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SYMMETRIC NUMERICAL SEMIGROUPS GENERATED BY FIBONACCI AND LUCAS TRIPLES

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Abstract

The symmetric numerical semigroups $\mathsf{S}\,(F_a, F_b, F_c)$ and $\mathsf{S}\,(L_k, L_m, L_n)$ generated by three distinct Fibonacci (F_a, F_b, F_c) and Lucas (L_k, L_m, L_n) numbers are considered. Based on divisibility properties of the Fibonacci and Lucas numbers we establish necessary and sufficient conditions for both semigroups to be symmetric and calculate their Hilbert generating series, Frobenius numbers, and genera.

1. Introduction

Recently, the numerical semigroups $\mathsf{S}\,(F_i, F_{i+2}, F_{i+k})$, $i, k \geq 3$, generated by three Fibonacci numbers F_j were discussed in [8]. It turns out that the remarkable properties of F_j in these triples suffice to calculate the Frobenius number $\mathcal{F}\,(\mathsf{S})$ and genus $G\,(\mathsf{S})$ of the semigroup. In this article we show that properties of the Fibonacci and Lucas numbers is sufficient not only to calculate the specific parameters of semigroups, but also to describe completely the structure of the symmetric numerical semigroups $\mathsf{S}\,(F_a, F_b, F_c)$, $a, b, c > 2$, and $\mathsf{S}\,(L_k, L_m, L_n)$, $k, m, n > 1$, generated by distinct Fibonacci[1] and Lucas numbers, respectively. Based on divisibility properties of these numbers we establish necessary and sufficient conditions for both semigroups to be symmetric and calculate their Hilbert generating series, Frobenius numbers, and genera.

2. Basic Properties of Numerical Semigroups Generated by Three Elements

We recall basic definitions and known facts about numerical semigroups generated by three integers. Let $\mathsf{S}\,(d_1, d_2, d_3) \subset \mathbb{Z}^+ \cup \{0\}$ be the additive numerical semigroup finitely generated by a minimal set of distinct positive integers $\{d_1, d_2, d_3\}$ such that $d_1, d_2, d_3 > 2$, $\gcd(d_1, d_2, d_3) = 1$. The semigroup $\mathsf{S}(d_1, d_2, d_3)$ is said to be be generated by the minimal set of three distinct natural numbers d_i irrespective of their

[1] We avoid to use the term "*Fibonacci semigroup*" because it has been already reserved for another algebraic structure [10].

natural ordering as integers if there are no nonnegative integers $b_{i,j}$ for which the following dependence holds:

$$d_i = \sum_{j \neq i}^{m} b_{i,j} d_j \ , \quad b_{i,j} \in \{0, 1, \ldots\} \quad \text{for any} \ \ i \leq m \ . \tag{1}$$

For short we denote the vector (d_1, d_2, d_3) by $\mathbf{d}^3$. Following Johnson [6], we define *the minimal relation* $\mathcal{R}_3$ for given $\mathbf{d}^3$ as follows:

$$\mathcal{R}_3 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \ \mathcal{R}_3 = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}, \left\{ \begin{array}{l} \gcd(a_{11}, a_{12}, a_{13}) = 1 \\ \gcd(a_{21}, a_{22}, a_{23}) = 1, \\ \gcd(a_{31}, a_{32}, a_{33}) = 1 \end{array} \right. \tag{2}$$

where

$$\begin{array}{rcl} a_{11} & = & \min\left\{v_{11} \mid v_{11} \geq 2, \ v_{11} d_1 = v_{12} d_2 + v_{13} d_3, \ v_{12}, v_{13} \in \mathbb{Z}^+ \cup \{0\}\right\} \ , \\ a_{22} & = & \min\left\{v_{22} \mid v_{22} \geq 2, \ v_{22} d_2 = v_{21} d_1 + v_{23} d_3, \ v_{21}, v_{23} \in \mathbb{Z}^+ \cup \{0\}\right\} \ , \\ a_{33} & = & \min\left\{v_{33} \mid v_{33} \geq 2, \ v_{33} d_3 = v_{31} d_1 + v_{32} d_2, \ v_{31}, v_{32} \in \mathbb{Z}^+ \cup \{0\}\right\} \ . \end{array} \tag{3}$$

The uniquely defined values of $v_{ij}, i \neq j$, that give a_{ii} will be denoted by $a_{ij}, i \neq j$. Note that due to the minimality of the set $\{d_1, d_2, d_3\}$, the elements $a_{ij}, i, j \leq 3$ satisfy

$$\begin{array}{lll} a_{11} = a_{21} + a_{31}, & a_{22} = a_{12} + a_{32}, & a_{33} = a_{13} + a_{23} \ , \\ d_1 = a_{22} a_{33} - a_{23} a_{32}, & d_2 = a_{11} a_{33} - a_{13} a_{31}, & d_3 = a_{11} a_{22} - a_{12} a_{21}. \end{array} \tag{4}$$

The smallest integer $C\left(\mathbf{d}^3\right)$ such that all integers $s, \ s \geq C\left(\mathbf{d}^3\right)$, belong to $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *the conductor* of $\mathsf{S}\left(\mathbf{d}^3\right)$;

$$C\left(\mathbf{d}^3\right) := \min\left\{s \in \mathsf{S}\left(\mathbf{d}^3\right) \ \mid s + \mathbb{Z}^+ \cup \{0\} \subset \mathsf{S}\left(\mathbf{d}^3\right)\right\} \ .$$

The number $\mathcal{F}\left(\mathbf{d}^3\right) = C\left(\mathbf{d}^3\right) - 1$ is referred to as *the Frobenius number*. Denote by $\Delta\left(\mathbf{d}^3\right)$ the complement of $\mathsf{S}\left(\mathbf{d}^3\right)$ in $\mathbb{Z}^+ \cup \{0\}$, i.e., $\Delta\left(\mathbf{d}^3\right) = \mathbb{Z}^+ \cup \{0\} \setminus \mathsf{S}\left(\mathbf{d}^3\right)$. The cardinality (#) of the set $\Delta\left(\mathbf{d}^3\right)$ is called *the number of gaps*, $G\left(\mathbf{d}^3\right) := \#\left\{\Delta\left(\mathbf{d}^3\right)\right\}$, or *genus* of $\mathsf{S}\left(\mathbf{d}^3\right)$.

The semigroup ring $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right] = \mathsf{k}\left[z^{d_1}, z^{d_2}, z^{d_3}\right]$ over a field k of characteristic 0 associated with the semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ is a graded subring of the commutative polynomial ring $\mathsf{k}\left[z\right]$. The Hilbert series $H(\mathbf{d}^3; z)$ of $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ is defined [12] by

$$H(\mathbf{d}^3; z) = \sum_{s \ \in \ \mathsf{S}(\mathbf{d}^3)} z^s = \frac{Q(\mathbf{d}^3; z)}{(1 - z^{d_1})\,(1 - z^{d_2})\,(1 - z^{d_3})} \ , \tag{5}$$

where $Q(\mathbf{d}^3; z)$ is a polynomial in z.

The semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *symmetric* if and only if for any integer s we have

$$s \in \mathsf{S}\left(\mathbf{d}^3\right) \quad \Longleftrightarrow \quad \mathcal{F}\left(\mathbf{d}^3\right) - s \notin \mathsf{S}\left(\mathbf{d}^3\right)\ . \tag{6}$$

Otherwise $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *non–symmetric*. The integers $G\left(\mathbf{d}^3\right)$ and $C\left(\mathbf{d}^3\right)$ are related [5] as follows:

$$\begin{aligned} & 2G\left(\mathbf{d}^3\right) = C\left(\mathbf{d}^3\right) \quad \text{if } \mathsf{S}\left(\mathbf{d}^3\right) \text{ is a symmetric semigroup} \\ \text{and} \quad & 2G\left(\mathbf{d}^3\right) > C\left(\mathbf{d}^3\right) \quad \text{otherwise.} \end{aligned} \tag{7}$$

Notice that $\mathsf{S}\left(\mathbf{d}^2\right)$ is always a symmetric semigroup [1]. The number of independent entries a_{ij} in (2) can be reduced if $\mathsf{S}\left(\mathbf{d}^3\right)$ is symmetric: at least one off-diagonal element of $\widehat{\mathcal{R}}_3$ vanishes, e.g., $a_{13} = 0$ and therefore $a_{11}d_1 = a_{12}d_2$. Due to the minimality of the last relation we have, by (2), the following equalities and consequently the matrix representation as well [4] (see also [3], Section 6.2):

$$\begin{aligned} & a_{11} = a_{21} = \mathsf{lcm}(d_1, d_2)/d_1, \\ & a_{12} = a_{22} = \mathsf{lcm}(d_1, d_2)/d_2, \\ & a_{33} = d_1/a_{22} = d_2/a_{11}, \\ & a_{23} = 0, \end{aligned} \qquad \widehat{\mathcal{R}}_{3s} = \begin{pmatrix} a_{11} & -a_{22} & 0 \\ -a_{11} & a_{22} & 0 \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix}, \tag{8}$$

where (the subscript) "s" stands for symmetric semigroup. Combining (8) with the formula for the Frobenius number of a symmetric semigroup (see [4]) $\mathcal{F}\left(\mathbf{d}_s^3\right) = a_{22}d_2 + a_{33}d_3 - \sum_{i=1}^{3} d_i$, we have

$$\mathcal{F}\left(\mathbf{d}_s^3\right) = e_1 + e_2 - \sum_{i=1}^{3} d_i\ , \quad e_1 = \mathsf{lcm}(d_1, d_2)\ , \quad e_2 = d_3\ \mathsf{gcd}(d_1, d_2)\ . \tag{9}$$

If $\mathsf{S}\left(\mathbf{d}^3\right)$ is a symmetric semigroup then $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ is a complete intersection [4] and the numerator $Q(\mathbf{d}^3; z)$ in the Hilbert series (5) reads (see [12])

$$Q(\mathbf{d}^3; z) = (1 - z^{e_1})(1 - z^{e_2})\ . \tag{10}$$

2.1. Structure of Generating Triples of Symmetric Numerical Semigroups

The following two statements, Theorem 1 and Corollary 3, give necessary and sufficient conditions for $\mathsf{S}\left(\mathbf{d}^3\right)$ to be symmetric.

Theorem 1. ([4] and [14, Proposition 3]) *If a semigroup $\mathsf{S}\left(d_1, d_2, d_3\right)$ is symmetric then there exist $i \neq j$ such that the minimal generating set of $\mathsf{S}\left(d_1, d_2, d_3\right)$ has the following presentation:*

$$\gcd(d_i, d_j) = \lambda > 1, \ \ \gcd(d_k, \lambda) = 1, \ \ d_k \in \mathsf{S}\left(\frac{d_i}{\lambda}, \frac{d_j}{\lambda}\right), \tag{11}$$

where $k \neq i, j$.

It turns out that (11) also gives sufficient conditions for $\mathsf{S}(\mathbf{d}^3)$ to be symmetric. This follows by Corollary 3 of the old lemma of Watanabe [14] for a semigroup $\mathsf{S}(\mathbf{d}^m)$.

Lemma 2. ([14, Lemma 1]) *Let $\mathsf{S}(d_1,\ldots,d_m)$ be a numerical semigroup and let a and b be positive integers such that: (i) $c \in \mathsf{S}(d_1,\ldots,d_m)$ and $c \neq d_i$, (ii) $\gcd(c,\lambda)=1$. Then the semigroup $\mathsf{S}(\lambda d_1,\ldots,\lambda d_m,c)$ is symmetric if and only if $\mathsf{S}(d_1,\ldots,d_m)$ is symmetric.*

Combining Lemma 2 with the fact that every semigroup $\mathsf{S}(\mathbf{d}^2)$ is symmetric we arrive at the following corollary.

Corollary 3. *Let $\mathsf{S}(d_1,d_2)$ be a numerical semigroup, and let c and λ be positive integers with $\gcd(c,\lambda)=1$. If $c \in \mathsf{S}(d_1,d_2)$, then the semigroup $\mathsf{S}(\lambda d_1,\lambda d_2,c)$ is symmetric.*

In Corollary 3 the requirement $c \neq d_1, d_2$ can be omitted since both semigroups $\mathsf{S}(\lambda d_1,\lambda d_2,d_1)$ and $\mathsf{S}(\lambda d_1,\lambda d_2,d_2)$ are generated by two elements $(d_1,\lambda d_2)$ and are also symmetric.

We finish this section with an important proposition adapted to the numerical semigroups generated by three elements.

Theorem 4. ([5, Proposition 1.14]) *The numerical semigroup $\mathsf{S}(3,d_2,d_3)$, with $\gcd(3,d_2,d_3)=1$, $3 \nmid d_2$ and $d_3 \notin \mathsf{S}(3,d_2)$, is never symmetric.*

3. Divisibility of Fibonacci and Lucas Numbers

We recall remarkable divisibility properties of Fibonacci and Lucas numbers which are necessary for further consideration. Theorem 5 dates back to E. Lucas [7, Section 11, p. 206].

Theorem 5. *Let F_m and F_n, $m>n$, be the Fibonacci numbers. Then*

$$\gcd(F_m,F_n) = F_{\gcd(m,n)}\ . \tag{12}$$

As for Theorem 6, its weak version was given by Carmichael [2][2]. We present here its modern form proved by Ribenboim [11] and McDaniel [9].

Theorem 6. *Let L_m and L_n be the Lucas numbers, and let $m=2^a m'$, $n=2^b n'$, where m' and n' are odd positive integers and $a,b \geq 0$. Then*

$$\gcd(L_m,L_n) = \begin{cases} L_{\gcd(m,n)} & \text{if} \quad a=b\ , \\ 2 & \text{if} \quad a\neq b\ , \quad 3 \mid \gcd(m,n) \\ 1 & \text{if} \quad a \neq b\ , \quad 3 \nmid \gcd(m,n)\ . \end{cases} \tag{13}$$

[2]Carmichael [2] (Theorem 7, p. 40) proved only the most difficult part of Theorem 6, namely, the first equality in (13).

We also recall another basic divisibility property of Lucas numbers:

$$L_m = 0 \pmod 2 , \quad \text{if and only if} \quad m = 0 \pmod 3 . \tag{14}$$

We will need a technical corollary, which follows from Theorem 6.

Corollary 7. *Let L_m and L_n be the Lucas numbers, and let $m = 2^a m'$, $n = 2^b n'$, where m' and n' are odd positive integers and $a, b \ge 0$. Then*

$$\gcd(L_m, L_n) = 1 , \quad \textit{if and only if} \quad \begin{cases} a = b = 0 , & \gcd(m', n') = 1 , \\ a \neq b , & \gcd(3, \gcd(m, n)) = 1 . \end{cases} \tag{15}$$

4. Symmetric Numerical Semigroups Generated by Fibonacci Triples

In this section we consider symmetric numerical semigroups generated by three distinct Fibonacci numbers F_a, F_b and F_c, $a, b, c > 2$, irrespective of their natural ordering as integers. The two first values 3 and 4 of index i in F_i are of special interest because of the Fibonacci numbers $F_3 = 2$ and $F_4 = 3$. First, the semigroup $\mathsf{S}(F_3, F_b, F_c)$, $\gcd(2, F_b, F_c) = 1$, is always symmetric and actually has only two generators. Next, according to Theorem 4 the semigroup $\mathsf{S}(F_4, F_b, F_c)$ is symmetric if and only if at least one of two requirements, $3 \nmid F_b$ and $F_c \notin \mathsf{S}(3, F_b)$, is broken. Avoiding those trivial cases we state:

Theorem 8. *Let F_a, F_b and F_c be Fibonacci numbers where $a, b, c > 4$. Then a numerical semigroup $\mathsf{S}(F_a, F_b, F_c)$ is symmetric if and only if there exist two indices a and b such that*

$$\lambda = \gcd(a, b) \ge 3 , \quad \gcd(\lambda, c) = 1, 2 , \quad F_c \in \mathsf{S}\left(\frac{F_a}{F_\lambda}, \frac{F_b}{F_\lambda}\right) . \tag{16}$$

Proof. By Theorem 1 and Corollary 3, a numerical semigroup $\mathsf{S}(F_a, F_b, F_c)$ is symmetric if and only if there exist F_a and F_b such that

$$g = \gcd(F_a, F_b) > 1 , \quad \gcd(g, F_c) = 1 , \quad F_c \in \mathsf{S}\left(\frac{F_a}{g}, \frac{F_b}{g}\right) . \tag{17}$$

By consequence of Theorem 5 and the definition of Fibonacci numbers we get

$$\begin{cases} g = F_\lambda > 1 & \rightarrow \quad \gcd(a, b) \ge 3 , \\ \gcd(F_\lambda, F_c) = F_{\gcd(\lambda, c)} = 1 & \rightarrow \quad \gcd(\lambda, c) = 1, 2 . \end{cases} \tag{18}$$

The last containment in (17) gives

$$F_c = A\frac{F_a}{g} + B\frac{F_b}{g} = A\frac{F_a}{F_\lambda} + B\frac{F_b}{F_\lambda} , \quad A, B \in \mathbb{Z}^+ ,$$

which finishes the proof. $\square$

By (9), (10) and (16) we get

Corollary 9. *Let F_a, F_b and F_c be Fibonacci numbers and let the numerical semigroup $\mathsf{S}(F_a, F_b, F_c)$ satisfy (16). Then its Hilbert series and Frobenius number are given by*

$$\begin{aligned} H(F_a, F_b, F_c) &= \frac{(1 - z^{f_1})(1 - z^{f_2})}{(1 - z^{F_a})(1 - z^{F_b})(1 - z^{F_c})} \\ \mathcal{F}(F_a, F_b, F_c) &= f_1 + f_2 - (F_a + F_b + F_c) . \end{aligned} \tag{19}$$

where

$$f_1 = \frac{F_a F_b}{F_{\gcd(a,b)}}, \quad f_2 = F_c \cdot F_{\gcd(a,b)}.$$

The next corollary gives only the sufficient condition for $\mathsf{S}(F_a, F_b, F_c)$ to be symmetric and is weaker than Theorem 8. However, instead of containment (16) it gives an inequality which is easy to check.

Corollary 10. *Let F_a, F_b and F_c be the Fibonacci numbers where $a, b, c > 4$. Then a numerical semigroup $\mathsf{S}(F_a, F_b, F_c)$ is symmetric if there exist two indices a and b such that*

$$\lambda = \gcd(a, b) \geq 3 , \quad \gcd(\lambda, c) = 1, 2 , \quad F_c F_\lambda > \mathsf{lcm}(F_a, F_b) - F_a - F_b . \tag{20}$$

The Hilbert series and Frobenius number are given by (20).

Proof. The first two relations in (20) are taken from Theorem 8 and were proven in (18). We also must use (16). For this purpose, take F_c exceeding the Frobenius number of the semigroup generated by the two numbers F_a/F_λ and F_b/F_λ. This number $\mathcal{F}(F_a/F_\lambda, F_b/F_\lambda)$ is classically known (due to Sylvester [13]). So, we get

$$F_c > \frac{F_a}{F_\lambda}\frac{F_b}{F_\lambda} - \frac{F_a}{F_\lambda} - \frac{F_b}{F_\lambda} = \frac{\mathsf{lcm}(F_a, F_b) - F_a - F_b}{F_\lambda} ,$$

where the Hilbert series $H(F_a, F_b, F_c)$ and Frobenius number $\mathcal{F}(F_a, F_b, F_c)$ are given by (20). Thus, the corollary is proven. □

We finish this section with Example 11 where a Fibonacci triple does satisfy the containment in (16) but does not satisfy inequality in (20). We calculate the elements of the numerical semigroup, its Hilbert generating series, the Frobenius number and genus.

Example 11. $\{d_1, d_2, d_3\} = \{F_6 = 8, F_8 = 21, F_9 = 34\}$

$$\gcd(F_6, F_9) = F_3 , \quad \gcd(F_3, F_8) = 1 , \quad F_8 \in \mathsf{S}\left(\frac{F_6}{F_3}, \frac{F_9}{F_3}\right) = \mathsf{S}(4, 17) ,$$

$$
\begin{aligned}
&f_1 = \mathsf{lcm}(F_6, F_9) = 136 \ , \quad f_2 = F_8 \cdot F_3 = 42 \\
&F_8 \cdot F_3 < \mathsf{lcm}(F_6, F_9) - F_6 - F_9 \ , \\
&H\left(F_6, F_8, F_9\right) = \frac{(1-z^{136})(1-z^{42})}{(1-z^8)\,(1-z^{21})\,(1-z^{34})} \ , \quad \mathcal{F}\left(F_6, F_8, F_9\right) = 115 \ , \\
&G\left(F_6, F_8, F_9\right) = 58 \ , \\
&\mathsf{S}\left(F_6, F_8, F_9\right) = \{0, 8, 16, 21, 24, 29, 32, 34, 37, 40, 42, 45, 48, 50, 53, 55, 56, 58, 61, \\
&\qquad 63, 64, 66, 68, 69, 71, 72, 74, 76, 77, 79, 80, 82, 84, 85, 87, 88, 89, \\
&\qquad 90, 92, 93, 95, 96, 97, 98, 100, 101, 102, 103, 104, 105, 106, 108, \\
&\qquad 109, 110, 111, 112, 113, 114, 116, \rightarrow\} \ ,
\end{aligned}
$$

where '$\rightarrow$' indicates that all integers greater than 116 are in $\mathsf{S}\left(F_6, F_8, F_9\right)$.

5. Symmetric Numerical Semigroups Generated by Lucas Triples

In this section we consider symmetric numerical semigroups generated by three distinct Lucas numbers L_k, L_m, and L_n, $k, m, n > 1$, irrespective of their natural ordering as integers. Note that the case $k = 2$ is trivial by Theorem 4 since $L_2 = 3$. The semigroup $\mathsf{S}\left(L_2, L_m, L_n\right)$ is symmetric if and only if at least one of two requirements, $3 \nmid L_m$ and $L_n \notin \mathsf{S}\left(3, L_m\right)$, is broken.

Theorem 12. *Let L_k, L_m, and L_n, $k, m, n > 2$, be Lucas numbers and let*

$$
\begin{aligned}
&m = 2^a m', \ \ n = 2^b n', \ \ k = 2^c k', \\
&\qquad \textit{where} \ \ m' = n' = k' = 1 \pmod 2, \ a, b, c \geq 0 \ , \\
&l = \gcd(m, n) = 2^d l' \ , \\
&\qquad \textit{where} \ \ l' = \gcd(m', n') = 1 \pmod 2 \ , \quad d = \min\{a, b\} \ .
\end{aligned}
\tag{21}
$$

Then a numerical semigroup generated by these numbers is symmetric if and only if L_k, L_m, and L_n satisfy

$$
L_k \in \mathsf{S}\left(\frac{L_m}{L_l}, \frac{L_n}{L_l}\right) \ , \quad \textit{if} \ a = b \ , \quad \textit{or} \quad L_k \in \mathsf{S}\left(\frac{L_m}{2}, \frac{L_n}{2}\right) \ , \quad \textit{if} \ a \neq b \ , \tag{22}
$$

and one of three following relations:

$$
\begin{array}{lll}
1) & a = b \neq 0 \ , & a = b \neq c \ , \quad 3 \nmid \gcd(k, l) \ , \\
2) & a = b = 0 \ , \ \ \gcd\left(m', n'\right) > 1 \ , & \left\{ \begin{array}{l} c = 0 \ , \quad \gcd\left(k', l'\right) = 1 \ , \\ c \neq 0 \ , \quad 3 \nmid \gcd(k, l) \ , \end{array} \right. \\
3) & a \neq b \ , \quad 3 \mid \gcd(m, n) \ , & 3 \nmid k \ .
\end{array}
\tag{23}
$$

Proof. By Theorem 1 and Corollary 3, a numerical semigroup $\mathsf{S}\left(L_k, L_m, L_n\right)$ is symmetric if and only if there exist two elements of its minimal generating set such that

$$\eta = \gcd(L_n, L_m) > 1 \ , \ \ \gcd(L_k, \eta) = 1 \ , \ \ L_k \in \mathsf{S}\left(\frac{L_n}{\eta}, \frac{L_m}{\eta}\right) \ . \tag{24}$$

Represent n and m as in (21) and substitute them into the first relation in (24). As a consequence of Theorem 6, it holds if and only if

$$1) \ \ a = b \ , \ \ \gcd(m, n) > 1, \qquad \text{or} \qquad 2) \ \ a \neq b \ , \ \ 3 \mid \gcd(m, n) \ . \tag{25}$$

First, assume that the first requirement in (25) holds. This results in $\eta = L_l$, by Theorem 6. For the second requirement in (24), apply Corollary 7. Here we consider two cases $a = b \neq 0$ and $a = b = 0$:

$$a = b \neq 0 \ , \quad a = b \neq c \ , \quad 3 \nmid \gcd(k, l) = 1 \ , \tag{26}$$

$$a = b = 0 \ , \quad \gcd\left(m', n'\right) > 1 \ , \quad \left\{ \begin{array}{ll} c = 0 \ , & \gcd\left(k', l'\right) = 1 \ , \\ c \neq 0 \ , & 3 \nmid \gcd(k, l) \ . \end{array} \right. \tag{27}$$

Now, assume that the second requirement in (25) holds that results by Theorem 6 in $\eta = 2$. Making use of the second requirement in (24) and applying (14) we get,

$$a \neq b \ , \ \ 3 \mid \gcd(m, n) \ , \quad 3 \nmid k \ . \tag{28}$$

Combining (26), (27), and (28) we arrive at (23). The last requirement in (24), together with Theorem 6, gives

$$L_k = A\frac{L_m}{\eta} + B\frac{L_n}{\eta} = \left\{ \begin{array}{lll} A \cdot L_m/L_l + B \cdot L_n/L_l & \text{if} & a = b \\ A \cdot L_m/2 + B \cdot L_n/2 & \text{if} & a \neq b \end{array} \right. \ , \quad A, B \in \mathbb{Z}^+ \ ,$$

that proves (22) and finishes the proof of this theorem. □

As a consequence of Theorem 12, the following corollary holds for the most simple Lucas triples and follows by applying Theorem 12 for $a = b = c = 0$; see (27).

Corollary 13. *Let $L_{k'}$, $L_{m'}$ and $L_{n'}$ be the Lucas numbers with odd indices such that*

$$\gcd(m', n') > 1 \ , \quad \gcd(m', n', k') = 1 \ . \tag{29}$$

Then a numerical semigroup generated by these numbers is symmetric if and only if

$$L_{k'} \in \mathsf{S}\left(\frac{L_{m'}}{L_{\gcd(m', n')}}, \frac{L_{n'}}{L_{\gcd(m', n')}}\right) \ . \tag{30}$$

We give, without derivation, the Hilbert series and the Frobenius number for the symmetric semigroup $\mathsf{S}\left(L_{k'}, L_{m'}, L_{n'}\right)$ under the assumption that (29) and (30) hold:

$$\begin{aligned} &H\left(L_{k'}, L_{m'}, L_{n'}\right) = \frac{(1-z^{l_1})(1-z^{l_2})}{\left(1-z^{L_{n'}}\right)\left(1-z^{m'}\right)\left(1-z^{L_{k'}}\right)} \ , \quad l_1 = \frac{L_{n'} \cdot L_{m'}}{L_{\gcd(m',n')}} \ , \\ &\mathcal{F}\left(L_{k'}, L_{m'}, L_{n'}\right) = l_1 + l_2 - \left(L_{n'} + L_{m'} + L_{k'}\right) \ , \quad l_2 = L_{k'} \cdot L_{\gcd(m',n')} \ . \end{aligned} \tag{31}$$

In general, containment (30) is hard to verify because it presumes an algorithmic procedure. Instead, one can formulate a simple inequality that provides only the sufficient condition for the semigroup $\mathsf{S}\left(L_{k'}, L_{m'}, L_{n'}\right)$ to be symmetric.

Corollary 14. *Let $L_{k'}$, $L_{m'}$ and $L_{n'}$ be Lucas numbers with odd indices such that (29) is satisfied and the following inequality holds:*

$$L_{k'}\ L_{\gcd(m',n')} > \frac{L_{n'}\ L_{m'}}{L_{\gcd(m',n')}} - L_{n'} - L_{m'} \ . \tag{32}$$

Then a numerical semigroup $\mathsf{S}\left(L_{k'}, L_{m'}, L_{n'}\right)$ is symmetric and its Hilbert series and Frobenius number are given by (31).

The proof of Corollary 14 is completely similar to the proof of Corollary 10 for symmetric semigroup generated by three Fibonacci numbers.

We finish this section with Example 15, where a Lucas triple does satisfy the containment in (30) but does not satisfy inequality (32). We calculate the Hilbert generating series of the numerical semigroup, the Frobenius number, and genus.

Example 15. $\{d_1, d_2, d_3\} = \{L_9 = 76, L_{15} = 1364, L_{17} = 3571\}$

$$\begin{aligned} &\gcd(L_9, L_{15}) = L_3 \ , \quad \gcd(L_3, L_{17}) = 1 \ , \\ &L_{17} \in \mathsf{S}\left(\frac{L_9}{L_3}, \frac{L_{15}}{L_3}\right) = \mathsf{S}\left(19, 341\right) \ , \\ &l_1 = \mathsf{lcm}(L_9, L_{15}) = 25916 \ , \quad l_2 = L_{17} \cdot L_3 = 14264 \ , \\ &L_{17} \cdot L_3 < \mathsf{lcm}(L_9, L_{15}) - L_9 - L_{15} \ , \\ &H\left(L_9, L_{15}, L_{17}\right) = \frac{(1-z^{25916})(1-z^{14264})}{\left(1-z^{76}\right)\left(1-z^{1364}\right)\left(1-z^{3571}\right)} \ , \\ &\mathcal{F}\left(L_9, L_{15}, L_{17}\right) = 35189 \ , \quad G\left(L_9, L_{15}, L_{17}\right) = 17595 \ . \end{aligned}$$

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