ON UNIVERSAL BINARY HERMITIAN FORMS

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Abstract

Earnest and Khosravani, Iwabuchi, and Kim and Park recently gave a complete classification of the universal binary Hermitian forms. We give a unified proof of the universalities of these Hermitian forms, relying upon Ramanujan's list of universal quadratic forms and the Bhargava-Hanke 290-Theorem. Our methods bypass the *ad hoc* arguments required in the original classification.

1. Introduction

The question of representing integers by quadratic forms dates back to the time of Fermat, whose *Two Squares Theorem* solved the question of which primes could be represented by the form $x^2 + y^2$ (see [6, p. 219]). This theorem was later generalized by Lagrange, who showed in his *Four Squares Theorem* [11] that every positive integer can be written as a sum of four squares of integers.

Lagrange's theorem has led to the modern study of *universal forms*, those forms which represent all positive integers. In the first half of the twentieth century, Ramanujan [13] identified the universal positive-definite classically integral quaternary diagonal quadratic forms, up to equivalence. Maass [12] and Chan, Kim, and Raghavan [3] gave analogous classification results leading to the full classification of the positive-definite classically integral ternary quadratic forms which are universal over real quadratic fields.

Motivated by the work on universal quadratic forms over real fields, Earnest and Khosravani [5] sought a classification of universal binary Hermitian forms over imaginary quadratic fields. Recently, Iwabuchi [7] and Kim and Park [10] finished Earnest and Khosravani's program, completing the list of universal binary Hermitian forms.

A different direction of recent research has focused on the search for *universality criteria*, simple tests which characterize the universality of positive-definite quadratic forms. The earliest-discovered result in this vein is Conway and Schneeberger's surprising *15-Theorem* (see [4] for statement and history and [1] for a proof):

15-Theorem. *A positive-definite classically integral quadratic form is universal if and only if it represents the nine "critical numbers"*

$$\{1, 2, 3, 5, 6, 7, 10, 14, 15\}.$$

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More recently, Bhargava and Hanke [2] showed an analogous criterion for the universality of positive-definite nonclassically integral quadratic forms:

290-Theorem. *A positive-definite nonclassically integral quadratic form is universal if and only if it represents the numbers*

$$S_{290} = \{1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21, 22, 23, 26, \\ 29, 30, 31, 34, 35, 37, 42, 58, 93, 110, 145, 203, 290\}.$$

While the criterion theorems reduce testing a form's universality to a simple computation, they have rarely been applied in practice. The reason for this somewhat curious fact is that the proofs of both the 15- and 290-Theorems rely on independent identification of many universal forms of low rank, called the *universal escalators*.

The results on Hermitian forms, however, give us a chance to greatly simplify prior work through an application of the 290-Theorem. Specifically, we apply the 290-Theorem to reduce the most difficult universality verifications in the classification of universal binary Hermitian forms to simple, finite computations.

2. Preliminaries

We let E be an imaginary quadratic field over $\mathbb{Q}$ and let $m > 0$ be a squarefree integer for which $E = \mathbb{Q}(\sqrt{-m})$. We denote the $\mathbb{Q}$-involution of E by $^-$ and the ring of integers of E by $\mathcal{O}_E$.

We let V/E be an n-dimensional Hermitian space over E with nondegenerate Hermitian form H. As shown by Jacobson [8], we may consider (V, H) as a $2n$-dimensional quadratic space $(\widetilde{V}, B)$ with the bilinear form B defined by the trace map

$$B(v, w) = \frac{1}{2}\mathrm{Tr}_{E/\mathbb{Q}}(H(v, w)).$$

An $\mathcal{O}_E$-lattice L is a finitely generated $\mathcal{O}_E$-module on the Hermitian space (V, H). We consider only positive-definite integral $\mathcal{O}_E$-lattices L, that is, those for which $H(v, w) \in \mathcal{O}_E$ for all $v, w \in L$ and $H(v, v) > 0$ for all $L \ni v \neq 0$. If an $\mathcal{O}_E$-lattice L is of the form $L = L_1 \oplus L_2$ for sublattices L_1, L_2 of L with $H(v_1, v_2) = 0$ for all $v_1 \in L_1$ and $v_2 \in L_2$, then we write $L \cong L_1 \perp L_2$.

When E has class number 1, the ring $\mathcal{O}_E$ is a principal ideal domain whereby every $\mathcal{O}_E$-lattice L is free. In this case, we may think of the Hermitian form H acting on L as a function $f : \mathcal{O}_E^n \to \mathbb{Z}$ defined by

$$f(x_1, \ldots, x_n) = H\left(\sum_{i=1}^{n} x_i v_i, \sum_{i=1}^{n} x_i v_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} H(v_i, v_j) x_i \bar{x}_j$$

for some suitable basis $\{v_i\}_{i=1}^n$ of L. If the basis $\{v_i\}_{i=1}^n$ is orthogonal, we write $L \cong \langle H(v_1), \ldots, H(v_n)\rangle$. (For example, the form $x\bar{x} + 2y\bar{y}$ is associated to the lattice $\langle 1, 2\rangle$.)

Similarly, we may associate a quadratic lattice $\widetilde{L}$ with every Hermitian $\mathcal{O}_E$-lattice L. The ring $\mathcal{O}_E$ has a basis $\{1, \omega_m\}$ as a $\mathbb{Z}$-module, where

$$\omega_m = \begin{cases} \frac{1+\sqrt{-m}}{2}, & m \equiv 3 \bmod 4, \\ \sqrt{-m}, & \text{otherwise.} \end{cases}$$

Then, $\tilde{f}(x_1, y_1, \ldots, x_n, y_n) = f(x_1 + \omega_m y_1, \ldots, x_n + \omega_m y_n)$ is a quadratic form in $2n$ variables corresponding to the lattice $\widetilde{L}$. From this construction, it is clear that the Hermitian form f is universal if and only if the quadratic form $\tilde{f}$ is. We write $\sim$ to denote the correspondence between a Hermitian lattice and its associated quadratic form.

3. Classification of Universal Hermitian Forms

Earnest and Khosravani [5], Iwabuchi [7], and Kim and Park [10] identified all potentially universal Hermitian forms over imaginary quadratic fields. This "screening process" is the more straightforward part of the classification, relying on a uniform computational method (see [5]).

The universality of the candidates identified was then shown by a variety of methods. Indeed, a total of eight different approaches were used. Six of these methods were "*ad hoc*" arguments, each an intricate method developed to prove the universality of an individual Hermitian form.

We give a unified proof of the universalities of the forms in the classification, relying upon Ramanujan's list of universal forms [13] and the 290-Theorem [2]. The universalities of the twenty-five universal binary Hermitian forms follow directly from our methods.

Main Theorem. *Up to equivalence, the integral positive-definite universal binary Hermitian lattices in imaginary quadratic fields are exactly the lattices in* (1)*:*

$$\begin{array}{c|l} \mathbb{Q}(\sqrt{-m}) & \textit{universal binary lattices} \\ \hline \mathbb{Q}(\sqrt{-1}) & \langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \\ \mathbb{Q}(\sqrt{-2}) & \langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \langle 1,4\rangle, \langle 1,5\rangle, \\ \mathbb{Q}(\sqrt{-3}) & \langle 1,1\rangle, \langle 1,2\rangle, \\ \mathbb{Q}(\sqrt{-5}) & \langle 1,2\rangle, \langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_5 \\ -1+\bar{\omega}_5 & 3 \end{pmatrix}, \\ \mathbb{Q}(\sqrt{-6}) & \langle 1\rangle \perp \begin{pmatrix} 2 & \omega_6 \\ \bar{\omega}_6 & 3 \end{pmatrix}, \\ \mathbb{Q}(\sqrt{-7}) & \langle 1,1\rangle, \langle 1,2\rangle, \langle 1,3\rangle, \\ \mathbb{Q}(\sqrt{-10}) & \langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{10} \\ \bar{\omega}_{10} & 5 \end{pmatrix}, \\ \mathbb{Q}(\sqrt{-11}) & \langle 1,1\rangle, \langle 1,2\rangle, \\ \mathbb{Q}(\sqrt{-15}) & \langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{15} \\ \bar{\omega}_{15} & 2 \end{pmatrix}, \\ \mathbb{Q}(\sqrt{-19}) & \langle 1,2\rangle, \\ \mathbb{Q}(\sqrt{-23}) & \langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{23} \\ \bar{\omega}_{23} & 3 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{23} \\ -1+\bar{\omega}_{23} & 3 \end{pmatrix}, \\ \mathbb{Q}(\sqrt{-31}) & \langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{31} \\ \bar{\omega}_{31} & 4 \end{pmatrix}, \langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{31} \\ -1+\bar{\omega}_{31} & 4 \end{pmatrix}. \end{array} \tag{1}$$

Proof. Earnest and Khosravani [5], Iwabuchi [7], and Kim and Park [10] showed that no binary Hermitian forms not in the list (1) can be universal over an imaginary quadratic field E. Therefore, we must only show the universality of each of these candidate forms.

First, we identify the diagonal lattices in the list (1) which correspond to diagonal quaternary quadratic forms:

$$\begin{array}{lcl} \langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-1}) & \sim & w^2+x^2+y^2+z^2, \\ \langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) & \sim & w^2+x^2+2y^2+2z^2, \\ \langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-1}) & \sim & w^2+x^2+2y^2+2z^2, \\ \langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) & \sim & w^2+2x^2+2y^2+4z^2, \\ \langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-5}) & \sim & w^2+2x^2+5y^2+10z^2, \\ \langle 1,3\rangle \text{ in } \mathbb{Q}(\sqrt{-1}) & \sim & w^2+x^2+3y^2+3z^2, \\ \langle 1,3\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) & \sim & w^2+3x^2+3y^2+6z^2, \\ \langle 1,4\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) & \sim & w^2+2x^2+4y^2+8z^2, \\ \langle 1,5\rangle \text{ in } \mathbb{Q}(\sqrt{-2}) & \sim & w^2+2x^2+5y^2+10z^2. \end{array} \tag{2}$$

The universality of each of the forms on the right-hand side of (2) was shown by Ramanujan [13]. Thus, we have the universality of the Hermitian forms on the left-hand side of (2) immediately.

This leaves only eight other diagonal Hermitian lattices in (1),

$$\begin{array}{lcl}
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-3}) & \sim & w^2+wx+x^2+y^2+yz+z^2, \\
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-7}) & \sim & w^2+wx+2x^2+y^2+yz+2z^2, \\
\langle 1,1\rangle \text{ in } \mathbb{Q}(\sqrt{-11}) & \sim & w^2+wx+3x^2+y^2+yz+3z^2, \\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-3}) & \sim & w^2+wx+x^2+2y^2+2yz+2z^2, \\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-7}) & \sim & w^2+wx+2x^2+2y^2+2yz+4z^2, \\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-11}) & \sim & w^2+wx+3x^2+2y^2+2yz+6z^2, \\
\langle 1,2\rangle \text{ in } \mathbb{Q}(\sqrt{-19}) & \sim & w^2+wx+5x^2+2y^2+2yz+10z^2, \\
\langle 1,3\rangle \text{ in } \mathbb{Q}(\sqrt{-7}) & \sim & w^2+wx+2x^2+3y^2+3yz+6z^2.
\end{array} \tag{3}$$

We may invoke the 290-Theorem to show the universality of the eight quadratic forms in (3); the check that each of these forms represents all of S_{290} is an easy computation. It then follows directly that the eight Hermitian forms in (3) are all universal.

Now, we turn to the non-diagonal Hermitian lattices in (1). These are the remaining eight lattices,

$$\begin{array}{rcl}
\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_5 \\ -1+\bar{\omega}_5 & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-5}) & \sim & w^2+2x^2+2xy+3y^2+5z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_6 \\ \bar{\omega}_6 & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-6}) & \sim & w^2+2x^2+3y^2+6z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{10} \\ \bar{\omega}_{10} & 5 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-10}) & \sim & w^2+2x^2+3y^2+10z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{15} \\ \bar{\omega}_{15} & 2 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-15}) & \sim & w^2+2x^2+xy+2y^2+wz+4z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{23} \\ \bar{\omega}_{23} & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-23}) & \sim & w^2+2x^2+xy+3y^2+wz+6z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{23} \\ -1+\bar{\omega}_{23} & 3 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-23}) & \sim & w^2+2x^2+xy+3y^2+wz+6z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & \omega_{31} \\ \bar{\omega}_{31} & 4 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-31}) & \sim & w^2+2x^2+xy+4y^2+wz+8z^2, \\
\langle 1\rangle \perp \begin{pmatrix} 2 & -1+\omega_{31} \\ -1+\bar{\omega}_{31} & 4 \end{pmatrix} \text{ in } \mathbb{Q}(\sqrt{-31}) & \sim & w^2+2x^2+xy+4y^2+wz+8z^2.
\end{array} \tag{4}$$

Now, all of the diagonal quadratic forms in (4) are found in the list of universal forms obtained by Ramanujan [13]. Furthermore, the universalities of the non-diagonal quadratic forms in (4) follow from the 290-Theorem. It then follows immediately that all the Hermitian forms in (4) are universal. □

4. Remarks

Kim, Kim, and Park [9] have recently announced a criterion which completely characterizes the universality of Hermitian forms.

15-Theorem for Hermitian Lattices. *A positive-definite integral Hermitian form is universal if and only if it represents the ten integers* $\{1, 2, 3, 5, 6, 7, 10, 13, 14, 15\}$.

Unfortunately, the proof of this result cites the original proof of the classification of universal binary Hermitian forms. Consequently, Kim, Kim, and Park's 15-Theorem for Hermitian Lattices cannot give a direct, unified proof of our Main Theorem.

Kim, Kim, and Park note that the 290-Theorem can be used to simplify some of the arguments in their proof of the 15-Theorem for Hermitian Lattices. Such simplifications would take the same form as those we have presented here to unify the classification of universal binary Hermitian forms.

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