have length L, then there is a critical move, $\alpha = L - 2(|A| - 1)$, after which the winning player *must* have at least one man in her opponent's base. In Section 2.2 we learned that for a quick game two ladders must be built. The first ladder can be built by both players in the moves before α, while the second ladder must be finished by the losing player in the moves after α.

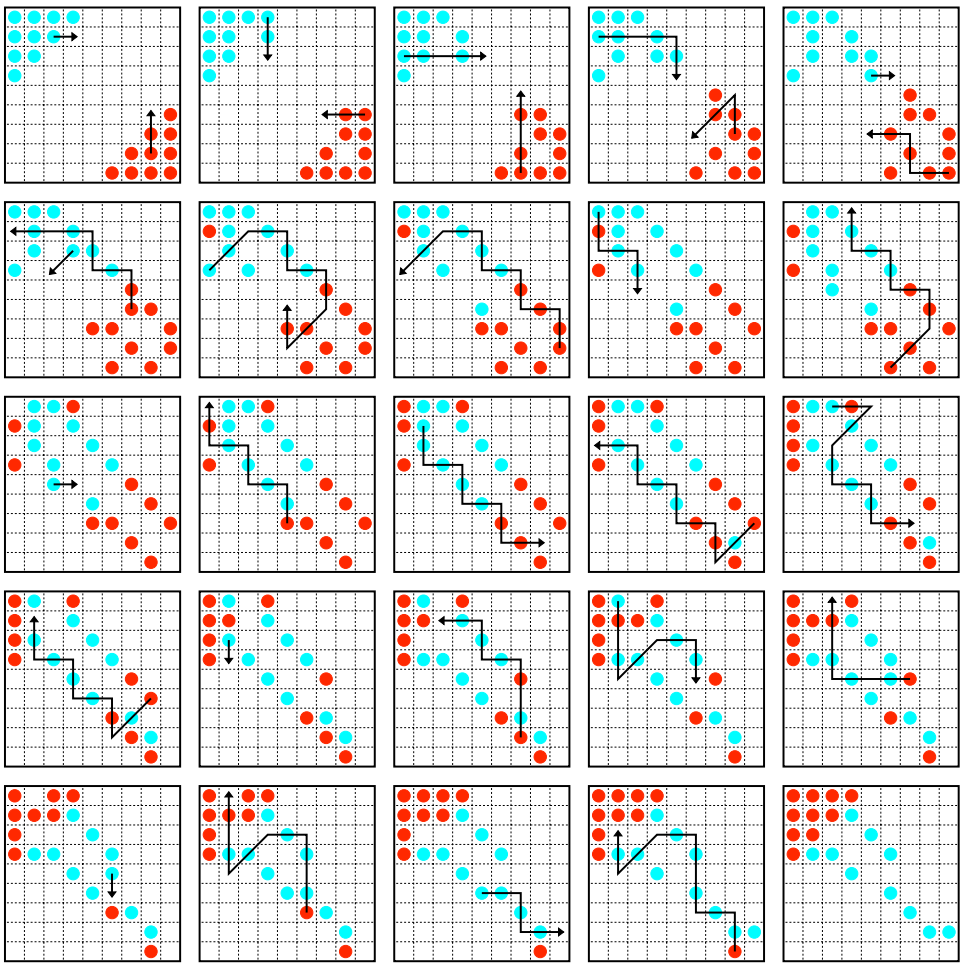

Figure 5: David Fabian's 30 move game of Chinese Checkers.

In summary, some of the properties that a short solution usually has are:

(i) The winning player only jumps (on odd boards).

(ii) The first ladder is built by both players during the first α moves.

(iii) The second ladder is completed by the losing player on moves after α.

(iv) After each move $n \geq \alpha$ by the winning player, there must be at least $(n–\alpha)/2+1$ men in the opposing base.

(v) In the middle of the game, each diagonal line $x-y=k$ between the bases must be occupied by at least one man, where $-4 \leq k \leq 4$ (this forces placement of two ladders).

Finally, Figure 5 shows David Fabian's 30-move solution, which has all these properties ($\alpha = 12$ in this case). We have modified the losing player's last move slightly, as there are many possibilities. In fact this freedom might suggest the existence of a 29-move solution, but we shall see that this is not the case.

2.5 Search Algorithms

One way to show that no 29-move solution exists is to do a search of the game tree. We perform a breadth-first search so that we can easily eliminate duplicate board positions. The search proceeds by **levels**, where the level set L_i consists of board positions that can be reached after i moves. The level set L_0 contains only the initial board position, while L_1 contains one element for each possible first move by the blue player. There are 14 possible first moves in Chinese Checkers, but 7 of these are equivalent by symmetry, so $|L_1| = 7$, $|L_2| = 7 \cdot 14 = 98$, and $|L_3| = 1253$. The total size of the state space for the game is $\binom{81}{10,10} \approx 8.67 \times 10^{23}$, and the level sets L_i grow much too rapidly to calculate L_{29}.

Fortunately a very good lower bound on the number of moves remaining exists, namely the bound given in Theorem 1. Although the bound in Theorem 1 assumes the board is in the initial position, it can be easily modified to apply to any board position. If the distance between the two armies $d(R, B) > 2$, we can use (5) unmodified. When this distance becomes two or less, we modify s in (5) to be the number of men in the winning army that are not in the opposing base. Given any board position P, we can use (5) as a lower bound on the number of moves remaining, $h(P)$.

To apply this in our search scheme we use a version of A* search called "breadth-first iterative deepening A*" [15]. Suppose we are searching for a solution with length at least m. At move i if the board position is P, no solution from this board position can be shorter than $i + h(P)$. Thus, we can terminate the search from this node if $i + h(P) > m$.

This gives a search tree that expands rapidly until a critical move α, the first move where the winning player must place at least one man in the opposing base. At this level the search tree contracts significantly. But the next move by the losing player is unconstrained so the search expands, only to contract on the next move by the winning player who again must place another man in the opposing base.

For Chinese Checkers, we know that a solution of length 30 exists. Therefore,we apply the search algorithm to look for all solutions of length 29 and 28. These searches come up empty, so the shortest solution has 30 moves. The solution in Figure 5 is not unique; the algorithm finds several hundred different 30-move games.

The same algorithm can find the shortest game for many other starting configurations. The search strategy in these cases proceeds follows: first, by incorporating the heuristic rules of Section 2.4, we find a solution of length N which we believe is the shortest possible. Then, we remove these heuristic rules and use only the bound on solution length (5) to show that no solution exists of length $m = N - 1$ and $m = N - 2$.

Game	Board	Move Rule	Army Size	$d(B, R)$	Lower Bound (5)	Shortest Game
Chinese Checkers	9×9	6-move	10	10	27	30
Chinese Checkers (15 man)	9×9	6-move	15	8	35	36
Halma	16×16	8-move	19	10	45	Unknown
Grasshopper	8×8	8-move	10	4	21	24
Chinese Checkers (8-move)	9×9	8-move	10	5	22	24

Table 2: Summary of shortest game lengths on various boards.

Table 2 shows the results of these runs. For a Chinese Checkers game with 15 men per side, David Fabian found a 38-move game [6, p. 309], but our algorithm finds that a 36-move game is the shortest possible. Grasshopper [4, p. 117] is Halma played on an 8×8 (even) board, with the initial configuration of 10 men the same shape as in Chinese Checkers. By the type analysis in Section 2.1, the winning player must make at least 2 step moves, and a game in 22 moves or less is impossible. In order to win in exactly 24 moves, the winning player must make 2 step moves and 10 jump moves, and each of the jump moves must finish either inside their opponents base or one cell short of it. For the same game on a 9×9 board (odd), the shortest game also has 24 moves, but the winning player only jumps.

3. Army Transfer Problems

We now consider the problem of moving one army quickly between bases, without any opposing men. On a Chinese Checkers board, this problem was discussed in a 1976 Martin Gardner column [6], although the earliest reference is a 1959 Canadian periodical on magic [7]. Octave Levenspiel worked by hand to find short transfers; in 1971 he found a 27-move solution [6, p. 73].

Figure 6 shows Levenspiel's 27-move solution. An interesting feature of this solution is that it is *palindromic*, meaning that moves 15–27 are mirror images of the moves 1–13, taken in the reverse order (where the mirror symmetry is about the line $x = y$), and move 14 is itself mirror symmetric. If the solution is interrupted in the middle of the 14th move (the last diagram of Figure 6), the board position is mirror symmetric about the line $x = y$. It is *not true* that every 27-move solution to this problem is palindromic, but palindromic solutions often seem to exist, and are the most elegant.

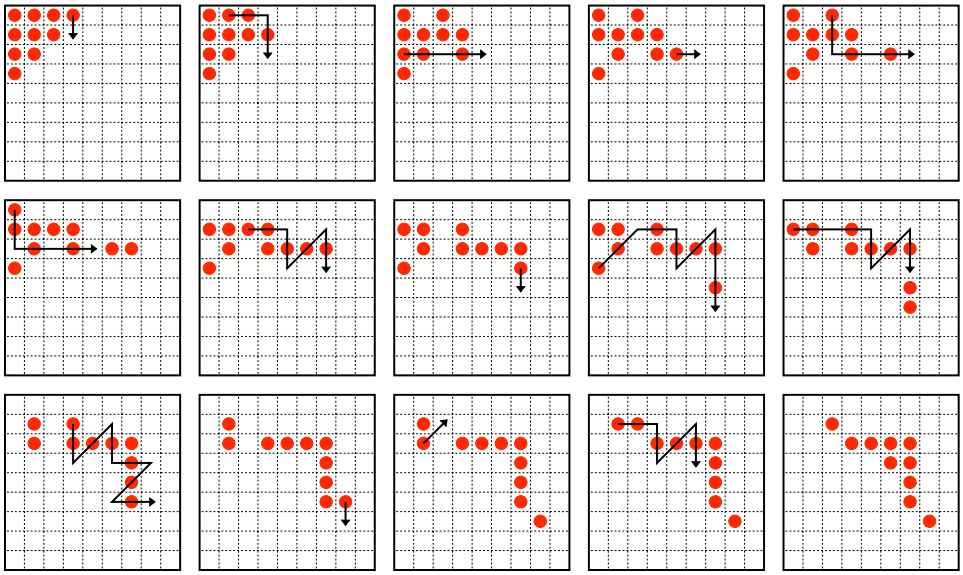

Figure 6: Octave Levenspiel's 27-move solution. Only the first half of the solution is shown, because the solution is a palindrome (only half of the 14th move is shown).

In 1973, Harry O. Davis sent Martin Gardner a proof that 27 moves was the shortest possible solution to this problem. Although this proof is mentioned in Gardner's book [6, p. 68], it was never published. This proof has been preserved in Martin Gardner's files [3]. Davis begins his one-page proof with an argument that the shortest solution to the problem must contain at least 10 step moves (note that Levenspiel's solution in Figure 6 contains 10 steps and 17 jump moves). However, we have found 27-move solutions to this problem with only 8 step moves (see Appendix A), so Davis' claim is false. Although the proof appears flawed, the theorem is nonetheless true—there is no solution shorter than 27 moves. This will be demonstrated numerically below.

3.1 Centroids and Bounds

The (diagonal) *centroid* of a man $\mathbf{a}$ with Cartesian coordinates (a_x, a_y) is defined as

$$c(\mathbf{a}) = a_x - a_y. \tag{6}$$

The centroid $c(\mathbf{a})$ is a measure of how far this man has progressed in his journey between the bases. In the blue player's army, the centroid of any man begins at -5 or less, and ends at $+5$ or greater.

The *centroid* of an army A is the integer-valued function defined by

$$c(A) = \sum_{\mathbf{a}\in A} c(\mathbf{a}) = \sum_{\mathbf{a}\in A} a_x - a_y. \tag{7}$$

This is related to the classical centroid, or center of gravity, in the following sense: the army will balance across the diagonal line $x - y = c(A)/|A|$, where $|A|$ is the size of army A.

This centroid gives a natural measure of an army's progress, and in competitive games which player is currently in the lead. In a game of Chinese Checkers, for example, where the origin is the center of the board, for the blue player $c(A)$ begins at -60 and ends at $+60$. Note that the starting army balances on the line $x-y = -6$. When $c(A) = 0$, the army is exactly half-way to their goal, and balances across the diagonal line $x = y$.

Theorem 2 *Consider an army A. Let $\mathbf{a}_{\min} \in A$ be a man with minimum centroid, i.e., $c(\mathbf{a}_{\min}) \leq c(\mathbf{a}), \forall \mathbf{a} \in A$. Similarly, let $\mathbf{a}_{\max} \in A$ be a man with maximum centroid. Then in one move, the centroid can increase at most δ, or decrease at most $-\delta$, where*

$$\delta = c(\mathbf{a}_{max}) - c(\mathbf{a}_{min}) + \ell, \tag{8}$$

where $\ell = 1$ for 4-move and 6-move play, and $\ell = 2$ for 8-move play.

Proof. Only one man can move, and the greatest centroid increase is achieved by taking a man $\mathbf{a}_{min}$ with minimum centroid and increasing his centroid as much as possible. The best $\mathbf{a}_{min}$ can do is finish with a jump over some man with centroid $c(\mathbf{a}_{max})$. Under 8-move play, if the last jump is diagonal, his final centroid can be be at most $c(\mathbf{a}_{max}) + 2$, otherwise it can be at most $c(\mathbf{a}_{max}) + 1$. □

Theorem 2 gives a relatively crude upper bound on centroid increase, which it is tempting to refine further. Under 4- and 6-move rules, if $c(\mathbf{a}_{max}) - c(\mathbf{a}_{min})$ is even, in order to reach $c(\mathbf{a}_{max}) + 1$, the man $\mathbf{a}_{min}$ must finish with a rightward or downward jump over a man $\mathbf{a}_{max}$, and this is not possible because this man has the wrong type. In this case the upper bound (8) can be reduced by one. More significantly, suppose the army can be partitioned into two pieces A_1 and A_2 with $\mathbf{a}_{min} \in A_1$, $\mathbf{a}_{max} \in A_2$ and $d(A_1, A_2) > 2$. Then the man $\mathbf{a}_{min}$ cannot reach the other half of the army, and the bound (8) can be reduced considerably. All such refinements result in a more complex formula for centroid increase.

We will also want to use Theorem 2 iteratively to obtain an upper bound on the centroid increase of an army after n moves. In this situation, the complexity of an improved bound increases greatly because we will have to consider the interaction of multiple moves. For example, in the case where the army can be partitioned into A_1 and A_2 with $d(A_1, A_2) = 3$, the first move could connect the two pieces, and then the second move could go between them. The crude bound in Theorem 2 is much easier to implement iteratively, because it is valid no matter how the moves interact.

We can also use the ideas in Theorem 2 to get a lower bound on the number of moves to accomplish the transfer. We assume that the initial army is A, and the location of the target base is B (a fixed set of cells). As the army A advances, the

distance $d(A, B)$ can decrease by at most one per move. Thus, the smallest number of moves to place a man in the target base B is $d(A, B)$. After this, the remaining $|A|-1$ men move into the base B at best one per move, so any solution to the transfer problem has length at least $d(A, B) + |A| - 1$. For the standard Chinese Checkers army, this gives a lower bound of 19 moves, not a very tight bound considering the minimum is 27 (as we shall see). Although these crude estimates don't get very close to the true optimum, we will see that Theorem 2 is useful during a numerical search.

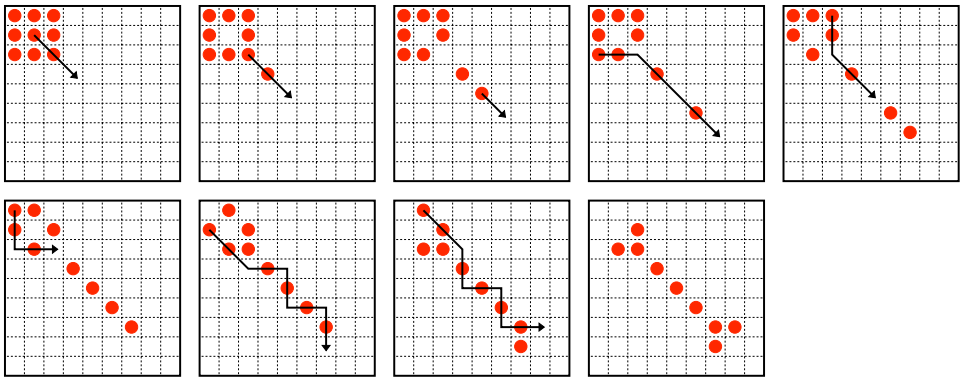

Figure 7: An elegant 16-move solution to a Halma transfer problem (H. Ajisawa and T. Maruyama [4, p. 118]). Again, only the first half of the solution is shown.

Figure 7 shows a 16-move solution to a Halma transfer problem [4, p.118]. The starting and ending armies in this case are 3×3 square arrays of men. Diagonal steps and jumps are allowed, in our terminology we have 8-move rules. The solution has even length and is palindromic, the board position after 8 moves is mirror symmetric about the line $x = y$. Although our simple lower bound indicates that any solution must be at least 12 moves, we will be able to show using the search algorithm in the next section that the 16-move solution in Figure 7 is the shortest possible.

3.2 Search Algorithms

As before our basic computational tool is breadth-first search, with the level sets L_i defined as before, except that only one player moves. Since the centroid must increase by 120 (from -60 to $+60$), clearly the most productive moves are those that increase the centroid. To cut down on the size of the level sets, it would seem we should *only* consider moves that increase the centroid. If we do so, however, there is no guarantee that we will find the shortest possible solution. Surprisingly, solutions of shortest possible length may contain moves that *decrease the centroid*. Examples of this counter-intuitive phenomenon will be found below.

There are several symmetries in this problem. The first is that the starting army is symmetric about the diagonal line $x = -y$. This effectively halves the number

of possible starting moves, and decreases the search space by a factor of two. The starting and finishing positions are also symmetric about the diagonal line $x = y$. Thus, the set of possible board positions one move before the finish is the same as the set of possible board positions one move from the start, reflected across the line $x = y$. This suggests that a good search technique is **bidirectional search** [11]. To search for a solution of length $2N$, we do a breadth-first search to N moves, and intersect the level set L_N with the set obtained by reflecting each element of L_N across the diagonal line $x = y$.

Suppose we have a solution of length $2N+1$, and want to prove that no solution of length $2N$ exists. Suppose we also know that the maximum possible centroid after *any* sequence of N moves is C_N^{max} (this is the maximum centroid of any board position in the level set L_N). For a solution of length $2N$ to exist, it must be that $C_N^{max} \geq 0$. The key observation is that *any solution* of length $2N$ must have centroid *at least* $-C_N^{max}$ at level N.

At any level $i \leq N$, we apply Theorem 2 iteratively to get an upper bound on the centroid after $N - i$ additional moves. If this upper bound is less than $-C_N^{max}$, then this board position cannot lead to a solution and we need not search further from this board position. This search combines aspects of bidirectional and A* search techniques, and will be called a minimum centroid constraint (MCC) search, because it eliminates boards with centroid too small to lead to a solution.

If you try to implement this MCC search, you will notice a problem. We assumed that we knew C_N^{max} at the start of the algorithm, yet this number is not determined until the algorithm is finished! In reality we must estimate C_N^{max}, then run the search algorithm using this estimate. If the search algorithm produces a board at level N with centroid greater than C_N^{max}, we must run it all over again with the correct C_N^{max}. It is only after the search finishes with a self-consistent value of C_N^{max}, and the bidirectional search comes up empty, that we are assured that no solution of length $2N$ exists.

One way to estimate C_N^{max} is to truncate the search at each level, keeping only the top M boards with the largest centroid. We have found that an M of a few million gives a good estimate of C_N^{max}. Short solutions can also be found quickly using this truncation technique. Bill Butler solved the Chinese Checker transfer problem (Figure 6) this way, as documented on his web site [2]. He found five different 27-move solutions, but because he truncated the number of boards at each level, his search is not exhaustive.

For the standard Chinese Checker transfer problem, we find that $C_{13}^{max} = 5$. We should note that although the MCC algorithm makes the search problem solvable, it is still not easy. The largest level set L_{10} contains 1.3×10^7 boards, and the complete (unsuccessful) search for a 26-move solution takes over two hours of CPU time[4].

Table 3 gives the results of such a search over a wide variety of problems, with solutions given in Appendix A. Except as noted, the MCC search runs to completion in

[4]On a 1 GHz PC with 512MB of RAM.

Problem	Configuration	# Men	Board	Shortest solution under 4-move	6-move	8-move
# 1	Triangle	6	9×9	25	23	16
# 2	Triangle	10	9×9	30	27 (Fig 6)	20
# 3	Triangle (jumps only)	10	9×9	46	35 (Fig 8)	21
# 4	Triangle	15	9×9	36†	31† (Fig 9)	26†
# 5	Square	4	9×9	15	15	12
# 6	Square	9	9×9	25	25	16 (Fig 7)
# 7	Checkers start	12	Fig 2b	20	16	16
† - MCC search did not terminate, a shorter solution may be possible.						

Table 3: Summary of shortest solution lengths for transfer problems; all are the shortest possible except as noted.

each case, so the number of moves given is the smallest possible. In particular, Octave Levenspiel's 27-move solution in Figure 6 is the shortest possible, and 16 moves is the shortest solution to the Halma transfer problem in Figure 7. Problem #7 was suggested by Gardner [5], and refers to moving a 12-man army across a checkers board. This is equivalent to moving one of the 12-man armies in Figure 2b to the opposite side under 4-move rules. The shortest solution to this problem has 20 moves.

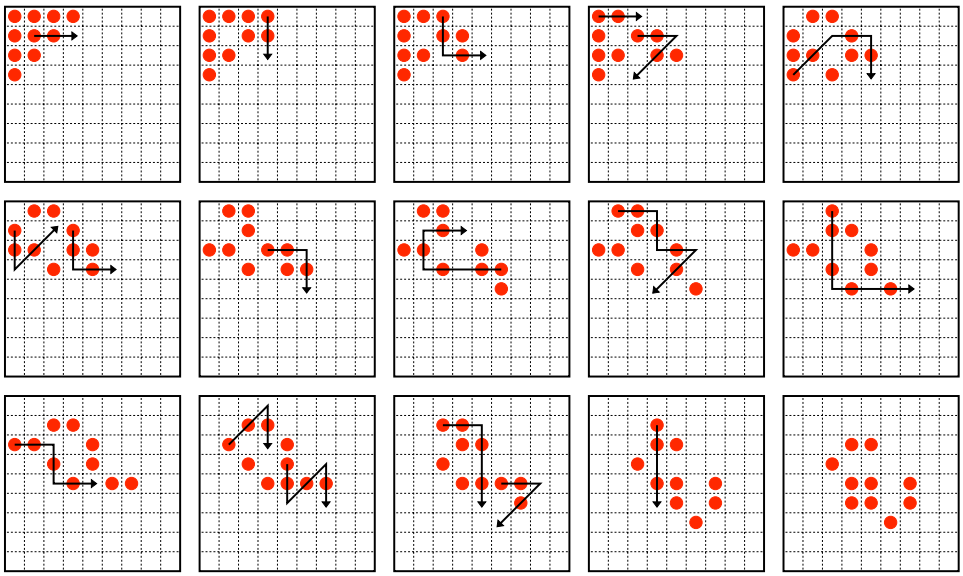

Figure 8: The first half of a 35-move solution with jumps only (the shortest possible). Note that the 10th move (middle diagram, second row) goes backwards!

If we allow only jump moves, how quickly can a 10-man triangular army be transferred? Table 3 shows that the answer (under 6-move rules) is 35 moves, with a sam-

ple solution shown in Figure 8. One interesting feature of this solution is a *backward* jump on the 10th move which reduces the army's centroid $c(A)$. This backward jump move can be explained by the fact that the man making it is of type 3. There is only one type 3 man in this army, and in order to utilize him he must be moved around a lot, even jumping backwards. This phenomenon is also observed in the 4-move version of the problem.

Appendix A includes solutions to problems in Table 3 with backward step moves. These exceptions indicate that we cannot eliminate backward jump and step moves when searching for the shortest possible solution.

One interesting problem is the 15-man Chinese Checkers army (problem #4 in Table 3). In 1974, Min-Wen Du of Taiwan sent Martin Gardner a letter with a solution to this problem in 35 moves[5] [9]. My search algorithm has found a solution to this problem in 31 moves (see Figure 9), but an MCC search doesn't finish, so it is not known if this is the shortest solution possible. This 31-move solution is interesting because it uses two ladders rather than one.

3.3 Symmetry and Palindromes

In many cases the shortest solution can be chosen to be a palindrome, and these solutions are arguably the most elegant. We can add further constraints to search specifically for palindromic solutions.

First, we define the level of symmetry of an army A. Given a man $\mathbf{a}$ with Cartesian coordinates (a_x, a_y), the coordinates of the cell reflected across the line $x = y$ are (a_y, a_x). We define the function

$$\mathrm{sym}(\mathbf{a}) = \begin{cases} +1, & \text{if } (a_y, a_x) \text{ is occupied,} \\ 0, & \text{otherwise.} \end{cases} \tag{9}$$

We then define the *army symmetry* of an army A as

$$\mathrm{sym}(A) = \sum_{\mathbf{a} \in A} \mathrm{sym}(\mathbf{a}), \tag{10}$$

so $\mathrm{sym}(A)$ varies from zero to $|A|$ depending on the symmetry of the army across the line $x = y$, and $\mathrm{sym}(A) = |A|$ if and only if the army is mirror symmetric across this line.

Starting with an army A_0, denote by A_i the board position after i moves. For a palindromic solution of odd length $2N + 1$ to exist, it must be that $\mathrm{sym}(A_N) = \mathrm{sym}(A_{N+1}) = |A| - 1$. For a palindromic solution of even length $2N$ to exist, we must have $\mathrm{sym}(A_N) = |A|$. These two situations can be seen in the second to the last diagram of Figures 6 and 8, and the final diagram of Figure 7.

[5]Note that [6, p. 68] describes this problem incorrectly. The letter [9] removes any ambiguity.

The symmetry of the starting army, $\mathrm{sym}(A_0)$, equals 0, but by the middle of a palindromic solution of length $2N$ or $2N+1$, $\mathrm{sym}(A_N) = T$, where $T = |A|$ for an even solution length and $T = |A| - 1$ for an odd solution length. But the function sym() cannot increase from 0 to T in one move. Since a single move only affects one man, $\mathrm{sym}(\cdot)$ can increase or decrease by at most two (the only terms that can increase in the sum (10) are the endpoint of the move and its mirror image). So for any $i \le N$, the board position A_i in a palindromic solution must satisfy

$$\mathrm{sym}(A_i) \ge T - 2(N - i). \tag{11}$$

Now consider the Chinese Checker transfer problem for the 15-man army (Problem #4 in Table 3 under 6-move rules). After $N = 15$ moves, for a palindromic solution of length $2N = 30$ to exist, we must have $\mathrm{sym}(A_{15}) = 15$. The symmetry constraint (11) is therefore

$$\mathrm{sym}(A_i) \ge 15 - 2(15 - i) = 2i - 15 \text{ for } 8 \le i \le 15. \tag{12}$$

The symmetry constraint (12) can be added to the MCC algorithm, and it speeds up the search after level 7, and we can run the search to level 15. The resulting bidirectional search finds that there does not exist a *palindromic* 30-move solution. It remains a possibility that a non-palindromic 30-move solution exists. The same search technique finds two palindromic solutions of length 31 (Figure 9).

In general, the symmetry constraint (11) can be added to our search algorithm, and it can dramatically increase the search speed. Once again, though, this search technique contains a "chicken-in-egg paradox", because in order to run the search, we must already know the solution's length. For a general problem, our search strategy is first to find a solution of any length, then try to find a shorter solution, and finally to prove that no shorter solution exists.

To find a solution of any length, we can truncate the level sets at each step, as explained in the previous Section 3.2. This technique can be viewed as a truncation based on the centroid "score" $c(A)$, but it does not place any extra value on palindromic solutions. If the truncated nth level set is R_n, then to find a solution of length $2N$ the set R_N must satisfy the conditions:

$$\max\{c(A) : A \in R_N\} \ge 0, \tag{13}$$

$$\min\{c(A) : A \in R_N\} \le 0. \tag{14}$$

Since the truncation technique keeps board positions with largest centroid, the first condition will eventually be satisfied as N increases. However, for large problems, the second condition may not be satisfied (because of the truncation), and then no solution will ever be found, no matter how large N is. For problems starting from 15-man armies, we instead truncate the level sets based on the modified score

$$c(A) + \beta\,\mathrm{sym}(A),$$

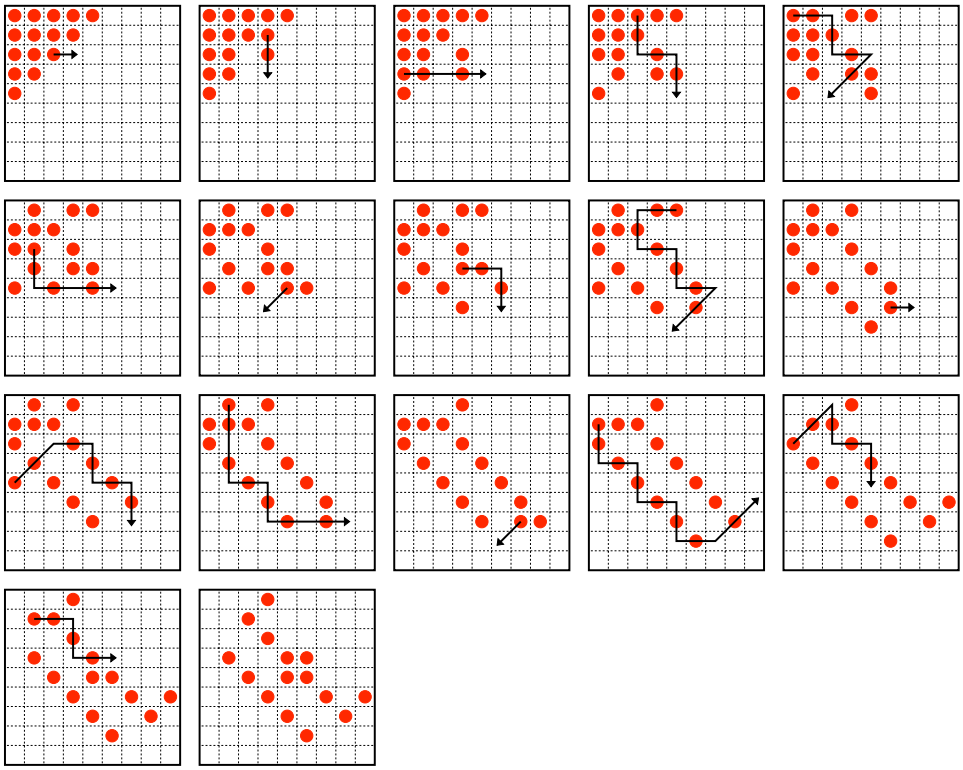

Figure 9: The first half of a 31-move solution starting with 15 men. The only other palindromic 31-move solution is obtained by extending the 15th move to end at c7.

where $\beta \geq 0$ is an arbitrary weight factor for symmetry. This modified score weights palindromic solutions more heavily. We have had good results using $\beta = 2$.

The palindromic solutions in Figures 6–9 are symmetric about the line $x = y$. Note that the Checkers board of Figure 2 is not symmetric about the line $x = y$; a palindromic solution on this board must instead be symmetric with respect to 180° rotation. Martin Gardner gives a 20-move palindromic solution on this board [5, p. 217], and our algorithm can find all palindromic solutions, using a version of $\mathrm{sym}(A)$ corresponding to a rotation rather than a reflection.

3.4 Fast Armies and Balanced Armies

Given an army A, we define the center of mass $\mathbf{m}(A) \in \mathbb{R}^2$ as the average coordinate over the army,

$$\mathbf{m}(A) = \frac{1}{|A|} \sum_{\mathbf{a} \in A} (a_x, a_y). \tag{15}$$

The diagonal centroid of Section 3.1 is related to $\mathbf{m}(A) = (m_x, m_y)$ by $c(A) = |A|(m_x - m_y)$. If an army A moves to B in n moves, we define its average **speed** as the distance traveled by the center of mass per move,

$$\sigma = \frac{d(\mathbf{m}(A), \mathbf{m}(B))}{n} = \frac{||\mathbf{m}(A) - \mathbf{m}(B)||}{n}, \tag{16}$$

using the norm (1) specific to 4-, 6-, or 8-move rules. Note that the same army may have a different speed measured under different rules. We call the army B a **translate** of A if B can be obtained by shifting each man in A by the same (2D) vector $\mathbf{k}$, $B = \{\mathbf{a} + \mathbf{k} : \mathbf{a} \in A\}$.

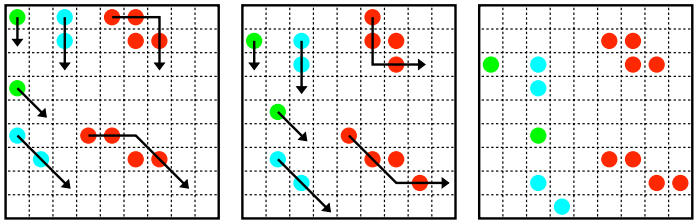

Figure 10: Speed 1 armies under 4- and 8-move rules: "atom" (1 man, green), "frog" (2 men, blue), "serpent" (4 men, red).

Figure 10 shows fast armies of size 1, 2, or 4. Under 4-move rules, the three armies in the top row each have speed 1. Auslander et al. [1] prove that if A moves to B under 4-move rules, and B is a translate of A, then the speed σ cannot exceed 1, and the only speed 1 armies are the three in the top row of Figure 10. The bottom row of Figure 10 shows armies under 8-move rules, all having speed 1. Under 8-move rules, all six armies in Figure 10 have speed 1, except for the top serpent which has speed 1/2.

We say that an army is **balanced** if the distribution of men over types is as uniform as possible. The smallest example of a perfectly balanced army are the serpents in Figure 10, which contain one man of each type. The balance of an army can only be changed by a step move. One reason why a balanced army may be fast is that the number of possible jump moves is large. For example, among all 4-man armies under 4-move rules, the largest number of possible jump moves is 8, and this can only be achieved by taking one man of each type, as in the serpent, or 4 men in a square configuration.

In the previous sections we have seen fast armies of larger size. The speed in each case can be computed and is given in Appendix A. One interesting aspect of the armies in Figures 6, 7 and 9 is that they are usually balanced. In the Chinese Checker problem of Figure 6, the army is initially unbalanced, because there are $(3, 3, 3, 1)$ men of types 0–3. The first move (d1-d2) converts a man of type 1 to type 3, resulting in the balanced army $(3, 2, 3, 2)$.

Finally, we note that the most common opening among experienced players of competitive Chinese Checkers is the balance restoring step d1-d2 [14]. Is this a coincidence? In the future, it may be productive to look into the concept of army balance in competitive Chinese Checkers games. Halma begins with the unbalanced army $(6, 5, 5, 3)$; restoring balance would suggest a diagonal step ending at the type 3 cells d4, f2, b6 or f6 as one of the opening moves.

4. Summary

Chinese Checkers and Halma seem to have a reputation as slow-moving games. Alternate (but more complex) jumping rules have even been devised in an attempt to speed them up [12]. However, increasing complexity does not necessarily improve a game. We have shown than an entire game of Chinese Checkers can take as few as 30 moves. While such short solutions rely on the two players cooperating, a competitive game can also move along more quickly than many people think.

We have demonstrated, using computational search, that 30 moves is the shortest possible Chinese Checkers game. We have also studied a solitaire version of the game where the goal is to transport a single army across the board as quickly as possible. These solutions often have palindromic symmetry, and pass through a board position which is mirror symmetric across the line $x = y$. We have obtained a bound on how quickly the centroid can increase, and applied it numerically to show that no standard Chinese Checkers army can cross the board in under 27 moves.

We have also studied the army transfer problem in a more general context, considering various army configurations under several different jumping rules, with results summarized in Table 3. Can we always find a shortest solution that is palindromic? The answer is probably no, but we have not seen a counterexample.

The problem of the fastest way to transport a 19-man Halma army across a 16×16 board is difficult to answer computationally. Nonetheless, this is an interesting problem to work on by hand, and we have found a solution in 47 moves, given in Appendix A. Can the reader find a shorter solution? The centroid arguments of Section 3.1 give a lower bound of only 28 moves, so there would appear to be ample room for improvement.

Finally, we have introduced the concept of a balanced army, which means to keep (as much as possible) the same number of men of each type. The fastest armies usually seem to be balanced, and this is an interesting area for further study.

Appendix A. Solutions

A.1 Shortest games

Chinese Checkers in 30 moves (Figure 5), 10 man armies, 6-move rules (by David Fabian): c2-d2, h8-h6, d1-d3, i6-g6, a3-c3-e3, g9-g7-g5, a2-c2-e2-e4, h7-h5-f7, e4-f4, i9-g9-

g7-e7, d3-c4, g6-g4-e4-e2-c2-a2, a4-c2-e2-e4-g4-g6-e8-e6, i8-i6-g6-g4-e4-e2-c2-a4, a1-a3-c3-c5, f9-h7-h5-f5-f3-d3-d1, c5-d5, e7-e5-c5-c3-a3-a1, b2-b4-d4-d6-f6-f8-h8, i7-g9-g7-e7-e5-c5-c3-a3, c1-e1-c3-c5-e5-e7-g7, h6-f8-f6-d6-d4-b4-b2, b3-b4, g8-g6-g4-e4-e2-c2, b1-b3-b5-d3-f3-f5, g5-e5-c5-c3-c1, f5-f6, f7-f5-f3-d3-b5-b3-b1, e6-g6-g8-i8, h9-h7-f7-f5-f3-d3-b5-b3 (red wins).

Chinese Checkers in 36 moves, 15 man armies, 6-move rules: e1-e2, g8-g6, c1-e1-e3, h6-f6, e3-e4, f9-f7-f5, a1-c1-e1-e3-e5-g5-e7, g7-g5-e5-e3-e1-c1-a1, a5-b5, i7-g7-g5-e5-e3-e1-c1, c3-a5-c5, g9-i7-g7-g5-e5-e3-e1-c3-a5, a3-c3-e1-e3-e5-g5-g7-i7-g9, i9-i7-g7-g5-e5-e3-e1-c3-a3, a4-c4-c6, e9-g7-g5-e5-e3-e1-c3, c6-d6, i5-i7-g7-g5-e5-e3-e1, a2-a4-c4-c6-e6-g4, i8-g8-e8-e6-c6-c4-a4-a2, d2-f2, i6-g8-e8-e6-c6-c4-a4, b3-d3-f1-f3-d5-d7-f7-h5, f5-f7-d7-d5-f3-f1-d3-b3, d1-f1-d3, h8-h6-h4-f4-d4-d2, b1-d1-f1-f3-d5-d7-f7-f5, h9-f9-f7-d7-d5-f3-f1-d1-b1, b4-b6-d4-f4-h4-h6-h8, h7-h9-f9-f7-d7-d5-f3-f1-d1, c2-c4-c6-e6-e8, g6-e6-c6-c4-c2, b2-b4-b6-d4-f4-h4-h6, f6-f4-d4-b6-b4-b2, d3-f1-f3-d5-d7-f7-f9-h9, f8-d8-f6-f4-d4-b6-b4 (red wins).

A.2 Short Solutions to Army Transfer Problems

Problem numbers refer to those in Table 3. All solutions given are palindromic, so only half the solution is given, followed by "(reflect)". This indicates to repeat the moves in reverse order, reflected about the line $x = y$. All solutions are the shortest possible, except as noted.

Problem #2 (Figure 6), 6-move rules in 27 moves (Octave Levenspiel), $C_{13}^{max} = 5$, $\sigma = 4/9$: d1-d2, b1-d1-d3, a3-c3-e3, e3-f3, c1-c3-e3-g3, a1-a3-c3-e3, c2-e2-e4-g2-g4, g4-g5, a4-c2-e2-e4-g2-g4-g6, a2-c2-e2-e4-g2-g4, d2-d4-f2-f4-h4-f6-h6, h6-h7, b3-c2, b2-d2-d4-f2-f4- (reflect). An alternate solution with only 8 step moves: c2-d2, a4-c2-e2, c1-e1-e3, e3-e4, a1-c1-e1-e3-e5, a3-c1-e1-e3, d1-d3-f3-d5-f5, f5-g5, a2-c2, b3-d1-d3-f3-d5-f5-h5, b1-b3-d1-d3-f3-d5-f5, d2-f2-d4-f4-f6-h4-h6, h6-h7, b2-d2-f2-d4-f4- (reflect).

Problem #2, 8-move rules in 20 moves, $C_{10}^{max} = 12$, $\sigma = 3/10$: a2-c4, c4-d4, a4-a2-c4-e4, b1-d3-f5, f5-f6, a3-c3-e5-g7, d1-b1-d3-f5-f7-h7, g7-h8, a1-c3-e5-g7-i7, b3-b1-d3-f5, (reflect).

Problem #2, 4-move rules in 30 moves, $C_{14}^{max} = 5$, $C_{15}^{max} = 11$, $\sigma = 2/5$: a4-b4, a2-a4-c4, b4-d4, d4-d5, c1-c3-c5-e5, e5-e6, b2-b4-d4-d6-f6, f6-f7, a3-c3-c5-e5-e7-g7, g7-h7, d1-c1[6], c1-c3-c5-e5-e7-g7-i7, i7-i8, b3-c3, c3-c5-e5-e7-g7-i7-i9, (reflect).

Problem #6 (Figure 7), 8-move rules in 16 moves, $C_{8}^{max} = 10$, $\sigma = 3/8$: b2-d4, c3-e5, e5-f6, a3-c3-e5-g7, c1-c3-e5, a1-a3-c3, a2-c4-e4-e6-g6-g8, b1-d3-d5-f5-f7-h7, (reflect).

Problem #3 (Figure 8), 6-move rules, jumps only in 35 moves, $C_{17}^{max} = 8$, $\sigma = 12/35$: b2-d2, d1-d3, c1-c3-e3, a1-c1, c2-e2-c4, a4-c2-e2-e4, d2-d4-f4, a2-a4-c2, d3-f3-f5, f4-d4-b4-b2-d2, b1-d1-d3-f3-d5, c1-c3-c5-e5-g5, a3-c3-c5-e5, e4-e6-g4-g6, b3-d1-d3, c2-e2-e4-e6, f5-h5-f7, d2-d4-d6- (reflect).

Problem #4 (Figure 9), 6-move rules in 31 moves, possibly not shortest, $\sigma = 32/93$: c3-d3, d2-d4, a4-c4-e4, c1-c3-e3-e5, a1-c1-c3-e3-c5, b3-b5-d5-f5, e5-d6, d4-f4-f6, e1-c1-c3-

[6]Note this backward step move.

e3-e5-g5-e7, f6-g6, a5-c3-e3-e5-g5-g7, b1-b3-b5-d5-d7-f7-h7, g7-f8, a2-a4-c4-c6-e6-e8-g8-i6, a3-c1-c3-e3-e5, b2-d2-d4-f4- (reflect).

A 19-man Halma army crosses the board in 47 moves, probably not shortest, $\sigma = 225/893$: d2-d4, c3-e5, e5-f6, a1-c3-e5-g7, g7-h8, c1-c3-e5-g7-i9, i9-j10, e1-c1-c3-e5-g7-i9, a3-c3-e5-g7, a5-c3-e5, b3-d5-f5-f7-h7-h9-j9-j11, j11-k11, b5-b3-d5-f5-f7-h7-h9-j9-j11-l11, l11-l12, d1-b3-d5-f5-f7-h7-h9-j9-j11-l11-l13, l13-m13, c4-e4-e6-g6-g8-i8-i10-k10-k12-m12-m14, m14-n14, e2-c4-e4-e6-g6-g8-i8-i10-k10-k12-m12-m14-o14, d3-d5-f5-f7-h7-h9-j9-j11-l11-l13-n13-n15, a4-c4-e4-e6-g6-g8-i8-i10-k10-k12-m12-m14-o16, b2-b3, b4-c3, b1-d3-d5-f5-f7-h7-h9- (reflect).

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