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CONVERGENCE IN MÖBIUS NUMBER SYSTEMS

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Abstract

The Möbius number systems use sequences of Möbius transformations to represent the extended real line or, equivalently, the unit complex circle. An infinite sequence of Möbius transformations represents a point x on the circle if and only if the transformations, in the limit, take the uniform measure on the circle to the Dirac measure centered at the point x. We present new characterizations of this convergence.

Moreover, we show how to improve a known result that guarantees the existence of Möbius number systems for some Möbius iterative systems.

As Möbius number systems use subshifts instead of the whole symbolic space, we can ask what is the language complexity of these subshifts. We offer (under some assumptions) a sufficient and necessary condition for a number system to be sofic.

1. Introduction

The theory of Möbius number systems was introduced in [2]. A Möbius number system assigns numbers to sequences of Möbius transformations obtained by composing a finite starting set of Möbius transformations.

Möbius number systems display complicated dynamical properties and have connections to other kinds of number representation systems. In particular, Möbius number systems can generalize continued fractions (see [3]).

As Möbius transformations are bijective on the complex sphere, we cannot use the contraction theorem of Barnsley to define convergence of a series of maps to a point. Instead, in [2], convergence of sequences of Möbius transformations to points was defined using convergence of measures. An infinite sequence of Möbius transformations represents a point x on the circle if and only if the images of the uniform measure on the circle converge to the Dirac measure centered on the point x. There are multiple equivalent ways to state this convergence, including the convergence of the images of point 0. We present several new equivalent definitions of convergence at the end of Section 3.

The existence of the number system is already known for a considerable class of sets of Möbius transformations (see [3], Theorem 9). In Section 4, we improve this

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result by slightly weakening the assumptions of the existence theorem.

As Möbius number systems are essentially subshifts, we can ask what the language of a given system is. Because these systems can be described in terms of point sets on the unit circle instead of the usual language-theoretic ways, we need tools for investigating the complexity of subshifts involved.

In Section 5, we present one such tool for some Möbius number systems, a sufficient and necessary condition for being sofic, similar to the classic Myhill–Nerode theorem for regular languages.

2. Preliminaries

Denote by $\mathbb{T}$ the unit circle and by $\mathbb{D}$ the closed unit disc in the complex plane. For $x, y \in \mathbb{T}$ denote by (x, y) the counterclockwise interval from x to y. To make notation more convenient, we define the sum $x+l$ for $x \in \mathbb{T}$ and $l \in \mathbb{R}$ by $\arg(x+l) = \arg x + l$ modulo 2π.

Let A be a finite alphabet. Denote by $A^\star$ the monoid of all finite words on A with the operation of concatenation and by A^ω the set of all one-sided infinite words. Let $|v|$ denote the length of the word v. We use the notations $w = w_0 w_1 w_2 \cdots$ and $w_{[i,j]} = w_i w_{i+1} \cdots w_j$. Let $v \in A^\star$ be a word of length n. Then we write $[v] = \{w \in A^\omega : w_{[0,n-1]} = v\}$ and call the resulting subset of A^ω the *cylinder* of v.

Let X be a metric space. We denote by ρ the metric function of X, by $\mathrm{Int}(V)$ interior of the set V and by $B_r(x)$ the open ball of radius r centered at x. The diameter of a nonempty set V, denoted by $\mathrm{diam}(V)$, is the supremum of $\{\rho(x, y) : x, y \in V\}$. If I is a finite union of intervals on $\mathbb{T}$, denote by $|I|$ the total length of I.

We equip $\mathbb{C}$ with the metric $\rho(x, y) = |x-y|$ and $\mathbb{T}$ with the circle distance metric (i.e., metric measuring distances along the circle). The shift space A^ω of one-sided infinite words comes equipped with the metric $\rho(u, v) = \max\left(\{2^{-k} : u_k \neq v_k\} \cup \{0\}\right)$. It is easy to see that the topology of A^ω is the product topology. A *subshift* $\Sigma \subset A^\omega$ is a set that is both topologically closed and invariant (i.e., $\sigma(\Sigma) \subset \Sigma$) under the *shift* map $\sigma(w)_i = w_{i+1}$. The *language of a subshift* $\mathcal{L}(\Sigma)$ is the set of all words $v \in A^\star$ such that there exists $w \in \Sigma$ and indices i and j satisfying $v = w_{[i,j]}$. See [5] for details.

Unlike in [2], we will only consider symbolic representations of $\mathbb{T}$. The extended real line $\mathbb{R} \cup \{\infty\}$ is homeomorphic to the unit complex circle using the map

$$u : \mathbb{T} \to \mathbb{R} \cup \{\infty\},\ u(z) = \frac{-iz + 1}{z - i}.$$

Therefore, as long as we are not interested in arithmetics, representing $\mathbb{T}$ is equivalent to representing the extended real line.

A Möbius transformation (MT for short) of the complex sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is any map of the form

$$F(x) = \frac{ax+b}{cx+d}$$

where $(a,b),(c,d)$ are linearly independent vectors from $\mathbb{C}^2$. We can associate with F the matrix $A_F = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ normalized by $\det F = 1$.

It is well-known (see [1]) that MTs take circles and lines to circles and lines (possibly turning a circle into a line or vice versa). Also, Möbius transformations form a group under addition and composing MTs corresponds to multiplying their respective matrices: $A_{FG} = A_F \cdot A_G$. In the following we identify F with A_F.

By default, we will consider Möbius transformations that map $\mathbb{D}$ onto $\mathbb{D}$. It is straightforward to prove that these are precisely the transformations of the form

$$F = \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix}$$

with the normalizing condition $|\alpha|^2 - |\beta|^2 = 1$. These transformations not only map $\mathbb{T}$ onto $\mathbb{T}$, but they also preserve orientation of intervals on the circle (clockwise versus counterclockwise).

Denote by μ the uniform measure on $\mathbb{T}$ such that $\mu(\mathbb{T}) = 1$. If ν is a measure on $\mathbb{T}$ and $F : \mathbb{T} \to \mathbb{T}$ an MT, we define the measure $F\nu$ by $F\nu(E) = \nu(F^{-1}(E))$ for all measurable sets E on $\mathbb{T}$. The Dirac measure centered at point x is the measure

$$\delta_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{otherwise} \end{cases}$$

for any E measurable subset of $\mathbb{T}$.

Definition 1. Let $\{F_i\}_{i=0}^{\infty}$ be a sequence of Möbius transformations. We say that the sequence $\{F_i\}_{i=1}^{\infty}$ represents the point $x \in \mathbb{T}$ if and only if $\lim_{i\to\infty} F_i\mu = \delta_x$, where the convergence of measures is taken in the weak* topology, i.e., $\nu_i \to \nu$ if and only if for all $f : \mathbb{T} \to \mathbb{R}$ continuous it is $\int f \mathrm{d}\nu_i \to \int f \mathrm{d}\nu$.

In [2], Lemma 4 gives a sufficient condition for a sequence of MTs to represent a point. Note that the original statement of this Lemma in [2] contains a mistake which we correct here.

Lemma 2. *Let $\{M_n\}_{n=1}^{\infty}$ be a sequence of Möbius transformations. Assume that there exists $t \in \mathbb{T}$ and $c > 0$ such that for each open interval I with $t \in I$ we have*

$$\lim \inf_{n\to\infty} M_n\mu(I) > c.$$

Then $\lim_{n\to\infty}(M_n\mu)(I) = 1$ and $\lim_{n\to\infty} M_n\mu = \delta_t$.

Note that the word "open" was missing in the original statement of Lemma 2. It is sufficient to consider only open intervals, while the equality $\lim_{n\to\infty}(M_n\mu)(I) = 1$ may fail when t is an endpoint of I: Consider the sequence of transformations $\{M_n\}_{n=1}^{\infty}$ where $M_n(z) = \frac{\left(n+\frac{1}{n}\right)z+\left(n-\frac{1}{n}\right)}{\left(n-\frac{1}{n}\right)z+\left(n+\frac{1}{n}\right)}$. Then $M_n\mu \to \delta_1$ but the interval $I = [-1, 1]$ is invariant under all M_n and so $\lim_{n\to\infty}(M_n\mu)(I) = \mu(I) = \frac{1}{2}$.

We will need a slightly modified Proposition 6 from [2]:

Lemma 3. *Let $\{F_i\}_{i=0}^{\infty}$ be a sequence of Möbius transformations. Then $\{F_i\}_{i=0}^{\infty}$ represents $x \in \mathbb{T}$ if and only if $\lim_{i\to\infty} F_i(0) = x$.*

3. General Properties of Möbius Transformations

We will now show several useful properties of circle-preserving MTs, as well as equivalent descriptions of what it means for a sequence of MTs to represent a point on $\mathbb{T}$.

Definition 4. A Möbius transformation of the form $R_\alpha(z) = e^{i\alpha}z$ is called a *rotation*. A *contraction to* 1 is a transformation of the form

$$C_r = \begin{pmatrix} \frac{1}{2}\left(r+\frac{1}{r}\right) & \frac{1}{2}\left(r-\frac{1}{r}\right) \\ \frac{1}{2}\left(r-\frac{1}{r}\right) & \frac{1}{2}\left(r+\frac{1}{r}\right) \end{pmatrix},$$

where $r \geq 1$.

Obviously, C_1 is the identity map and the identity transformation is both a contraction to 1 and a rotation.

Observe that any contraction to 1 fixes the points ± 1. For $r > 1$, C_r acts on $\mathbb{T}$ by making all points (with the exception of -1) "flow" towards 1 and so the sequence $\{C_n\}_{n=1}^{\infty}$ represents the point 1.

Lemma 5. *Let F be a Möbius transformation. Then there exist two rotations R_{ϕ_1}, R_{ϕ_2} and C_r, a contraction to 1, such that $F = R_{\phi_1} \circ C_r \circ R_{\phi_2}$. Moreover, if F is not a rotation then $R_{\phi_1}, R_{\phi_2}, C_r$ are uniquely determined by F.*

Proof. We want to satisfy the equation

$$\begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} = F = R_{\phi_1} \circ C_r \circ R_{\phi_2} = \begin{pmatrix} \frac{1}{2}\left(r+\frac{1}{r}\right)e^{i\frac{\phi_1+\phi_2}{2}} & \frac{1}{2}\left(r-\frac{1}{r}\right)e^{i\frac{\phi_1-\phi_2}{2}} \\ \frac{1}{2}\left(r-\frac{1}{r}\right)e^{-i\frac{\phi_1-\phi_2}{2}} & \frac{1}{2}\left(r+\frac{1}{r}\right)e^{-i\frac{\phi_1+\phi_2}{2}} \end{pmatrix}.$$

Choose r so that $\frac{1}{2}\left(r+\frac{1}{r}\right) = |\alpha|$. It is easy to see that we will then have $\frac{1}{2}\left(r-\frac{1}{r}\right) = |\beta|$.

Now choose the parameters of the rotations ϕ_1 and ϕ_2 so that we satisfy the conditions $\phi_1 + \phi_2 = 2\arg\alpha$ and $\phi_1 - \phi_2 = 2\arg\beta$ and we are done. Obviously, as long as $\beta \neq 0$, this linear system has a unique solution. Because $\beta = 0$ if and only if F is a rotation, the uniqueness follows. $\square$

Denote by $F^\bullet(x)$ the modulus of the derivative of F at x. Direct calculation gives us that

$$F^\bullet(x) = \frac{1}{\left|\overline{\beta}x + \overline{\alpha}\right|^2}$$

Definition 6. Let F be a Möbius transformation. Then, inspired by [2] and [1], we define the four point sets

$$\begin{aligned} U &= \{x \in \mathbb{T} : F^\bullet(x) < 1\}, \\ V &= \{x \in \mathbb{T} : (F^{-1})^\bullet(x) > 1\}, \\ C &= \{x \in \overline{\mathbb{C}} : F^\bullet(x) \geq 1\}, \\ D &= \{x \in \overline{\mathbb{C}} : (F^{-1})^\bullet(x) \geq 1\}. \end{aligned}$$

Call U the *contraction interval of* F, V the *expansion interval of* F^{-1} and C resp. D the *expansion sets* of F resp. F^{-1}.

Obviously, $U = \mathbb{T} \setminus C$ and $V = \mathbb{T} \cap \operatorname{Int}(D)$.

By Lemma 5 we have that for every F there exist ϕ_1, ϕ_2 and r such that $F = R_{\phi_1} \circ C_r \circ R_{\phi_2}$. As $R_\phi^\bullet = 1$, we have $F^\bullet(x) = {C_r}^\bullet(R_{\phi_2}(x))$ and $F^{-1^\bullet}(x) = C_r^{-1^\bullet}(R_{-\phi_1}(x))$. Because the sets U and C are defined using $F^\bullet(x) = {C_r}^\bullet(R_{\phi_2}(x))$, the value of r determines the shapes and sizes of U and C while ϕ_2 rotates U and C clockwise around the point 0. Similarly, the shapes of V and D depend on r while ϕ_1 determines positions of V and D, rotating them (counterclockwise) around 0.

Lemma 7. *Let F be a Möbius transformation that is not a rotation. Then the following hold:*

1. $F(\overline{\mathbb{C}} \setminus C) = \operatorname{Int} D$.

2. $F(U) = V$.

3. *C and D are circles with the same radius $|\beta|^{-1}$ and centers c, d such that $|c| = |d| = \sqrt{|\beta|^{-2} + 1}$.*

4. $|V| < \pi$.

5. $|U| + |V| = 2\pi$.

6. *If $x \neq y$ are points in V then I, the shorter of the two intervals joining x, y, lies in V.*

Proof. Part (1) follows from the formula for the derivative of composite function. We can write $F^{-1^\bullet}(F(z)) = \frac{1}{F^\bullet(z)}$ and so $F^\bullet(z) < 1$ if and only if $F^{-1^\bullet}(F(z)) > 1$. Part (2) is a consequence of (1) and the definition of U and V.

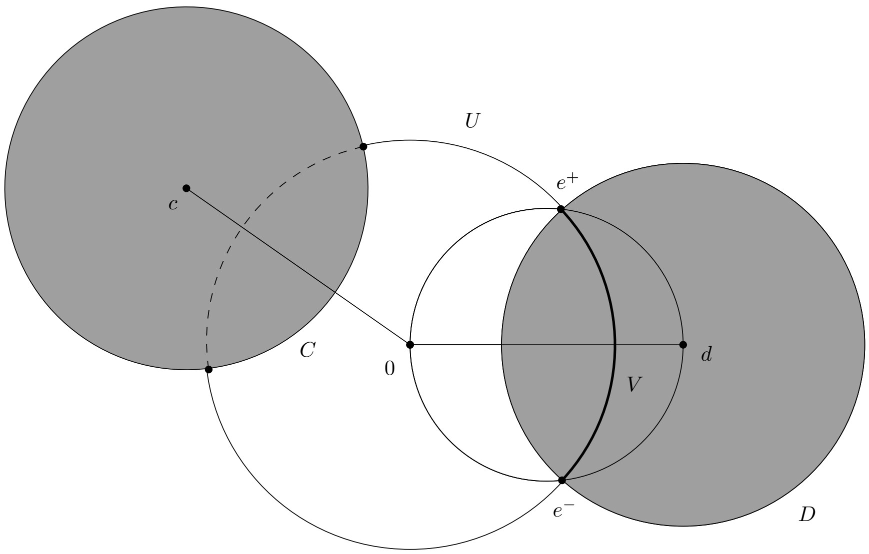

Figure 1: The geometry of C and D.

We prove (3) by direct calculation. We have $F^{\bullet}(x) = \frac{1}{|\overline{\beta}x+\overline{\alpha}|^2}$ and therefore $x \in C$ if and only if

$$\left| x + \frac{\overline{\alpha}}{\overline{\beta}} \right| < |\beta|^{-1}.$$

This is the equation of a circle with the center $c = -\frac{\overline{\alpha}}{\overline{\beta}}$ and radius $|\beta|^{-1}$. Also, it is

$$|c| = \frac{|\alpha|}{|\beta|} = \frac{\sqrt{1+|\beta|^2}}{|\beta|} = \sqrt{|\beta|^{-2}+1}.$$

The case of F^{-1} is similar.

Elementary analysis of the situation yields (4) and (5), with (6) being a direct consequence of (4) (see Figure 3). □

Remark 8. Observe that the triangles $0de^+$ and $0de^-$ in Figure 3 are right by Pythagoras' theorem. Also, we can compute that the length of V is equal to $2\arccos\left(\frac{|\beta|}{\sqrt{1+|\beta|^2}}\right)$ and the distance of d from V is $\sqrt{1+|\beta|^{-2}}-1$. Therefore, the size of D, length of V and the distance of d and $\mathbb{T}$ are all decreasing functions of $|\beta|$.

Theorem 9. *Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of MTs. Denote by V_n the expansion interval of F_n^{-1}, D_n the expansion set of F_n^{-1} and d_n the center of D_n. Then the following statements are equivalent:*

(1) *The sequence* $\{F_n\}_{n=1}^{\infty}$ *represents* $x \in \mathbb{T}$.

(2) $\lim_{n\to\infty} \overline{V}_n = \{x\}$

(3) $\lim_{n\to\infty} d_n = x$

(4) $\lim_{n\to\infty} D_n = \{x\}$

(5) *For all* $K \subset \mathrm{Int}(\mathbb{D})$ *compact we have* $\lim_{n\to\infty} F_n(K) = \{x\}$.

(6) *For all* $z \in \mathrm{Int}(\mathbb{D})$ *we have* $\lim_{n\to\infty} F_n(z) = x$.

(7) *There exists* $z \in \mathrm{Int}(\mathbb{D})$ *such that* $\lim_{n\to\infty} F_n(z) = x$.

In (2), (4), *and* (5), *we take convergence in the Hausdorff metric on the space of nonempty compact subsets of* $\mathbb{T}$, $\mathbb{C}$ *and* $\mathbb{D}$, *respectively. In particular,* $E_n \to \{x\}$ *if and only if for every* $\varepsilon > 0$ *there exists* n_0 *such that* $\forall n > n_0$ *we have* $E_n \subset B_\varepsilon(x)$.

Proof. Assume (1). To prove (2), we show that the length $|V_n|$, as well as the distance $\rho(x, V_n)$ (measured along $\mathbb{T}$), go to zero.

Consider the interval $I_\varepsilon = (x - \varepsilon, x + \varepsilon)$ of $\mathbb{T}$. It is easy to see that we have $\lim_{n\to\infty} |F_n^{-1}(I_\varepsilon)| = 2\pi$. Therefore, there exists n_0 such that $|F_n^{-1}(I_\varepsilon)| > 1$ for all $n > n_0$. If now $\varepsilon < \frac{1}{2}$ and $n > n_0$, then $|F_n^{-1}(I_\varepsilon)| > |I|$ and so there exists $z \in I_\varepsilon$ such that $F_n^{-1\bullet}(z) > 1$. Therefore $z \in I \cap V_n \neq \emptyset$, so $\rho(x, D_n) \leq \varepsilon$.

Now assume that there exists $r > 0$ such that for any n_0 there exists $n > n_0$ with $|V_n| > 2r$. For any interval I we have the inequality

$$|F_n^{-1}(I)| = |F_n^{-1}(I \cap V_n)| + |F_n^{-1}(I \setminus V_n)| \leq |U_n| + |I| = 2\pi - |V_n| + |I|,$$

as the part of I inside V_n will expand at most to the length $|U_n|$ and the part of I outside V_n will only get shorter.

Now consider $I_{\frac{1}{2}r}$. If $|V_n| > 2r$ then $|F_n^{-1}(I_{\frac{1}{2}r})| < 2\pi - 2r + r = 2\pi - r$. Therefore, it cannot be true that $F_n^{-1}(I_{\frac{1}{2}r}) \to 2\pi$, a contradiction.

An elementary examination of the geometry of V_n and D_n shows that (2), (3) and (4) are all equivalent. The distance of d_n from V_n is an increasing function of $|V_n|$ and $\rho(d_n, V_n) \to 0$ as $|V_n| \to 0$, proving (3). In a similar way, the diameter of D_n goes to zero when $\rho(d_n, \mathbb{T}) \to 0$.

Denote by C_n the expansion set of F_n^{-1} and assume (4). Recall that the diameter of C_n is always equal to the diameter of D_n. This means that $\mathrm{diam}(C_n) \to 0$ and so for any compact $K \subset \mathrm{Int}(\mathbb{D})$ there exists n_0 such that $K \cap C_n = \emptyset$ holds whenever $n > n_0$. Then $F_n(K) \subset D_n$ and so $F_n(K) \to \{x\}$, proving (5).

As $\{z\}$ is a compact set, (6) easily follows from (5). Also, statement (7) is an obvious consequence of (6).

It remains to prove (7) $\Rightarrow$ (1). Consider the circle-preserving MT M that sends 0 to z. Then the sequence $\{F_n \circ M\}$ represents the point x by Lemma 3. Therefore, if I is an open interval on $\mathbb{T}$ containing x, then $\lim |(F_n \circ M)^{-1}(I)| \to 2\pi$. It remains to see that

$$|(F_n \circ M)^{-1}(I)| = |M^{-1} \circ F_n^{-1}(I)| \geq m|F_n^{-1}(I)|$$

where $m > 0$ is the minimum of $M^\bullet(y)$ for $y \in \mathbb{T}$. Therefore, $\{F_n\}_{n=0}^\infty$ represents x by Lemma 2 and we are done. □

Remark 10. To see that we cannot expect to have convergence for all $z \in \mathbb{T}$, let $\{m_n\}_{n=1}^\infty$ be a sequence of real numbers such that the set

$$E = \{z : z = R_{-m_n}(-1) \text{ for infinitely many } n\}$$

is dense in $\mathbb{T}$. Consider the sequence of transformations $F_n = C_n R_{m_n}$. Obviously, $\{F_n\}_{n=1}^\infty$ represents the point 1, but the set $\{x \in \mathbb{T} : \neg \lim_{n\to\infty} F_n(x) = 1\}$ contains the whole E and therefore is dense in $\mathbb{T}$.

Corollary 11. *Let $\{F_k\}_{k=1}^\infty$ be a sequence of Möbius transformations such that for some $x_0 \in \mathbb{T}$ we have $\lim_{k\to\infty} (F_k^{-1})^\bullet(x_0) = \infty$. Then $\{F_k\}_{k=1}^\infty$ represents x_0.*

Proof. As

$$(F_k^{-1})^\bullet(x_0) = \frac{1}{|\overline{\beta}_k x_0 + \overline{\alpha}_k|^2},$$

we have $-\frac{\overline{\alpha}_k}{\overline{\beta}_k} \to x_0$. But this means precisely that $d_k \to x_0$ and so $\{F_k\}$ must represent x_0 by part (2) of Theorem 9. □

4. Möbius Number Systems

In an attempt to put Möbius number systems in a more abstract context we give one possible definition of symbolic representation. Afterwards, we prove Theorem 21, which strengthens part of Theorem 9 of [2].

Definition 12. Let X be a compact topological space. A mapping $P : \mathcal{L}(\Sigma) \to 2^X$ together with a subshift $\Sigma \subset A^\omega$ form a *symbolic representation of* X if the following are true:

1. $P(v)$ is a closed subset of X for each $v \in \mathcal{L}(\Sigma)$.
2. $\bigcap_{i=1}^\infty P(w_{[0,i]})$ is a singleton for every $w \in \Sigma$. Denote the element of this set by $p(w)$.
3. The map $p : \Sigma \to X$ is surjective.

The above definition is reminiscent of the definition of a generating cover from [4]. The following lemma is an exercise in the use of compactness.

Lemma 13. *Let P be a symbolic representation of X. Then p is continuous in Σ.*

Proof. Given an open set $U \subset X$ we want to show that $p^{-1}(U)$ is open in Σ. Let $w \in p^{-1}(U)$ and assume that for all k there exists w' such that $w_{[0,k]} = w'_{[0,k]}$ and $p(w') \notin U$. Then we must have $\bigcap_{i=1}^{k} P(w_{[0,i]}) \cap U^c \neq \emptyset$ and from compactness of X we obtain that $\bigcap_{i=1}^{\infty} P(w_{[0,i]}) \cap U^c \neq \emptyset$. But then $\bigcap_{i=1}^{\infty} P(w_{[0,i]})$ cannot be a singleton as $p(w) \notin U^c$, a contradiction. □

In the rest of this section, let us have a set $\{F_a : a \in A\}$ of MTs where A is a finite alphabet. Such a set is called a *Möbius iterative system.* Assign to each word $v \in A^\star$ the transformation $F_v = F_{v_0} F_{v_1} \cdots F_{v_{n-1}}$ where $n = |v|$.

Denote by V_v the expansion interval of F_v^{-1}. For $a \in A$, label e_a^+ the counterclockwise and e_a^- the clockwise endpoint of the interval V_a (that is, in our notation we have $V_a = (e_a^-, e_a^+))$.

Definition 14. The map Φ assigns to each $w \in A^\omega$ the point $x \in \mathbb{T}$ such that $\{F_{w_{[0,n)}}\}_{n=1}^{\infty}$ represents x. If $\{F_{w_{[0,n)}}\}_{n=1}^{\infty}$ does not represent any point in $\mathbb{T}$, let $\Phi(w)$ be undefined. Denote the domain of Φ by $\mathbb{X}_F$.

Definition 15. The subshift $\Sigma \subset A^\omega$ is a *Möbius number system* if and only if $\Sigma \subset \mathbb{X}_F$, $\Phi(\Sigma) = \mathbb{T}$ and $\Phi_{|\Sigma}$ is continuous.

It is easy to observe that if $\Phi(w) = x$ then $\Phi(\sigma(w)) = F_{w_0}^{-1}(x)$. We will use this simple property later.

Originally, the following theorem comes from [3]. Here, we slightly modified it so that it refers to $\mathbb{T}$ instead of the extended real line. It gives some sufficient and some necessary conditions for a Möbius iterative system to represent $\mathbb{T}$.

Theorem 16. (Theorem 9, [3]). *Let $F : A^+ \times \mathbb{T} \to \mathbb{T}$ be a Möbius iterative system.*

1. *If $\overline{\bigcup_{u \in A^+} V_u} \neq \mathbb{T}$, then $\Phi(\mathbb{X}_F) \neq \mathbb{T}$.*
2. *If $\{V_u : u \in A^+\}$ is a cover of $\mathbb{T}$, then $\Phi(\mathbb{X}_F) = \mathbb{T}$ and there exists a subshift $\Sigma \subset \mathbb{X}_F$ on which Φ is continuous and $\Phi(\Sigma) = \mathbb{T}$.*

Observe that (by compactness of $\mathbb{T}$) if $\{V_u : u \in A^+\}$ covers $\mathbb{T}$, then there exists a finite B with $\{V_b : b \in B\}$ covering $\mathbb{T}$. We show that in part two of Theorem 16 it suffices to demand that there exists a finite $B \subset A^\star$ such that the set of *closures* $\{\overline{V}_b : b \in B\}$ covers $\mathbb{T}$.

First, we prove several technical lemmas needed to nail down Theorem 21.

The following lemma is stated in a different form in [2] as Lemma 2.

Lemma 17. *Let $\{F_a : a \in A\}$ be a Möbius iterative system and $B \subset A^\star$ be finite. Let L be the length of the longest of intervals in $\{\overline{V}_b : b \in B\}$. Then there exists an increasing continuous function $\psi : [0, L] \to \mathbb{R}$ such that $\psi(0) = 0$, $\psi(l) > l$ for*

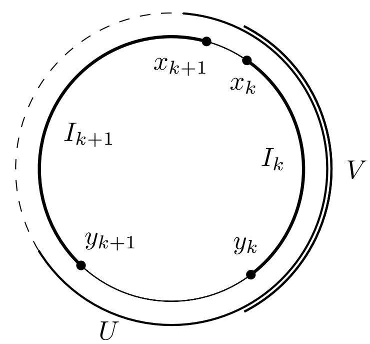

Figure 2: A twist

$l > 0$ and if I is an interval and $b \in B$ a word such that $I \subset \overline{V}_b$ then $|F_b^{-1}(I)| \geq \psi(|I|)$.

Proof. Thanks to Lemma 5 we can without loss of generality assume that all F_b are contractions to 1. Let $\psi(l) = \inf\{|F_b^{-1}(I)| \, : \, I \subset \overline{V}_b \text{ and } |I| = l\}$. By analyzing contractions, it is easy to see that ψ is increasing, continuous and $\psi(l) > l$ for $l > 0$. □

Lemma 18. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Let x, y be points and $w \in A^\omega$ a word such that for all k we have $F^{-1}_{w_{[0,k)}}(x), F^{-1}_{w_{[0,k)}}(y) \in \overline{V}_{w_k}$. Then $x = y$ or there exists a k_0 such that $F_{w_{[k,k+1]}}$ is a rotation for all $k > k_0$.*

Proof. Denote $x_k = F^{-1}_{w_{[0,k)}}(x), y_k = F^{-1}_{w_{[0,k)}}(y)$. Assume that $x \neq y$. Then for each k there exists the closed interval I_k with endpoints x_k, y_k such that $I_k \subset \overline{V}_{w_k}$. Moreover, $|I_k| > 0$ for all k, as Möbius transformations are bijective on $\mathbb{T}$.

Observe first that the sequence $\{|I_k|\}_{k=0}^{\infty}$ is a nondecreasing one. For any particular k we have two possibilities: Either $F^{-1}_{w_k}(I_k) = I_{k+1}$ and therefore $|I_{k+1}| \geq \psi(|I_k|) > |I_k|$ (by Lemma 17), or $I_{k+1} = \overline{\mathbb{T} \setminus F^{-1}_{w_k}(I_k)}$; see Figure 2. In the second case, $\mathbb{T} \setminus U_{w_k} \subset I_{k+1}$ and so $|I_{k+1}| \geq |V_{w_k}| \geq |I_k|$. Call the second case a *twist*.

Assume first that the number of twists is infinite. Whenever a twist happens for some k, we have $|V_{w_{k+1}}| \geq |I_{k+1}| \geq |V_{w_k}|$. There must exist a k_0 such that $|I_k|$ is constant for all $k > k_0$. Otherwise, the letter set $\{w_k : k = 0, 1, 2, \dots\}$ would need to contain infinitely many letters, as $|I_{k+1}| > |V_{w_k}|$ means we cannot use the letter w_k again. Therefore, $|I_{k+1}| = |V_{w_k}|$ for all but finitely many twists and the finiteness of A gives us $|I_k| = |I_l|$ for all k, l sufficiently large.

Assume that $k > k_0$. Simple case analysis shows that $|I_k| = |I_{k+1}|$ only when x_k, y_k are the endpoints of $\overline{V}_{w_k}$ and the transition is a twist. Therefore, x_{k+1}, y_{k+1} are also endpoints of $\overline{V}_{w_{k+1}}$ and $|I_{k+2}| = |I_k|$. By applying R, the rotation sending I_{k+2} to I_k, we obtain that $R \circ F^{-1}_{w_{[k,k+1]}}$ has two fixed points x_k, y_k and that

$(R \circ F^{-1}_{w_{[k,k+1]}})^{\bullet}(x_k) = (R \circ F^{-1}_{w_{[k,k+1]}})^{\bullet}(y_k) = 1$. This can happen only when $R \circ F^{-1}_{w_{[k,k+1]}} = \mathrm{id}$. Therefore, $F^{-1}_{w_{[k,k+1]}}$ is a rotation for all $k > k_0$ and we are done.

It remains to investigate the case when the number of twists is finite, i.e., $F^{-1}_{w_k}(I_k) = I_{k+1}$ for all $k \geq k_0$. By Lemma 17 we obtain $|I_{k+1}| \geq \psi(|I_k|)$ for all $k \geq k_0$ and therefore $|I_k| \geq \psi^{k-k_0}(|I_{k_0}|)$ for all $k \geq k_0$. Let $l = |I_{k_0}|$ so that if $x \neq y$, then $l > 0$.

Consider the sequence $\{\psi^k(l)\}_{k=0}^{\infty}$. Assume that $\psi^k(l) \leq L$ for all k. Then this sequence is increasing and bounded and therefore it has a limit. But the only fixed point of ψ is 0, a contradiction.

Therefore, there always exists a k such that $\psi^k(l) > L$. Then I_{k+k_0} cannot possibly fit into any of the intervals $\overline{V}_b$, a contradiction with $x_k, y_k \in \overline{V}_{w_k}$. This means that we must have $x = y$. □

The following easy observation is going to be useful when constructing our number system.

Lemma 19. *Let $I_1, \ldots, I_k$ be open intervals on $\mathbb{T}$. Then $x \in \overline{\bigcap_{i=1}^k I_i}$ if and only if both of the following conditions hold:*

(1) $x \in \bigcap_{i=1}^k \overline{I_i}$.

(2) *If x is an endpoint of I_i and I_j, then it is either a counterclockwise endpoint of both intervals or a clockwise endpoint of both intervals.*

Proof. Obviously, if $x \in \overline{\bigcap_{i=1}^k I_i}$ then $x \in \bigcap_{i=1}^k \overline{I_i}$. Were x a clockwise endpoint of I_i and counterclockwise endpoint of I_j then there would exist a neighborhood E of x such that $E \cap I_i \cap I_j = \emptyset$ and so x does not belong to $\overline{\bigcap_{i=1}^k I_i}$.

In the other direction, assume that x has all the required properties. Then a simple case analysis shows that for any neighborhood E of x we have $E \cap \bigcap_{i=1}^k I_i \neq \emptyset$ and so $x \in \overline{\bigcap_{i=1}^k I_i}$. □

Assume that $\{F_b : b \in B\}$ is a Möbius iterative system. Given $(x, w) \in \mathbb{T} \times B^\omega$, we use the shorthand notation $x_i = F^{-1}_{w_{[0,i)}}(x)$.

Define $X \subset \mathbb{T} \times B^\omega$ to be the set of all pairs (x, w) such that:

(1) For all $i = 1, 2, \ldots$ we have $x_i \in \overline{V}_{w_i}$.

(2) For no i it is true that $x_i = e^+_{w_i}$ and $x_{i+1} = e^-_{w_{i+1}}$, neither it is true that $x_i = e^-_{w_i}$ and $x_{i+1} = e^+_{w_{i+1}}$.

Note that the second condition says that endpoints cannot "alternate": If x_i, x_{i+1} are endpoints of their respective intervals then they are both of the same type (clockwise or counterclockwise). The situation is illustrated in Figure 3.

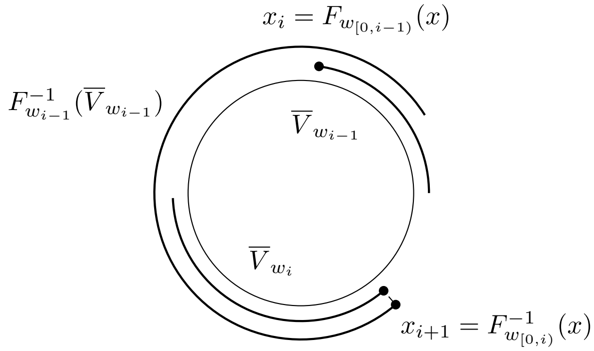

Figure 3: The situation in part (2)

We show that the set X is actually a closed subspace of $\mathbb{T}\times B^\omega$. Using Lemma 19, it is easy to see that $(x, w) \in X$ if and only if $x \in \overline{F_{w_{[0,i)}}(V_{w_i}) \cap F_{w_{[0,i+1)}}(V_{w_{i+1}})}$ for all i

Denote $P(v) = \bigcap_{i=0}^{n-2} \overline{F_{v_{[0,i)}}(V_{v_i}) \cap F_{v_{[0,i+1)}}(V_{v_{i+1}})}$ for $v \in B^\star$ such that $|v| = n > 1$. For formal reasons, let $P(b) = \overline{V}_b$ for $b \in B$ and $P(\lambda) = \mathbb{T}$ where λ is the empty word.

Rewriting the above observation, $(x, w) \in X$ if and only if $x \in \bigcap_{i=0}^{\infty} P(w_{[0,i)})$, and therefore

$$X = \bigcap_{k=0}^{\infty} \bigcup_{|v|=k} P(v) \times [v].$$

As the set $\bigcup_{|v|=k} P(v) \times [v]$ is closed for every k, we have that X is a closed (and hence compact) subset of $\mathbb{T} \times B^\omega$.

Observation 20. *If $(x, w) \in X$, then $F^{-1}_{w_{[k,k+1]}}$ is not a rotation for any k.*

Proof. From (1) and (2) we obtain that the open set $W = V_{w_k} \cap F_{w_k}(V_{w_{k+1}})$ is nonempty. Now it is easy to see that $F^{-1}_{w_{[k,k+1]}}$ is an expansion on W and therefore $F^{-1}_{w_{[k,k+1]}}$ cannot be a rotation. □

We are now ready to improve Theorem 16.

Theorem 21. *Let $\{F_a : a \in A\}$ be a Möbius iterative system. Assume that there exists a finite subset B of A^+ such that $\{\overline{V}_b : b \in B\}$ covers $\mathbb{T}$. Then there exists a subshift $\Sigma \subset A^\omega$ that, together with the iterative system $\{F_a : a \in A\}$, forms a Möbius number system.*

Proof. Take B as our new alphabet. We will show that there exists $\Sigma \subset B^\omega$ with the required properties. Once we have Σ, we can define the subshift $\Sigma' \subset A^\omega$ as

$\Sigma' = \bigcup_{i=0}^{d} \sigma^d(\Sigma)$ where $d = \max\{|b| : b \in B\}$. It is easy to see that if Σ is a Möbius number system then Σ' is also a Möbius number system.

Consider the set $X \subset \mathbb{T} \times B^\omega$ introduced above. We have already shown that X is closed and therefore compact. Taking projections π_1, π_2 of X to first and second elements, we obtain the set of points $\pi_1(X) \subset \mathbb{T}$ and the set of words $\pi_2(X) \subset B^\omega$. To conclude our proof, we verify that:

(i) $\pi_1(X) = \mathbb{T}$.

(ii) $\pi_2(X) = \Sigma$ is a subshift of B^ω.

(iii) For $(x, w) \in X$ we have $\Phi(w) = x$.

(iv) The map $P(v)$ together with Σ forms a symbolic representation of $\mathbb{T}$.

(v) $\Phi_{|\Sigma}$ is continuous.

(i) This follows from the fact that the system $\{\overline{V}_b : b \in B\}$ covers $\mathbb{T}$. Given $x \in \mathbb{T}$ we can construct $w \in B^\omega$ with $(x, w) \in X$ by induction.

(ii) As X is compact and π_2 is continuous, Σ is compact. To show σ-invariance, consider $(x, w) \in X$. Then it is easy to see that $(F_{w_0}^{-1}(x), \sigma(w)) \in X$.

(iii) Let $(x, w) \in X$. Recall the notation $x_i = F_{w_{[0,i)}}^{-1}(x)$. We will divide the proof into several cases.

Assume first that x_i is an endpoint of $\overline{V}_{w_i}$ for all i. Then either x_i is always the clockwise endpoint or always the counterclockwise endpoint of $\overline{V}_{w_i}$. Assume without loss of generality that $x_i = e_i^+$ and denote $l = \min\{|V_b| : b \in B\}$.

We will use Lemma 2. Let $I \subset \overline{V}_{w_0}$ be a nondegenerate closed interval with x as the counterclockwise endpoint. We want to show that $\liminf_{i\to\infty} |F_{w_{[0,i)}}^{-1}(I)| \geq l$, that is, there exists j such that $i > j \Rightarrow |F_{w_{[0,i)}}^{-1}(I)| \geq l$. This will be enough to prove $\Phi(w) = x$ since for every J open interval, $x \in J$, there exists a suitable $I \subset J$.

Observe first that if $|F_{w_{[0,i)}}^{-1}(I)| \geq l$ for some i then $|F_{w_{[0,i)}}^{-1}(I) \cap \overline{V}_{w_i}| \geq l$ and, because $F_{w_i}^{-1}$ expands all intervals in $\overline{V}_{w_i}$, we have $|F_{w_{[0,i+1)}}^{-1}(I)| > l$. All we have to show is that there exists i such that the length $|F_{w_{[0,i)}}^{-1}(I)|$ is at least l. But this follows from Lemma 18; all we have to do is choose $x \neq y \in I$ and observe that $F_{w_{[k,k+1]}}^{-1}$ is not a rotation for any k by Observation 20. Therefore, in this case we have $\Phi(w) = x$.

If there exists k such that x_i is an endpoint of V_{w_i} for all $i \geq k$ then $\Phi(\sigma^k(w)) = F_{w_{[0,k)}}^{-1}(x)$ by the above argument, and so $\Phi(w) = x$, which was to be proven.

It remains to consider the case when x_i is not endpoint for infinitely many values of i. We want to show that then $\lim_{i\to\infty} (F_{w_{[0,i)}}^{-1})^{\bullet}(x) = \infty$. This will be enough to finish the proof, thanks to Lemma 11.

First of all, note that there exists a number $\xi > 0$ with the following property: For all b, the intervals $\left(F_b^{-1}(e_b^-), F_b^{-1}(e_b^- + \xi) + \xi\right)$ and $\left(F_b^{-1}(e_b^+ - \xi) - \xi, F_b^{-1}(e_b^+)\right)$ do not contain any endpoint e_c^-, e_c^+ of any interval V_c (where $c \in B$, see Figure 4). The existence of ξ follows from the fact that the set of all endpoints of all intervals

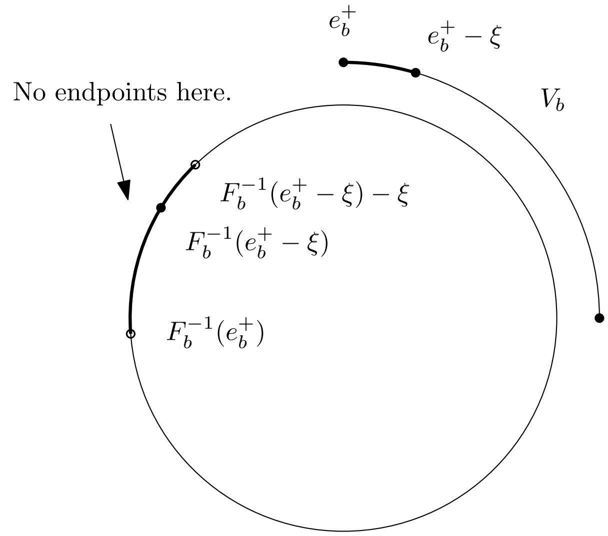

Figure 4: The condition for ξ

$\{V_c : c \in B\}$ is finite. We find such a ξ_b for every $b \in B$ and then choose the minimum ξ_b as the global ξ.

Observe that

$$(F^{-1}_{w_{[0,i)}})^\bullet(x) = \prod_{k=0}^{i-1} (F^{-1}_{w_k})^\bullet(F^{-1}_{w_{[0,k)}}(x)) = \prod_{k=0}^{i-1} (F^{-1}_{w_k})^\bullet(x_k). \qquad (\heartsuit)$$

As $x_k \in \overline{V}_{w_k}$, we have that $(F^{-1}_{w_k})^\bullet(x_k) \geq 1$. Moreover, there exists a $\delta > 0$ such that $(F_b^{-1})^\bullet(y) > 1 + \delta$ whenever $y \in [e_b^- + \xi, e_b^+ - \xi]$.

Therefore, the only way the limit of the product ($\heartsuit$) can be finite is when there exists an i_0 such that

$$\rho(x_i, \{e^-_{w_i}, e^+_{w_i}\}) < \xi$$

for all $i \geq i_0$. But this leads to a contradiction: Assume (without loss of generality) $\rho(x_{i_0}, e^+_{w_{i_0}}) < \xi$. Then we have $\rho(x_{i_0+1}, e^-_{w_{i_0+1}}) \geq \xi$ (because $e^-_{w_{i_0+1}} \notin [x_{i_0+1}, x_{i_0+1} + \xi)$ by the choice of ξ) and so $\rho(x_{i_0+1}, e^+_{w_{i_0+1}}) < \xi$. By induction, it is then true that $\rho(x_i, e^+_{w_i}) < \xi$ for all $i \geq i_0$ and we obtain that $F^{-1}_{w_{[i_0,i)}}(e^+_{w_{i_0}}) \in \overline{V}_{w_i}$ (because by the choice of ξ, we have $e^+_{w_{i_0+1}} \notin (F^{-1}_{w_{[i_0,i)}}(e^+_{w_{i_0}}), x_{i_0+1}]$, see Figure 5), which is a contradiction with Lemma 18 and Observation 20.

(iv) All we need to show is that $P(w) = \bigcap_{i=0}^{\infty} P(w_{[0,i)})$ is a singleton for every $w \in \Sigma$. We know that when $(x, w) \in X$ then $x \in P(w)$. Let $y \in P(w)$. Then for all k we have $F^{-1}_{w_{[0,k)}}(\{x, y\}) \in \overline{V}_{w_k}$ and applying Lemma 18 together with Observation 20 we obtain $x = y$, which is what we needed.

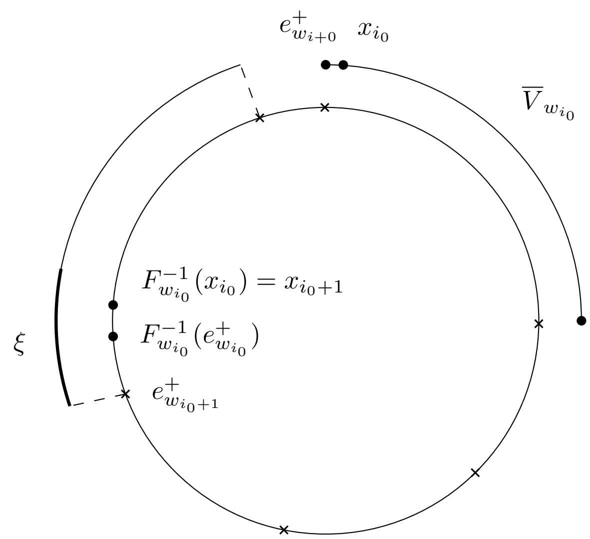

Figure 5: The situation brought about by the properties of ξ

(v) Follows from the previous point and Lemma 13. □

From the proof of Theorem 21, we obtain that there are alternative subshifts that also produce a Möbius number system.

Assume that we replace all the sets $V_b, b \in B$ with some smaller open intervals $W_b, b \in B$ satisfying $W_b \subset V_b$ and $\bigcup_{b\in B} \overline{W}_b = \mathbb{T}$. Then we can define a Möbius number system derived from the set $X' \subset \mathbb{T} \times A^\omega$.

Let $(x, w) \in X'$ if and only if:

(3) For all $i = 1, 2, \ldots$ we have $x_i \in \overline{W}_{w_i}$.

(4) There are no i, j indices such that x_i is the counterclockwise endpoint of $\overline{W}_{w_i}$ and x_j is the clockwise endpoint of $\overline{W}_{w_j}$.

Note that the new conditions are at least as restrictive as the original conditions (1) and (2), therefore $X' \subset X$.

The set X' is closed, which follows from the observation that $(x, w) \in X'$ if and only if

$$\forall k,\ x \in \overline{\bigcap_{i=0}^{k} F_{w_{[0,i)}}(W_{w_i})}$$

and therefore

$$X' = \bigcap_{n=0}^{\infty} \bigcup_{|v|=n} \left[\overline{\bigcap_{i=0}^{n-1} F_{v_{[0,i)}}(W_{v_i})} \times [v] \right].$$

To see that $\pi_1(X') = \mathbb{T}$, consider $x \in \mathbb{T}$. Then there exists w_0 with $x \in \overline{W}_{w_0}$ such that x is not the clockwise endpoint of $\overline{W}_{w_0}$. Similarly, there always is a set $\overline{W}_{w_1}$ such that $F_{w_0}^{-1}(x) \in \overline{W}_{w_1}$ and $F_{w_0}^{-1}(x)$ is not the clockwise endpoint of $\overline{W}_{w_1}$. We continue in this manner, producing $w_2, w_3, \ldots$ until we have the whole word w.

By the same arguments as in proof of Theorem 21, we then obtain that $\Sigma' = \pi_2(X')$ is a subshift. And $X' \subset X$ gives us that $\Phi_{|\Sigma'}$ is continuous and $\Phi(w) = x$ for $(x, w) \in X'$. We conclude that Σ' together with $\{F_b : b \in B\}$ forms a Möbius number system.

Corollary 22. *Let $\{W_b : b \in B\}$ be a set of open intervals such that $W_b \subset V_b$ for every b and $\bigcup_{b \in B} \overline{W}_b = \mathbb{T}$. Let $\Sigma_{\mathcal{W}} \subset B^\omega$ be a subshift such that $w \in \Sigma_{\mathcal{W}}$ if and only if*

$$\forall k, \bigcap_{i=0}^{k} F_{w_{[0,i)}}(W_{w_i}) \neq \emptyset.$$

Then $\Sigma_{\mathcal{W}}$ together with $\{F_b : b \in B\}$ is a Möbius number system.

Proof. It suffices to show that $\Sigma_{\mathcal{W}} = \Sigma'$.

Let $w \in B^\omega$ and denote $E_k = \overline{\bigcap_{i=0}^{k} F_{w_{[0,i)}}(W_{w_i})}$. By compactness of $\mathbb{T}$, $w \in \Sigma_{\mathcal{W}}$ if and only if there exists $x \in \bigcap_{k=0}^{\infty} E_k$. But from Lemma 19, we obtain that $x \in \bigcap_{k=0}^{\infty} E_k$ if and only if $(x, w) \in X'$. So

$$w \in \Sigma_{\mathcal{W}} \Leftrightarrow \exists x, \ (x, w) \in X',$$

which means that $\Sigma_{\mathcal{W}} = \pi_2(X') = \Sigma'$. □

5. Sofic Representations

A subshift Σ is called *sofic* if and only if the language of Σ is regular (recognizable by a finite automaton). Papers [2] and [3] contain several examples of such subshifts.

In the following, we offer a tool for testing whether a Möbius number system is sofic. The tool is far from universal, but should cover most interesting systems.

Define the *follower set* of a word v as $\mathcal{F}_v = \{w \in \Sigma : vw \in \Sigma\}$ and denote the image of $\mathcal{F}_v$ by $Z_v = \Phi(\mathcal{F}_v)$.

Observation 23. *Let Σ be a Möbius number system such that Σ is sofic. Then $\{Z_v : v \in A^\star\}$ is a finite set.*

Proof. It is a well-known fact that a subshift is sofic if and only if $\{\mathcal{F}_v : v \in A^\star\}$ is finite (see [5]). Therefore, if Σ is sofic then $\{Z_v : v \in A^\star\} = \{\Phi(\mathcal{F}_v) : v \in A^\star\}$ is finite. □

We state the converse statement only for one special class of Möbius number systems. We shall see in a moment that the shift from Theorem 21 is precisely such a shift.

Theorem 24. *Let t be a positive integer. Let $X \subset \mathbb{T} \times A^\omega$ be a set such that for all $x \in \mathbb{T}, v \in A^\star$ with $|v| \geq t$ we have the equivalence*

$$\exists w \in A^\omega,\ (x, vw) \in X \Leftrightarrow \left[\forall k = 0, 1, \ldots, |v| - t,\ \tau\left(v_{[k,k+t)}, F^{-1}_{v_{[0,k)}}(x)\right)\right]$$

where τ is some predicate (the same condition for all k). Assume moreover that $\Phi(w) = x$ for all $(x, w) \in X$. Then $\mathcal{L}(\pi_2(X))$ is regular if and only if $\{Z_v : v \in A^\star\}$ is finite.

Proof. The "only if" direction of the claim was proved in Observation 23.

To prove the other direction, assume that X satisfies all the requirements. We construct a finite automaton recognizing the language of Σ.

The states of our automaton will be $T_v = \{(Z_v, v_{[|v|-t,|v|)}) : v \in A^\star\}$. The transitions are of the form "$T_v \xrightarrow{a} T_{va}$". Designate (Z_λ, λ) as the initial state and let all states be accepting except for states of the form $(\emptyset, u)$. As $v \in \mathcal{L}(\Sigma) \Leftrightarrow Z_v \neq \emptyset$, our automaton indeed recognizes $\mathcal{L}(\Sigma)$.

To complete the proof, we need to show that the automaton is defined correctly, i.e., if $T_v = T_u$ then $T_{va} = T_{ua}$. We do this by showing that $x \in Z_{va}$ if and only if

$$x \in F_a^{-1}(Z_v)\ \&\ \tau(v_{[|v|+1-t,|v|)}a, x).$$

This will be enough, as then T_{va} is a function of T_v. We can assume that $|v| > t$, as we can add all the finitely many short words as special states.

By rewriting the conditions, we obtain:

$$\begin{aligned}
Z_v &= \left\{x : \exists w,\ (F_v(x), vw) \in X\right\},\\
Z_v &= \left\{x : \forall k = 0, 1, \ldots, |v| - t,\ \tau\left(v_{[k,k+t)}, F^{-1}_{v_{[0,k)}}(F_v(x))\right)\right\},\\
Z_{va} &= \left\{x : \forall k = 0, 1, \ldots, |v| - t + 1,\ \tau\left((va)_{[k,k+t)}, F^{-1}_{va_{[0,k)}}(F_v F_a(x))\right)\right\}.
\end{aligned}$$

A simple inspection of the conditions for $x \in Z_{va}$ shows that we can write them in the form:

$$\begin{aligned}
Z_{va} &= \left\{x : F_a(x) \in Z_v\ \&\ \tau\left(va_{[|v|+1-t,|v|+1)}, F^{-1}_{va}(F_v F_a(x))\right)\right\},\\
Z_{va} &= \left\{x : F_a(x) \in Z_v\ \&\ \tau\left(v_{[|v|+1-t,|v|)}a, x\right)\right\}.
\end{aligned}$$

Therefore, Z_{va} indeed depends only on Z_v and the word $v_{[|v|+1-t,|v|)}a$. $\square$

As a sample application of this theorem, we prove a criterion for soficity of the Möbius number system introduced in Theorem 21.

Corollary 25. *Assume that* Σ *is the Möbius number system set up in Theorem 21. Then the shift* Σ *is sofic if and only if* $\{F_v^{-1}(P(v)) : v \in A^\star\}$ *is a finite set.*

Proof. We can assume that $B = A$, as the transition from subshift of B^ω to a subshift of A^ω does not affect soficity due to B being finite.

We know that $(x, w) \in X$ if and only if $x \in \bigcap_{i=0}^{\infty} P(w_{[0,i)})$. Moreover, given v, we can find for each $x \in P(v)$ a w such that $(x, vw) \in X$ because $\{\overline{V}_a : a \in A\}$ covers $\mathbb{T}$. Therefore $Z_v = F_v^{-1}(P(v))$. Setting $t = 1$ and $\tau(u, y)$ equal to "$y \in P(u)$" satisfies the requirements of Theorem 24 and so Σ is sofic if and only if $\{Z_v : v \in A^\star\} = \{F_v^{-1}(P(v)) : v \in A^\star\}$ is finite. □

Recall that after Theorem 21, we have introduced an additional Möbius number system. This shift Σ' would require extra effort, due to the nonlocal character of (3) and (4).

On the other hand, it is easy to see that if we take some system of open intervals $\{V'_b : b \in B\}$ such that $V'_b \subset V_b$ and $\{\overline{V'}_b : b \in B\}$ covers $\mathbb{T}$ and write conditions (1') and (2') with V_b replaced by V'_b then the proof of Theorem 21 as well as of Corollary 25 goes verbatim.

Therefore, one strategy to produce sofic Möbius number systems could be to find some cover of $\mathbb{T}$, say $\{\overline{V}_b : b \in B\}$, and then try to adjust the sets $\{V'_b : b \in B\}$ to obtain a system such that $\{Z'_v : v \in A^\star\}$ is finite and so the resulting shift is sofic. At present, we are not able to show any concrete examples of sofic systems obtained in this way.

6. Conclusions

In the whole paper we have built upon the results of [2] and [3]. In particular, Theorem 21 narrows the gap between the Möbius iterative systems that admit a Möbius number system and the iterative systems that do not. However, there are quite a few open problems, practical as well as theoretical, in this area.

Most obviously, we would like to have a sufficient and necessary condition for the existence of a Möbius number system for a given iterative system.

Having such a characterization, we would like to know if it is effective, i.e., given a Möbius iterative system with coefficients in some suitable computable field, can we decide whether there is a corresponding Möbius number system?

For manipulation with number systems, it would be nice to have a sofic Möbius number system. There are several known examples of such systems, but we do not know whether existence of a Möbius number system for a given iterative system implies the existence of a sofic system or not. Neither can we at the moment tell if a given set of MTs admits a sofic Möbius number system.

We have offered a tool to bring light to the sofic problem in Theorem 24. Unfortunately, our result does not work for all systems and requires us to generate possibly infinitely many sets Z_v (although the generation process eventually stops if the system in question is sofic).

A large part the complexity of these problems seems to come not from the number systems themselves but from the fact that we don't properly understand how do large numbers of MTs compose (or, equivalently, how long sequences of matrices multiply). This suggests that maybe the way forward lies in studying the limits of products of matrices.

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