#A42 INTEGERS 9 (2009), 555-567

ON THE AVERAGE ASYMPTOTIC BEHAVIOR OF A CERTAIN TYPE OF SEQUENCE OF INTEGERS

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Received: 10/12/08, Revised: 4/14/09, Accepted: 5/18/09, Published: 11/19/09

Abstract

In this paper, we prove the following result: Let $\mathcal{A}$ be an infinite set of positive integers. For all positive integer n, let τ_n denote the smallest element of $\mathcal{A}$ which doesn't divide n. Then we have

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\tau_n=\sum_{n=0}^{\infty}\frac{1}{\operatorname{lcm}\{a\in\mathcal{A}\mid a\leq n\}}.$$

In the two particular cases when $\mathcal{A}$ is the set of all positive integers and when $\mathcal{A}$ is the set of the prime numbers, we give a more precise result for the average asymptotic behavior of $(\tau_n)_n$. Furthermore, we discuss the irrationality of the limit of τ_n (in the average sense) by applying a result of Erdős.

1. Introduction and Results

In Number Theory, it is frequent that a sequence of positive integers does not have a regular asymptotic behavior but has a simple and regular asymptotic average behavior. As examples, we can cite the following:

(i) The sequence $(d_n)_{n\geq1}$, where d_n denotes the number of divisors of n.

(ii) The sequence $(\sigma(n))_{n\geq1}$, where $\sigma(n)$ denotes the sum of divisors of n.

(iii) The Euler totient function $(\varphi(n))_{n\geq1}$, where $\varphi(n)$ denotes the number of positive integers, not exceeding n, that are relatively prime to n.

We refer the reader to [3] for many other examples.

In this paper, we give another type of sequence which we describe as follows: Let $a_1<a_2<\dots$ be an increasing sequence of positive integers which we denote by $\mathcal{A}$. For all positive integers n, let τ_n denote the smallest element of $\mathcal{A}$ which doesn't divide n. Then, we shall prove the following theorem.

Theorem 1 *We have:*

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\tau_n=\sum_{n=0}^{\infty}\frac{1}{\operatorname{lcm}\{a\in\mathcal{A}\mid a\le n\}}$$

in both cases when the series on the right-hand side converges or diverges.

In the particular cases when $\mathcal{A}$ is the sequence of all positive integers and when it is the sequence of the prime numbers, we refine the proof of Theorem 1 to obtain the following more precise results:

Corollary 2 *For all positive integers n, let e_n denote the smallest positive integer which doesn't divide n. Then, we have*

$$\frac{1}{N}\sum_{n=1}^{N}e_n=\ell_1+O_N\left(\frac{(\log N)^2}{N\log\log N}\right),$$

where

$$\ell_1:=\sum_{n\in\mathbb{N}}\frac{1}{\operatorname{lcm}(1,2,\ldots,n)}<+\infty.$$

Corollary 3 *For all positive integers n, let q_n denote the smallest prime number which doesn't divide n. Then, we have*

$$\frac{1}{N}\sum_{n=1}^{N}q_n=\ell_2+O_N\left(\frac{(\log N)^2}{N\log\log N}\right),$$

where

$$\ell_2:=\sum_{n\in\mathbb{N}}\frac{1}{\prod_{p\text{ prime},\,p\le n}p}<+\infty.$$

Further, by applying a result of Erdős [1], we derive a sufficient condition for the average limit of $(\tau_n)_n$ to be an irrational number. We have the following.

Proposition 4 *Let $\underline{\mathbf{d}}(\mathcal{A})$ denote the lower asymptotic density of $\mathcal{A}$; that is,*

$$\underline{\mathbf{d}}(\mathcal{A}):=\liminf_{N\to+\infty}\frac{1}{N}\sum_{\substack{a\in\mathcal{A}\\ a\le N}}1.$$

Suppose that $\underline{\mathbf{d}}(\mathcal{A})>1-\log 2$. Then the average limit of $(\tau_n)_n$ is an irrational number. In particular, the numbers ℓ_1 and ℓ_2, appearing respectively in Corollaries 2 and 3, are irrational.

2. The Proofs

2.1. Some Preparations and Preliminary Results

Throughout this paper, we let $\mathbb{N}^*$ denote the set $\mathbb{N}\setminus\{0\}$ of all positive integers. For a given real number x, we let $\lfloor x\rfloor$ and $\langle x\rangle$ denote respectively the integer part and the fractional part of x. Further, we adopt the natural convention that the least common multiple and the product of the elements of an empty set are equal to 1.

We fix an increasing sequence of positive integers $a_1 < a_2 < \cdots$, which we denote by $\mathcal{A}$, and for all positive integer n, we let τ_n denote the smallest element of $\mathcal{A}$ which doesn't divide n.

For all $\alpha \in \mathcal{A}$, we let $L(\alpha)$ denote the positive integer defined by

$$L(\alpha) := \frac{\operatorname{lcm}\{a \in \mathcal{A} \mid a \le \alpha\}}{\operatorname{lcm}\{a \in \mathcal{A} \mid a < \alpha\}}. \tag{1}$$

We then let $\mathcal{B}$ denote the subset of $\mathcal{A}$ defined by

$$\mathcal{B} := \{a \in \mathcal{A} \mid L(a) > 1\}. \tag{2}$$

We shall see later that $\mathcal{B}$ is just the set of the values of the sequence $(\tau_n)_n$. We begin with the following lemma.

Lemma 5 *For all positive integer n, we have:*

$$\operatorname{lcm}\{a \in \mathcal{A} \mid a \le n\} = \operatorname{lcm}\{b \in \mathcal{B} \mid b \le n\}.$$

Proof. Let $n \ge 1$ and let $a_1, \ldots, a_k$ be the elements of $\mathcal{A}$ not exceeding n. By using the following well-known property of the least common multiple:

$$\operatorname{lcm}(a_1, \ldots, a_i, \ldots, a_k) = \operatorname{lcm}(\operatorname{lcm}(a_1, \ldots, a_i), a_{i+1}, \ldots, a_k) \quad (\text{for } i = 1, 2, \ldots, k),$$

we remark that when $a_i \notin \mathcal{B}$, we have $L(a_i) = 1$ and then $\operatorname{lcm}(a_1, \ldots, a_i) = \operatorname{lcm}(a_1, \ldots, a_{i-1})$. So, each a_i not belonging to $\mathcal{B}$ can be eliminated from the list $a_1, \ldots, a_k$ without changing the value of the least common multiple of that list. The lemma follows. □

From Lemma 5, we derive another formula for $L(\alpha)$ ($\alpha \in \mathcal{A}$). For all $\alpha \in \mathcal{A}$ we have

$$L(\alpha) = \frac{\operatorname{lcm}\{b \in \mathcal{B} \mid b \le \alpha\}}{\operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\}}. \tag{3}$$

The next lemma gives a useful characterization for the terms of the sequence $(\tau_n)_n$.

Lemma 6 *For all positive integers n and all $\alpha \in \mathcal{A}$, we have:*

$$\tau_n = \alpha \Longleftrightarrow \exists k \in \mathbb{N}^*, k \not\equiv 0 \bmod L(\alpha) \text{ such that } n = k \cdot \operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\}.$$

Proof. Let $n \in \mathbb{N}^*$ and $\alpha \in \mathcal{A}$. By the definition of the sequence $(\tau_n)_n$, the equality $\tau_n = \alpha$ amounts to saying that n is a multiple of each element $a \in \mathcal{A}$ satisfying $a < \alpha$ and that n is not a multiple of α. Equivalently, $\tau_n = \alpha$ if and only if n is a multiple of $\mathrm{lcm}\{a \in \mathcal{A} \mid a < \alpha\}$ without being a multiple of $\mathrm{lcm}\{a \in \mathcal{A} \mid a \leq \alpha\}$. So, it suffices to set $k := \frac{n}{\mathrm{lcm}\{a \in \mathcal{A} \ \mid \ a < \alpha\}}$ to obtain the equivalence:

$$\tau_n = \alpha \Longleftrightarrow \exists k \in \mathbb{N}^*, k \not\equiv 0 \bmod L(\alpha) \text{ such that } n = k \cdot \mathrm{lcm}\{a \in \mathcal{A} \mid a < \alpha\}.$$

The lemma then follows from Lemma 5. □

Corollary 7 *The sequence $(\tau_n)_n$ takes its values in the set $\mathcal{B}$. Besides, any element of $\mathcal{B}$ is taken by $(\tau_n)_n$ infinitely often.*

Proof. Let $\alpha \in \mathcal{A}$. If $\alpha \notin \mathcal{B}$, then we have $L(\alpha) = 1$ and thus there is no $k \in \mathbb{N}^*$ such that $k \not\equiv 0 \mod (L(\alpha))$. It follows, according to Lemma 6, that α cannot be a value of $(\tau_n)_n$.
Next, if $\alpha \in \mathcal{B}$, then $L(\alpha) \geq 2$ and thus there are infinitely many $k \in \mathbb{N}^*$ such that $k \not\equiv 0 \mod (L(\alpha))$. This implies (according to Lemma 6) that there are infinitely many $n \in \mathbb{N}^*$ satisfying $\tau_n = \alpha$. The corollary is proved. □

Actually, given $\alpha \in \mathcal{B}$, Lemma 6 even gives an estimation for the number of solutions of the equation $\tau_n = \alpha$ in an interval $[1, x]$ ($x \in \mathbb{R}^+$). For $x > 0$ and $\alpha \in \mathcal{B}$, define

$$\varphi(\alpha; x) := \#\{n \in \mathbb{N}^*, n \leq x \mid \tau_n = \alpha\}.$$

Then, we have the following:

Corollary 8 *Let $\alpha \in \mathcal{B}$ and $x > 0$. Then we have*

$$\varphi(\alpha; x) = \frac{L(\alpha) - 1}{\mathrm{lcm}\{b \in \mathcal{B} \mid b \leq \alpha\}}.x + \mathrm{c}_{\alpha,x},$$

where $|\mathrm{c}_{\alpha,x}| < 1$. Furthermore, if $\mathrm{lcm}\{b \in \mathcal{B} \mid b < \alpha\} > x$, then we have $\varphi(\alpha; x) = 0$.

Proof. Let $\alpha \in \mathcal{B}$ and $x > 0$. By Lemma 6, we have

$$\begin{aligned}
\varphi(\alpha; x) &= \#\left\{k \in \mathbb{N}^* \mid k \leq \frac{x}{\mathrm{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} \text{ and } k \not\equiv 0 \bmod (L(\alpha))\right\} \\
&= \left\lfloor \frac{x}{\mathrm{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} \right\rfloor - \left\lfloor \frac{x}{L(\alpha).\mathrm{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} \right\rfloor \\
&= \frac{x}{\mathrm{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} - \frac{x}{L(\alpha).\mathrm{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} + \mathrm{c}_{\alpha,x},
\end{aligned}$$

where $\mathrm{c}_{\alpha,x} := -\left\langle \frac{x}{\mathrm{lcm}\{b \in \mathcal{B} \ \mid \ b < \alpha\}} \right\rangle + \left\langle \frac{x}{L(\alpha).\mathrm{lcm}\{b \in \mathcal{B} \ \mid \ b < \alpha\}} \right\rangle$. So, it is clear that $|\mathrm{c}_{\alpha,x}| < 1$.

Next, we have

$$\frac{x}{\operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} - \frac{x}{L(\alpha).\operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\}} = \frac{L(\alpha)-1}{L(\alpha).\operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\}}.x$$
$$= \frac{L(\alpha)-1}{\operatorname{lcm}\{b \in \mathcal{B} \mid b \leq \alpha\}}.x,$$

by using (3). This confirms the first part of the corollary.

Now, if $\operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\} > x$, then none of the integers of the range $[1, x]$ is a multiple of $\operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\}$. It follows, according to Lemma 6, that the equation $\tau_n = \alpha$ doesn't have any solution in the range $[1, x]$. Hence $\varphi(\alpha, x) = 0$. This confirms the second part of the corollary and ends this proof. □

Now, let $b_1 < b_2 < \dots$ be the elements of $\mathcal{B}$. To prove our main result, we will need some properties of the sequence $(b_n)_n$. For simplicity, for all $n \geq 1$, let

$$L_n := L(b_n) \geq 2.$$

Then, by (3), we have:

$$L_n = \frac{\operatorname{lcm}\{b \in \mathcal{B} \mid b \leq b_n\}}{\operatorname{lcm}\{b \in \mathcal{B} \mid b < b_n\}} = \frac{\operatorname{lcm}(b_1, b_2, \dots, b_n)}{\operatorname{lcm}(b_1, b_2, \dots, b_{n-1})},$$

which gives

$$\operatorname{lcm}(b_1, b_2, \dots, b_n) = L_n \cdot \operatorname{lcm}(b_1, b_2, \dots, b_{n-1}) \qquad (\forall n \geq 1).$$

By iteration, we obtain for all $n \geq 1$:

$$\operatorname{lcm}(b_1, b_2, \dots, b_n) = L_1 L_2 \cdots L_n \geq 2^n. \tag{4}$$

For the simplicity of some formulas in what follows, it is useful to set $b_0 := 0$ and $L_0 := 1$. Note that $b_0 \notin \mathcal{B}$. We have the two following lemmas:

Lemma 9 *We have*

$$\sum_{n \in \mathbb{N}} \frac{1}{\operatorname{lcm}\{a \in \mathcal{A} \mid a \leq n\}} = \sum_{k=1}^{\infty} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}}.$$

Proof. According to Lemma 5, we have:

$$\sum_{n \in \mathbb{N}} \frac{1}{\operatorname{lcm}\{a \in \mathcal{A} \mid a \leq n\}} = \sum_{n \in \mathbb{N}} \frac{1}{\operatorname{lcm}\{b \in \mathcal{B} \mid b \leq n\}}$$
$$= \sum_{k=1}^{\infty} \sum_{b_{k-1} \leq n < b_k} \frac{1}{\operatorname{lcm}\{b \in \mathcal{B} \mid b \leq n\}}$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty} \frac{b_k - b_{k-1}}{\operatorname{lcm}(b_1, b_2, \ldots, b_{k-1})} \\
&= \sum_{k=1}^{\infty} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}} \qquad \text{(according to (4))}.
\end{aligned}
$$

The lemma is proved. □

Lemma 10 *Let r be a positive integer and N be an integer such that*

$$\operatorname{lcm}(b_1, b_2, \ldots, b_{r-1}) \leq N < \operatorname{lcm}(b_1, b_2, \ldots, b_r).$$

Then, we have

$$\frac{1}{N} \sum_{n=1}^{N} \tau_n = S_1(N) + S_2(N) f(N, r),$$

where

$$S_1(N) := -\frac{b_r}{L_1 L_2 \cdots L_r} + \sum_{k=1}^{r} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}} \quad , \quad S_2(N) := \frac{1}{N} \sum_{k=1}^{r} b_k$$

and $f(N, r)$ is a function of N and r, satisfying $|f(N, r)| < 1$.

Proof. According to Corollaries 7 and 8, and to relation (4), we have

$$
\begin{aligned}
\frac{1}{N} \sum_{n=1}^{N} \tau_n &= \frac{1}{N} \sum_{\alpha \in \mathcal{B}} \sum_{1 \leq n \leq N, \tau_n = \alpha} \alpha \\
&= \frac{1}{N} \sum_{\alpha \in \mathcal{B}} \alpha \varphi(\alpha; N) \\
&= \frac{1}{N} \sum_{\substack{\alpha \in \mathcal{B} \\ \operatorname{lcm}\{b \in \mathcal{B} \mid b < \alpha\} \leq N}} \alpha \varphi(\alpha; N) \\
&= \frac{1}{N} \sum_{k=1}^{r} b_k \varphi(b_k; N) \\
&= \frac{1}{N} \sum_{k=1}^{r} b_k \left(\frac{L_k - 1}{L_1 L_2 \cdots L_k} N + \mathrm{c}_{k,N} \right) \qquad \text{(where } |\mathrm{c}_{k,N}| < 1) \\
&= \sum_{k=1}^{r} b_k \left(\frac{1}{L_1 L_2 \cdots L_{k-1}} - \frac{1}{L_1 L_2 \cdots L_k} \right) + \frac{1}{N} \sum_{k=1}^{r} b_k \mathrm{c}_{k,N}.
\end{aligned}
$$

Then, the lemma follows by remarking that:

$$\begin{aligned}\sum_{k=1}^{r} b_k \left(\frac{1}{L_1 L_2 \cdots L_{k-1}} - \frac{1}{L_1 L_2 \cdots L_k}\right) &= \sum_{k=1}^{r} \left(\frac{b_{k-1}}{L_1 L_2 \cdots L_{k-1}} - \frac{b_k}{L_1 L_2 \cdots L_k}\right) \\ &\quad + \sum_{k=1}^{r} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}} \\ &= -\frac{b_r}{L_1 L_2 \cdots L_r} + \sum_{k=1}^{r} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}}\end{aligned}$$

and by defining

$$f(N,r) := \frac{\sum_{k=1}^{r} b_k \mathrm{c}_{k,N}}{\sum_{k=1}^{r} b_k},$$

which satisfies $|f(N,r)| < 1$ (since $|\mathrm{c}_{k,N}| < 1$ for all $k \geq 1$). This completes the proof. □

We will finally need two lemmas on the convergence of sequences and series.

Lemma 11 *Let $(x_n)_{n\geq 1}$ be a real non-increasing sequence. Suppose that the series $\sum_{n=1}^{\infty} x_n$ converges. Then, we have*

$$x_n = o\left(\frac{1}{n}\right) \qquad \textit{(as } n \textit{ tends to infinity).}$$

Lemma 12 *Let $(\theta_n)_{n\geq 1}$ be a sequence of real numbers. Suppose that θ_n tends to 0 as n tends to infinity. Then the sequence with general term given by*

$$\frac{1}{2^n} \sum_{k=1}^{n} 2^k \theta_k$$

also tends to 0 as n tends to infinity.

Remark. Note that Lemma 12 is in fact a particular case of a more general theorem in summability theory, called Silverman-Toeplitz theorem (see e.g., [2]).

2.2. Proofs of the Main Results

Proof of Theorem 1. Let N be a positive integer. Since (according to (4)) the sequence $(\mathrm{lcm}(b_1, b_2, \ldots, b_n))_n$ increases and tends to infinity with n, N must lie somewhere between two consecutive terms of this sequence. So, let $r \geq 1$ such that

$$\mathrm{lcm}(b_1, b_2, \ldots, b_{r-1}) \leq N < \mathrm{lcm}(b_1, b_2, \ldots, b_r).$$

Then, by using Lemma 10, we have

$$\frac{1}{N}\sum_{n=1}^{N}\tau_n = S_1(N) + S_2(N)f(N), \tag{5}$$

where

$$S_1(N) := -\frac{b_r}{L_1L_2\cdots L_r} + \sum_{k=1}^{r}\frac{b_k - b_{k-1}}{L_1L_2\cdots L_{k-1}}, \tag{6}$$

$$S_2(N) := \frac{1}{N}\sum_{k=1}^{r} b_k \tag{7}$$

and $|f(N)| < 1$.

Next, set

$$S := \sum_{n=0}^{\infty}\frac{1}{\operatorname{lcm}\{a \in \mathcal{A} \mid a \le n\}}.$$

To prove Theorem 1, we distinguish two cases according to whether S converges or diverges.

1st case: $S < +\infty$. In this case, since the sequence $(1/\operatorname{lcm}\{a \in \mathcal{A} \mid a < n\})_{n\ge 1}$ is clearly non-increasing, then by Corollary 11, we have

$$\lim_{n\to+\infty}\frac{n}{\operatorname{lcm}\{a \in \mathcal{A} \mid a < n\}} = 0.$$

By specializing in this limit n to the integers b_k ($k \ge 1$), we obtain (according to Lemma 5 and Formula (4)) that

$$\lim_{k\to+\infty}\frac{b_k}{L_1L_2\cdots L_{k-1}} = 0. \tag{8}$$

On the one hand, according to (8) and to Lemma 9, we have (because r tends to infinity with N)

$$\lim_{N\to+\infty} S_1(N) = \sum_{k=1}^{\infty}\frac{b_k - b_{k-1}}{L_1L_2\cdots L_{k-1}} = S \tag{9}$$

and, on the other hand, we have

$$\begin{aligned} S_2(N) &:= \frac{1}{N}\sum_{k=1}^{r} b_k \le \frac{1}{\operatorname{lcm}(b_1,\ldots,b_{r-1})}\sum_{k=1}^{r} b_k = \sum_{k=1}^{r}\frac{b_k}{L_1L_2\cdots L_{r-1}} \\ &\le \sum_{k=1}^{r}\frac{b_k}{L_1L_2\cdots L_{k-1}}2^{k-r} \quad \text{(since } L_i \ge 2 \text{ for all } i \ge 1) \\ &= \frac{1}{2^r}\sum_{k=1}^{r}2^k\frac{b_k}{L_1L_2\cdots L_{k-1}}. \end{aligned}$$

But by applying Lemma 12 for $\theta_k := \frac{b_k}{L_1 L_2 \cdots L_{k-1}}$ which is seen (from (8)) to tend to 0 as k tends to infinity, we have

$$\lim_{r \to +\infty} \frac{1}{2^r} \sum_{k=1}^{r} 2^k \frac{b_k}{L_1 L_2 \cdots L_{k-1}} = 0.$$

So, it follows (because r tends to infinity with N) that

$$\lim_{N \to +\infty} S_2(N) = 0. \tag{10}$$

Finally, by inserting (9) and (10) into (5), we get

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{n=1}^{N} \tau_n = S,$$

as required.

2nd case: $S = +\infty$. In this case, by using (5), we are going to bound from below $\frac{1}{N} \sum_{n=1}^{N} \tau_n$ by an expression tending to infinity with N.

On the one hand, we have

$$\begin{aligned} S_1(N) &:= -\frac{b_r}{L_1 L_2 \cdots L_r} + \sum_{k=1}^{r} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}} \\ &= -\frac{b_r}{L_1 L_2 \cdots L_r} + \frac{b_r - b_{r-1}}{L_1 L_2 \cdots L_{r-1}} + \sum_{k=1}^{r-1} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}} \\ &\geq \frac{b_r}{L_1 L_2 \cdots L_{r-1}} - 2 + \sum_{k=1}^{r-1} \frac{b_k - b_{k-1}}{L_1 L_2 \cdots L_{k-1}} \end{aligned} \tag{11}$$

(because we obviously have $b_i \leq \mathrm{lcm}(b_1, \ldots, b_i) = L_1 L_2 \cdots L_i$, for all $i \geq 1$).

On the other hand, we have

$$\begin{aligned} S_2(N) &:= \frac{1}{N} \sum_{k=1}^{r} b_k \leq \sum_{k=1}^{r} \frac{b_k}{L_1 L_2 \cdots L_{r-1}} \\ &\leq \frac{b_r}{L_1 L_2 \cdots L_{r-1}} + \sum_{k=1}^{r-1} \frac{L_1 L_2 \cdots L_k}{L_1 L_2 \cdots L_{r-1}} \quad (\text{since } b_i \leq L_1 L_2 \cdots L_i, \text{ for all } i) \\ &= \frac{b_r}{L_1 L_2 \cdots L_{r-1}} + \sum_{k=1}^{r-1} \frac{1}{L_{k+1} L_{k+2} \cdots L_{r-1}} \\ &\leq \frac{b_r}{L_1 L_2 \cdots L_{r-1}} + \sum_{k=1}^{r-1} \frac{1}{2^{r-k-1}} \qquad (\text{since } L_i \geq 2 \text{ for all } i) \\ &< \frac{b_r}{L_1 L_2 \cdots L_{r-1}} + 2. \end{aligned} \tag{12}$$

It follows, by inserting (11) and (12) into (5), that

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N}\tau_n &= S_1(N)+S_2(N)f(N)\\ &\geq S_1(N)-S_2(N) \quad (\text{since } |f(N)|<1)\\ &\geq \sum_{k=1}^{r-1}\frac{b_k-b_{k-1}}{L_1L_2\cdots L_{k-1}}-4.\end{aligned}$$

But since (according to Lemma 9) $\sum_{k=1}^{\infty}\frac{b_k-b_{k-1}}{L_1L_2\cdots L_{k-1}}=S=+\infty$, we conclude that

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{n=1}^{N}\tau_n=+\infty.$$

This completes the proof of the theorem. □

Proof of Corollary 2. In the situation of Corollary 2, $\mathcal{A}$ is the set of all positive integers and then $\mathcal{B}$ is the set of the powers of prime numbers. We must repeat the proof of Theorem 1 and give more precision to the two quantities $\frac{b_r}{L_1L_2\cdots L_r}$ and $\sum_{k=1}^{r}b_k$. First let us show that

$$b_r\sim\log N \qquad (\text{as } N \text{ tends to infinity}). \tag{13}$$

By the definition of r, recall that

$$\operatorname{lcm}(b_1,b_2,\ldots,b_{r-1})\leq N<\operatorname{lcm}(b_1,b_2,\ldots,b_r). \tag{14}$$

Next, by Lemma 5, we have (since $\mathcal{A}=\mathbb{N}^*$):

$$\begin{aligned}\operatorname{lcm}(b_1,b_2,\ldots,b_r)&=\operatorname{lcm}(1,2,\ldots,b_r)\ ,\\ \operatorname{lcm}(b_1,b_2,\ldots,b_{r-1})&=\operatorname{lcm}(1,2,\ldots,b_{r-1})\end{aligned}$$

and by the prime number theorem (see, e.g., [3]), we have, on the one hand,

$$\log\operatorname{lcm}(1,2,\ldots,n)\sim n \qquad (\text{as } n \text{ tends to infinity})$$

and, on the other hand (because $\mathcal{B}$ is the set of the powers of prime numbers),

$$b_n\sim b_{n-1} \qquad (\text{as } n \text{ tends to infinity}).$$

Taking into account all these facts, we derive from (14) that effectively $b_r\sim\log N$, confirming (13).

Now, we are going to be precise the order of magnitude of $\frac{b_r}{L_1L_2\cdots L_r}$ and $\sum_{k=1}^{r} b_k$. We have

$$\frac{b_r}{L_1L_2\cdots L_r} = \frac{b_r}{\operatorname{lcm}(b_1, b_2, \ldots, b_r)} < \frac{b_r}{N} \qquad \text{(according to (14))}.$$

It follows, according to (13), that

$$\frac{b_r}{L_1L_2\cdots L_r} = O\left(\frac{\log N}{N}\right). \tag{15}$$

Next, because $\mathcal{B}$ is the set of the powers of the prime numbers, we have

$$\sum_{k=1}^{r} b_k = \sum_{\substack{e\geq 1, p \text{ prime} \\ p^e \leq b_r}} p^e = \sum_{\substack{p \text{ prime} \\ p\leq b_r}} \left(\sum_{1\leq e\leq \lfloor \frac{\log b_r}{\log p}\rfloor} p^e \right) = \sum_{\substack{p \text{ prime} \\ p\leq b_r}} \frac{p}{p-1}\left(p^{\lfloor \frac{\log b_r}{\log p}\rfloor} - 1\right).$$

Since for any prime number p, we have $\frac{p}{p-1} \leq 2$ and $p^{\lfloor \log b_r/\log p\rfloor} \leq p^{\log b_r/\log p} = b_r$, it follows that

$$\sum_{k=1}^{r} b_k \leq 2b_r\pi(b_r),$$

where π denotes the prime-counting function. But, by using again the prime number theorem and (13), we have $b_r\pi(b_r) = O(\frac{b_r^2}{\log b_r}) = O(\frac{(\log N)^2}{\log\log N})$. Hence

$$\sum_{k=1}^{r} b_r = O\left(\frac{(\log N)^2}{\log\log N}\right). \tag{16}$$

It finally remains to insert (15) and (16) into (6) and (7) respectively to obtain (according to (5) and to Lemma 9) that

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{n=1}^{N} \tau_n = \sum_{n\in\mathbb{N}} \frac{1}{\operatorname{lcm}(1, 2, \ldots, n)} + O\left(\frac{(\log N)^2}{N\log\log N}\right).$$

The corollary is proved. □

Proof of Corollary 3. In the situation of Corollary 3, $\mathcal{A}$ is the set of the prime numbers and then $\mathcal{B} = \mathcal{A}$. So, for all $n \geq 1$, we have $a_n = b_n = L_n = p_n$, where p_n denotes the nth prime number. Consequently, we have (in the context of the proof of Theorem 1)

$$p_1p_2\cdots p_{r-1} \leq N < p_1p_2\cdots p_r. \tag{17}$$

So, by the prime number theorem (see, e.g., [3]), we have

$$p_r \sim \log N \qquad \text{(as } N \text{ tends to infinity)}.$$

From this last, it follows that

$$\frac{b_r}{L_1 L_2 \cdots L_r} = \frac{p_r}{p_1 p_2 \cdots p_r} < \frac{p_r}{N} \sim \frac{\log N}{N} \qquad \text{(as } N \text{ tends to infinity)},$$

which gives

$$\frac{b_r}{L_1 L_2 \cdots L_r} = O\left(\frac{\log N}{N}\right) \tag{18}$$

and that

$$\sum_{k=1}^{r} b_k = \sum_{k=1}^{r} p_k \sim \frac{p_r^2}{2\log p_r} \sim \frac{(\log N)^2}{2\log\log N} \qquad \text{(as } N \text{ tends to infinity)},$$

which gives

$$\frac{1}{N}\sum_{k=1}^{r} b_k = O\left(\frac{(\log N)^2}{N\log\log N}\right). \tag{19}$$

To conclude, it suffices to insert (18) and (19) into (6) and (7) respectively and use (5) and Lemma 9. The result of the corollary follows. □

Now, we are going to prove Proposition 4. To do so, we need the following result of Erdős [1].

Theorem (Erdős [1]) *Let $u_1 < u_2 < \cdots$ be an infinite sequence of positive integers. Set $\mathcal{U} := \{u_1, u_2, \dots\}$ and suppose that*

$$\underline{\mathbf{d}}(\mathcal{U}) > 1 - \log 2 = 0.306\dots$$

Then, the real positive number

$$\sum_{n=1}^{\infty} \frac{1}{\operatorname{lcm}\{u \in \mathcal{U} \mid u \le n\}}$$

is irrational.

Proof of Proposition 4. The first part of Proposition 4 which concerns a general set $\mathcal{A}$ is clearly an immediate consequence of the Main Theorem 1 and the above theorem of Erdős. Next, since the set $\mathbb{N}^*$ of all positive integers has asymptotic density $1 > 1 - \log 2$, the irrationality of the constant ℓ_1 of Corollary 2 is a direct application of the first part of the proposition. Now, let us prove the irrationality of the constant ℓ_2 appearing in Corollary 3. We must notice that this is not a direct application of the first part of the proposition, because the set of the prime numbers has asymptotic density $0 < 1 - \log 2$.

Let $\mathcal{F}$ denote the set of square-free numbers, that is the set of all positive integers which are a product of pairwise distinct prime numbers. It is known that $\mathcal{F}$ has asymptotic density $\frac{6}{\pi^2} > 1 - \log 2$ (see, e.g., [3]). So, it follows by Erdős' result that the number

$$\ell_3 := \sum_{n \in \mathbb{N}} \frac{1}{\operatorname{lcm}\{f \in \mathcal{F} \mid f \leq n\}}$$

is irrational.

But we remark that for all $n \in \mathbb{N}$, we have:

$$\operatorname{lcm}\{f \in \mathcal{F} \mid f \leq n\} \;=\; \prod_{\substack{p \text{ prime} \\ p \leq n}} p \;=\; \operatorname{lcm}\{p \text{ prime} \mid p \leq n\},$$

which shows that actually $\ell_3 = \ell_2$. Consequently ℓ_2 is an irrational number. This completes the proof of the proposition. $\square$

Remark. The irrationality of the constants ℓ_1 and ℓ_2 appearing in Corollaries 2 and 3 respectively can be shown by a more elementary way than that presented in Erdős' paper for the general case.

Acknowledgments

I would like to thank the referee for his/her comments and suggestions which certainly improved the readability of this paper.

References

[1] P. Erdős, On the irrationality of certain series: problems and results, in *New Advances in Transcendence Theory (Durham, 1986)*, p. 102-109, Cambridge Univ. Press, Cambridge, 1988.

[2] G.H. Hardy, Divergent Series, Oxford Univ. Press, London, 1949.

[3] G.H. Hardy and E.M. Wright, The Theory of Numbers, 5th ed., Oxford Univ. Press, London, 1979.