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SOME OBSERVATIONS AND SOLUTIONS TO SHORT AND LONG GLOBAL NIM

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Abstract

Short global and long global nim are variations on the rules of the classic game Nim. In short global nim, the move previously played cannot be immediately mirrored by the other player in any pile. In long global Nim, any move played in the game cannot be repeated for the rest of the game. The $\mathcal{N}$ and $\mathcal{P}$ positions are explored in this paper, as well as the Grundy numbers of positions in both games.

1. Introduction

Short global and long global nim are listed under Problem 22 in Guy and Nowakowski's list of unsolved problems in combinatorial game theory [1]. Both of these are variations on the classic game of nim. In short global nim, a move by the previous player may not be immediately repeated by the next player. In long global nim, any move made cannot be repeated in that game. So, for example, if one stone is taken from a pile during the game, thereafter no one can take one stone from any pile. The term 'global' refers to the fact that these rules apply to all piles. The terms 'short' and 'long' refer to how long the restrictions last. The game short global nim is also known as D.U.D.E., as it can be viewed as a variant of D.U.D.E.N.E.Y [2].

Because of the restrictions, the typical analysis using nim addition does not work to solve either game. We will analyze both of these games in search of the $\mathcal{N}$ and $\mathcal{P}$ positions for games of any number of piles of any size. We will also look at the Grundy numbers for selected positions in these games.

2. $\mathcal{N}$ and $\mathcal{P}$ Positions for Short Global Nim

Definition 1 In the short global nim game $(p_1, p_2, \ldots, p_n, k)$, p_i represents the size of the ith pile, and k is the previous move. By convention, $p_1 \leq p_2 \leq \ldots \leq p_n$. For the first move in a game, we let $k = 0$.

We begin small with a game of only one pile of size n, denoted (n, k). Clearly, if $k \neq n$ then the game is an $\mathcal{N}$ position. If $n = k$, then we have Theorem 2.

Theorem 2 For $k \geq 1$, the position (k, k) is $\mathcal{N}$ if and only if k ends in an odd number of zeros in its binary expansion.

Proof. First note that the moves from (k, k) are $(1, k-1), (2, k-2), \ldots, (k-1, 1)$, that is, they have the form (i, j) where $i + j = k$. Note that unless $i = j$, each of these positions is $\mathcal{N}$. Thus, if k is odd, (k, k) is $\mathcal{P}$ (and clearly k ends with an even number of zeros in its binary expansion). If k is even, (k, k) is $\mathcal{N}$ if and only if $(\frac{k}{2}, \frac{k}{2})$ is $\mathcal{P}$.

Therefore, we will show the theorem is true for even integers by induction. The case $k = 2$ is easily shown to be $\mathcal{N}$. Assume the position (i, i), where $i < k$ and i is an even integer, is $\mathcal{N}$ if and only if i ends in an odd number of zeros in its binary expansion. Given an even integer k, if the binary expansion of k ends in an odd number of zeros, the binary expansion of $\frac{k}{2}$ ends in an even number of zeros. Thus, by induction, $(\frac{k}{2}, \frac{k}{2})$ is $\mathcal{P}$ and (k, k) is $\mathcal{N}$. The other case (when the binary expansion of k ends in an even number of zeros) is similar. $\square$

For two pile short global nim, we will find that any game where $k = 1$ and the pile sizes are not equal is a $\mathcal{P}$ position. We will then expand this theorem to account for a game with any number of piles of any size.

Theorem 3 In short global nim, the game $(1, n, 1)$ is a $\mathcal{P}$ position and $(1, n, k)$, $k \neq 1$ is an $\mathcal{N}$ position.

Proof. In the case $n = 0$, we can clearly see that the game $(0, 1, 1)$ is $\mathcal{P}$ and the game $(0, 1, k)$ is $\mathcal{N}$ (when $k \neq 1$). Given n, assume $(1, y, 1)$ is a $\mathcal{P}$ position for all $y < n$, and $(1, y, k)$, $k \neq 1$, is an $\mathcal{N}$ position for all $y < n$.

Take the game $(1, n, 1)$. Given any $j > 1$, assume the next player removes j stones from the larger pile, leaving $(1, n-j, j)$. This is an $\mathcal{N}$ position by our induction hypothesis, so the game $(1, n, 1)$ is a $\mathcal{P}$ position. Take the game $(1, n, k)$, $k \neq 1$. The next player will remove one stone from the larger pile, leaving $(1, n-1, 1)$. This is a $\mathcal{P}$ position by our induction hypothesis, so the game $(1, n, k)$ is an $\mathcal{N}$ position. $\square$

Theorem 4 In short global Nim, when $m \neq n$ and $m \neq 0$, the game $(m, n, 1)$ is a $\mathcal{P}$ position and (m, n, k), $k \neq 1$, is an $\mathcal{N}$ position.

Proof. The game $(1, n, 1)$ is a $\mathcal{P}$ position and $(1, n, k)$, $k \neq 1$, is an $\mathcal{N}$ position by Theorem 3. Given m and n, assume $(x, y, 1)$ is a $\mathcal{P}$ position, for all $0 < x < m$ and $y \geq x$; (x, y, k), $k \neq 1$, is an $\mathcal{N}$ position for all $0 < x < m$ and $y \geq x$; and (m, y, k), $k \neq 1$, is an $\mathcal{N}$ position for all y, $m \leq y < n$.

Take the game $(m, n, 1)$ where $m < n$. Given $j > 1$, assume the first player removes j stones from a pile, leaving $(m - j, n, j)$ or $(m, n - j, j)$. Since $m < n$, $m - j \neq n$. So the game $(m - j, n, j)$, if $m - j > 0$, is an $\mathcal{N}$ position by our induction hypothesis. If $m - j = 0$, $(0, n, j)$ is an $\mathcal{N}$ position, because the next player simply takes n stones. If $n - j > m$, the game $(m, n - j, j)$ is an $\mathcal{N}$ position by our induction hypothesis. If $n - j < m$, the game $(n - j, m, j)$ is an $\mathcal{N}$ position by our induction hypothesis. If $n - j = 0$, $(0, m, j)$ is an $\mathcal{N}$ because the next player takes m stones.

Take the game (m, n, k). The first player will remove one stone from the smaller pile, leaving $(m - 1, n, 1)$. This is a $\mathcal{P}$ position by our hypothesis; therefore, (m, n, k) is an $\mathcal{N}$ position. □

There is a special case where this theorem does not apply. This is the game $(n, n, 1)$, with two equal piles and a last move of 1.

Corollary 5 Given $n \geq 1$, the position $(n, n, 1)$ is $\mathcal{P}$ if and only if $n = 1$ or the binary representation of n ends in an odd number of zeros.

Proof. The case $n = 1$ is obvious. Given $n > 1$, note that the next moves have the form $(i, n, n - i)$ for $0 \leq i \leq n - 2$. All next moves except $(0, n, n)$ are $\mathcal{N}$ by Theorem 4. Thus, $(n, n, 1)$ is $\mathcal{P}$ if and only if $(0, n, n)$ is $\mathcal{N}$. The proof then follows from Theorem 2. □

For the following theorem, let $S = \{(p_1, p_2, \ldots, p_n, k), p_1 \leq p_2 \leq \ldots \leq p_n\}$, the set of all short global nim game positions with n or fewer piles. Define $<$ to be the lexicographical ordering on S, ignoring the last term in the tuples.

Theorem 6 In short global nim with at least three piles, the game $(p_1, p_2, \ldots, p_n, k)$ where $k \neq 1$ is an $\mathcal{N}$ position and the game $(p_1, p_2, \ldots, p_n, 1)$ is a $\mathcal{P}$ position.

Proof. We first consider the three pile case where the smallest pile has size 1. Given the position $(1, p_2, p_3, k)$ where $k \neq 1$, if $p_2 < p_3$, then taking one stone from the smallest pile reduces to the position $(p_2, p_3, 1)$ which is $\mathcal{P}$ by Theorem 4. If $p_2 = p_3$, then taking one stone from the p_2 pile produces the position $(1, p_3 - 1, p_3, 1)$. Since the next player must then produce two nonequal larger piles, the winning play is then to take one stone from the smallest pile, producing an $\mathcal{P}$ position by Theorem 4. (If the next player takes all of one pile, we can cite Theorem 3.) Thus $(1, p_2, p_3, k)$ where $k \neq 1$ is $\mathcal{P}$.

Given the position $(1, p_2, p_3, 1)$, the next player will produce a position of the form

$(1, p_2', p_3', k)$ where $k \neq 1$. Since this is $\mathcal{N}$, the original position is $\mathcal{P}$.

Now letting $n \geq 3$, assume any game with at least three piles $(p_1', p_2', \ldots, p_n', 1) < (p_1, p_2, \ldots, p_n, 1)$ is a $\mathcal{P}$ position and any game with at least three piles $(p_1', p_2', \ldots, p_n', k) < (p_1, p_2, \ldots, p_n, m)$ is an $\mathcal{N}$ position, when $m \neq 1$ and $k \neq 1$.

Take the game $(p_1, p_2, \ldots, p_n, 1)$. If the next player takes j stones, where $j > 1$, from a pile, leaving n piles, by induction, the smaller game is an $\mathcal{N}$ position, and so the original game is a $\mathcal{P}$ position. If the next player removes a pile entirely and we have two unequal piles left, by Theorem 4, the original game is a $\mathcal{P}$ position. If we have two equal piles left, the smaller game is again an $\mathcal{N}$ position since the next player can remove one stone and we can again cite Theorem 4. If we have more than two piles left, we have the same result by induction.

Take the game $(p_1, p_2, \ldots, p_n, k)$ where $k \neq 1$. The next player will take one stone from any pile containing more than one stone. (If all remaining piles contain only one stone, then the next player takes one stone and wins). By induction, we know this game is a $\mathcal{P}$ position, and so the original game must be an $\mathcal{N}$ position. □

Therefore, the winning move in any short global nim game is to remove one stone from the smallest pile, as long as that pile has more than one stone. Following this strategy, the player will never reach a game where all the piles are the same size. These results match unpublished results of Richard Nowakowski and his students [3].

3. Grundy Numbers for Short Global Nim

In this section, we will attempt to find the Grundy numbers of positions for one pile short global nim. We will use the notation $*(n, k)$ to represent the Grundy number of the game (n, k).

As an example, we will consider the Grundy number of the game $(6, 0)$. This game has the options $(0, 6), (1, 5), (2, 4), (3, 3), (4, 2)$, and $(5, 1)$. The first two options have the Grundy numbers 0 and 1, respectively. By Theorem 2, we know $(3, 3)$ is an $\mathcal{P}$ position with a Grundy number of 0. The option $(2, 4)$ has a Grundy number of 1, since its only options are $(1, 1)$ and $(0, 2)$. Finally, the reader can check that $*(4, 2) = 3$ and $*(5, 1) = 3$. Thus $*(6, 0) = mex\{0, 1, 1, 0, 3, 3\} = 2$.

The table below shows computed values of $*(n, k)$ for small values of n and k. To recreate the above calculation on the table, note that $*(n, 0)$ equals the mex of the numbers on the diagonal extending from $(n, 0)$ to the upper right. Similarly, $*(n, k)$ equals the mex of the same set of numbers except that the number on the kth column is deleted.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1
3	2	2	2	0	2	2	2	2	2	2
4	3	2	3	3	0	3	3	3	3	3
5	3	3	3	3	3	0	3	3	3	3
6	2	2	2	2	2	2	2	2	2	2
7	4	4	4	4	4	4	4	0	4	4
8	5	4	5	3	5	5	5	5	5	5
9	5	5	5	5	5	5	5	5	5	5

Table 1: Some Small Values of $*(n, k)$

While we now have a recursive definition for $*(n, k)$, our goal is to describe some of the patterns in the above table. To do so, given a game position $(n, 0)$, we will define the set S_n of all possible next moves as $S_n = \{(0, n), (1, n-1), \ldots, (n-2, 2), (n-1, 1)\}$. Given a set S of game positions, we define $*S$ to be the set of corresponding Grundy numbers. Thus, $*(n, 0) = mex(*S_n)$ and $*(n, k) = mex(*(S_n - \{(n-k, k)\}))$.

Proposition 7 Given any n and k, $*(n, 0) \geq *(n, k)$.

Proof. This follows from the fact that $mex(*S_n) \geq mex(*(S_n - \{(n-k, k)\}))$. □

Proposition 8 If $k > n$, then $*(n, k) = *(n, 0)$.

Proof. If the last move was greater than the remaining pilesize, then the last move cannot affect the Grundy number of the game. □

Proposition 9 Given any n and k, $*(n, 0) = *(n, k)$ if and only if $*(n-k, k) > *(n, 0)$ or $*(n-k, k) = *(n-j, j)$ for some $j \neq k$.

Proof. The only way excluding $(n-k, k)$ will not change the minimum excluded value of S_n is if $*(n-k, k)$ appears elsewhere in the set $*S_n$ (and thus $*(n-k, k) = *(n-j, j)$ for some $j \neq k$) or the minimum excluded value of S_n is less than $*(n-k, k)$ (and thus $*(n-k, k) > *(n, 0)$). □

Corollary 10 If $*(n, 0) \neq *(n, k)$, then $*(n, k) = *(n-k, k)$.

Proof. Note that $*(n-k, k) \neq *(n-j, j)$ for all $j \neq k$ by the above proposition. Thus $*(n-k, k)$ is an excluded value in the set $*(S - \{(n-k, k)\})$ which implies that $*(n-k, k) \geq *(n, k)$. However, if $*(n-k, k) > *(n, k)$, the minimum excluded value in the set $*(S - \{(n-k, k)\})$ is also an excluded value in $*S_n$, which implies that the sets share the same minimum excluded value (and thus $*(n, 0) = *(n, k)$). Thus $*(n-k, k) = *(n, k)$. □

Note that the last proposition implies that all the values in $*S_n$ have the form $*(m, 0)$ for some $m < n$. For the value $*(n-i, i)$ either equals $*(n-i, 0)$ or $*(n-2i, i)$. If it equals the latter, then note that $*(n-2i,i) = *(n-2i,0)$ or $(n-3i,i)$. Eventually, either the pilesize is less than the last move (in which case we use Proposition 7) or we have $*(n-i,i) = *(m,0)$ for some m. We record this information in a corollary.

Corollary 11 For all n, $*(n,0) \leq mex(*(0,0), *(1,0), \ldots, *(n-1,0))$.

To better motivate why (n,k) has a particular Grundy number, we will use an intuition from regular nim. When we say a pile of 4 stones has a Grundy number of $*4$, we imply that the longest the game can last without reaching a $\mathcal{P}$-position is 4 moves. In addition, we also know that the game can last $0,1,2$ and 3 moves as well. We will use this idea of the length of the game to create series that substantiate the Grundy numbers in short global nim.

Definition 12 Given $n \geq 1$, a g-series for (n,k) is a finite series $s_1 + s_2 + \cdots + s_p$ summing to n such that $s_1 \neq k$, and, for all i where $1 \leq i < p$, $*(s_{i+1} + \ldots + s_p, s_i) = p - i$.

For example, the series $1+2+3+1+2+1$ is a g-series for $(10,0)$ because (as can be seen in the above table) $*(9,1) = 5$, $*(7,2) = 4$, $*(4,3) = 3$, $*(3,1) = 2$, and $*(1,2) = 1$. However, $1+2+3$ is not a g-series for $(6,0)$ since $*(5,1) \neq 2$. In fact, the only g-series for $(6,0)$ are 6, $5+1$, $4+2$, $1+2+1+2$, and $2+1+2+1$. That is, the longest g-series for $(6,0)$ where g-series of all shorter lengths exist is the g-series of length 2. After some work, this will explain why $*(6,0) = 2$.

Proposition 13 If $s_1 + \cdots + s_t$ is a g-series for $(n-j,j)$ and $*(n-j,j) = t$, then $j + s_1 + \cdots + s_t$ is a g-series for (n,k) where $k \neq j$.

Proof. Clearly the series sums to n and has a first term not equal to k. Moreover, $*(s_1 + \cdots + s_t, j) = *(n-j,j) = t$. All other necessary conditions are inherited from the fact that $s_1 + \cdots + s_t$ is a g-series for $(n-j,j)$. □

Theorem 14 For $k \neq n$ and $m > 0$, $*(n,k) \geq m$ if and only if there exist g-series for (n,k) of length $1, \ldots, m$.

Proof. If $n = 1$, note that $*(1,k) = 1$ for all $k \neq 1$ and that the one-term series 1 is the only possible g-series. If $n = 2$, then $*(2,k) = 1$ for all $k \neq 2$ and the one-term series 2 is the only g-series.

Assume that for all $r < n$, all $p > 0$, and all $k \neq r$, $*(r,k) \geq p$ if and only if there exist g-series for (r,k) of length $1, \ldots, p$. If we assume $*(n,k) \geq m$, then for all p such that $0 < p < m$ there exists $j \neq k$ such that $*(n-j,j) = p$. Fix p and its corresponding j.

If $j \neq \frac{n}{2}$, then by the induction hypothesis, there exists a g-series of length p for $(n-j,j)$. If $j = \frac{n}{2}$, note that by Corollary 10 either $*(\frac{n}{2}, \frac{n}{2}) = 0$ (in which case $p = 0$ when we assumed $p > 0$) or $*(\frac{n}{2}, \frac{n}{2}) = *(\frac{n}{2}, 0)$. By the induction hypothesis, there exists a g-series of length p for $(\frac{n}{2}, 0)$. Without loss of generality, we may assume that the series does not begin with $\frac{n}{2}$. (If it did, then $p = 1$ and we can always find g-series of length 1 by looking at $(1, n-1)$ instead for $n > 2$.)

Thus for all $p < m$ there exists j and a g-series of length p for $(n-j, j)$. Using the above proposition and adding the appropriate j to the front of each series, for all $p < m$ there exists a g-series of length $p+1$ for (n, k). Since n is a g-series of length 1 for (n, k), we have produced the required g-series.

Conversely, if there exist g-series for (n, k) of length $1, \ldots, m$, choose p such that $2 \leq p \leq m$ and find the corresponding g-series $s_1 + \ldots + s_p$. By the definition of g-series, $*(s_2 + \ldots + s_p, s_1) = *(n - s_1, s_1) = p - 1$. Therefore the set $*(S_n - (n-k, k))$ contains the value $p-1$ for all p such that $2 \leq p \leq m$. Since we know the set contains the value 0 (since $(0, n) = 0$), there is no excluded value less than m. Thus, $*(n, k) \geq m$. $\square$

Corollary 15 If $*(n, k) < *(n, 0)$ and $*(n, k) = p$, then all g-series for $(n, 0)$ of length $p+1$ start with k.

Proof. We already know that there exist g-series of length $1, .., p$ for (n, k). If a g-series of length $p+1$ for $(n, 0)$ exists with a starting term not equal to k, the g-series is also a g-series for (n, k), implying that $*(n, k) \geq p+1$. $\square$

Given our g-series machinery, we can now state some bounds on the sequence $*(n, 0)$.

Proposition 16 $*(n, 0) \leq \begin{cases} \frac{2}{3}n & \text{if } n \bmod 3 = 0 \\ 2\lfloor \frac{n}{3} \rfloor + 1 & \text{otherwise} \end{cases}$

Proof. This is an easy consequence of the fact that the longest possible g-series for $(n, 0)$ either has the form $1 + 2 + 1 + 2 + \ldots$ or $2 + 1 + 2 + 1 + \ldots\ldots$ $\square$

However, the above bound is not a good one for large n. Below, Figure 1 (generated from data from Richard J. Nowakowski [3]) shows that, for large n, $*(n, 0)$ looks much like $\frac{n}{2}$ with some notable exceptions where the sequence is much lower. In fact, for n up to 1000, we have observed the bound that $*(n, 0) \leq \frac{n}{2} + 2$ for n even and $*(n, 0) \leq \frac{n-1}{2} + 3$ for n odd. While we have no proof for this yet, we can prove some facts about how quickly the sequence grows.

Theorem 17 If $*(n, 0) = *(n, k)$, then $*(n+k, 0) \neq *(n, 0)$.

Proof. Let $m = *(n, 0)$. By Theorem 14, we know that there exists a g-series $s_1 + \ldots + s_m$ of length m for (n, k). By Proposition 13, since $*(n+k-k, k) = *(n, 0)$, the series $k + s_1 + \ldots + s_m$ is a g-series of length $m+1$ for $*(n+k, 0)$. Thus, if $*(n+k, 0) \geq m$, we have g-series of length p for $p = 1, \ldots, m+1$ and $*(n+k, 0) \geq m+1$. $\square$

Corollary 18 In the sequence $*(n, 0)$, any value occurs finitely many times.

Proof. This follows easily from the previous theorem and the fact that $*(n, k) = *(n, 0)$ for all $k > n$. $\square$

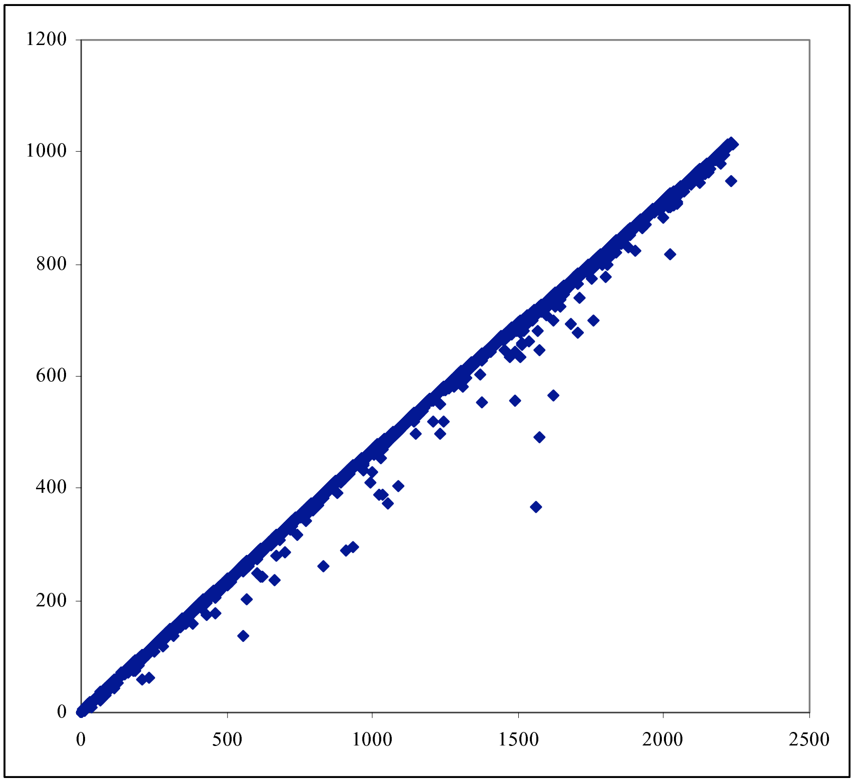

Figure 1: Short global nim Grundy numbers

In fact, we can find an upper limit for the last appearance of a value. For the next proposition, define $f(v)$ to be the least n such that $*(n, 0) = v$.

Proposition 19 For all $m \geq 2f(v) + 1$, $*(m, k) \neq v$.

Proof. Assume $*(m, 0) > v$. If we assume that $*(m, k) = v$, then all g-series of length $v + 1$ for $(m, 0)$ start with k. Thus for all $j < m$ where $j \neq k$, $*(m - j, j) \neq v$. However, consider the value $*(f(v), m - f(v))$. If $m - f(v) \neq k$, by our assumption that $m \geq 2f(v) + 1$, we know that $m - f(v) > f(v)$. Thus, $*(f(v), m - f(v)) = v$, leading to a contradiction. If $m - f(v) = k$, then $m = f(v) + k \geq 2f(v) + 1$, implying $k \geq f(v) + 1$. Thus $*(f(v), m - f(v)) = *(f(v), k) = v$, leading to a contradiction.

If $*(m, 0) = v$, then $*(m, 0) = *(f(v), 0)$ implies that $*(f(v), 0) \neq *(f(v), m - f(v))$. But as above, since $m - f(v) \geq f(v) + 1$, we know $*(f(v), m - f(v)) = *(f(v), 0)$. □

We now move to proving some facts about when the sequence increases.

Proposition 20 If $*(n + 1, 0) > *(n, 0)$, then either $*(n, 0) = *(n, 1)$ or $*(n, 0) \leq *(m, 0)$ for some $m < n$.

Proof. By Theorem 14, there exists a g-series of length $*(n, 0) + 1$ for $(n + 1, 0)$. Denote the g-series by $k + s_1 + \cdots + s_{*(n,0)}$. By the definition of g-series, $*(s_1 + \cdots + s_{*(n,0)}, k) = *(n, 0)$ and thus $*(n - k + 1, k) = *(n, 0)$. If $k = 1$, then $*(n, 1) = *(n, 0)$. Otherwise, $*(n, 0) =$

$*(n-k+1,k) \leq *(n-k+1,0)$ where $n-k+1 < n$. □

Corollary 21 If $*(n+1,0) > *(n,0)$ and $*(n,0)$ is the unique maximum of the set $\{*(0,0), *(1,0), \ldots, *(n,0)\}$, then $*(n+2,0) \leq *(n+1,0)$.

Proof. By the above proposition, we know that $*(n,0) = *(n,1)$. Moreover, by the definition of minimum excluded value, since $*(n+1,0) > *(n,0)$, we know that $*(n+1,0) = *(n,0)+1$. Given any g-series for $(n+1,0)$ of length $*(n,0)+1$, if the first term is k, then $*(n-k+1,k) = *(n,0)$ by the definition of g-series. Since $*(n,0)$ is the unique maximum of the set $\{*(0,0), *(1,0), \ldots, *(n,0)\}$, we know that $k=1$. Thus, any g-series for $(n+1,0)$ of length $*(n,0)+1$ must begin with 1. This implies that $*(n+1,1) < *(n+1,0)$ since there is no g-series of length $*(n,0)+1$ for $(n+1,1)$. By Corollary 10, $*(n+1,1) = *(n+1-1,1) = *(n,0)$.

If $*(n+2,0) > *(n+1,0)$, then there exists a g-series of length $*(n,0)+2$ for $(n+2,0)$. If the g-series is $k + s_1 + \cdots + s_{*(n,0)+1}$, then $*(s_1 + \cdots + s_{*(n,0)+1}, k) = *(n,0)+1 = *(n+1,0)$. So for some k, $*(n-k+2,k) = *(n+1,0)$. If $k=1$, then $*(n+1,1) = *(n+1,0)$ which contradicts our finding above. If $k \neq 1$, then $*(n+1,0) \leq *(n-k+2,0)$ which contradicts the fact that $*(n+1,0)$ is the unique maximum of the set $\{*(0,0), *(1,0), \ldots, *(n+1,0)\}$. Thus $*(n+2,0) \leq *(n+1,0)$. □

Note that the last corollary implies that the sequence $*(0,0), *(1,0), \ldots$ contains at most three consecutive increasing terms.

4. $\mathcal{N}$ and $\mathcal{P}$ Positions for LGN

We now begin a quest to find the $\mathcal{N}$ and $\mathcal{P}$ positions in long global nim. Recall that because no move can be repeated, each time a move is made we remove it from a set of possible moves.

Definition 22 A game of long global nim is described by $[(p_1, \ldots, p_n), S\}]$, where $S = \{s_1, \ldots, s_k\}$, the set of all available moves, and $s_1 < \cdots < s_k$.

All of the following theorems assume that the game is one pile long global nim and the pile has at least one element. If the pile size is an available move, then every game of this type is an $\mathcal{N}$ position. Therefore we will assume that the pile size is never an available move. For Theorem 23, we will let $S_n = \{1, 2, \ldots, n-1\}$, where S_n represents all possible moves except the pile size itself.

Theorem 23 The position $[n, S_n]$ is $\mathcal{N}$ iff n ends in an odd number of zeros in its binary expansion.

Proof. The moves from $[n, S_n]$ have the form $[i, S_n - \{j\}]$, where $i = n-j$. Note that unless $i = j$, all of these positions are $\mathcal{N}$. If n is odd, the position $[n, S_n]$ is $\mathcal{P}$, and n ends in an even number of zeros in its binary expansion. If n is even, the position $[n, S_n]$ is $\mathcal{N}$ iff $[\frac{n}{2}, S_n - \{\frac{n}{2}\}]$ is $\mathcal{P}$.

Therefore we will show the Theorem is true for even integers by induction. Take the case $[2, \{1\}]$. Clearly the next player takes one stone and wins. Given an even integer n, if the binary expansion of n ends with an odd number of zeros, then the binary expansion of $\frac{n}{2}$ ends in an even number of zeros. If $\frac{n}{2}$ is odd, then we know $[\frac{n}{2}, S_n - \{\frac{n}{2}\}]$ is $\mathcal{P}$ by our explanation above. If $\frac{n}{2}$ is even, then by induction we know $[\frac{n}{2}, S_n - \{\frac{n}{2}\}]$ is $\mathcal{P}$, so $[n, S_n]$ is $\mathcal{N}$. □

Theorem 24 Let $S = \{i, j\}$. The position $[n, S]$ is $\mathcal{N}$ iff $i + j > n$, and $\mathcal{P}$ iff $i + j \leq n$.

Proof. This proof is left to the reader. □

Theorem 25 Let $S = \{i, j, k\}$. The position $[n, S]$ is $\mathcal{N}$ if $n - k < i$ or if $n - k \geq i + j$. Otherwise it is a $\mathcal{P}$ position.

Proof. Take the game $[n, S]$. After taking k stones from n, if there are less than i stones remaining in the pile, then the move $n - k$ is a $\mathcal{P}$ position and so $[n, S]$ is $\mathcal{N}$ position. If after taking k stones from n there are more than $i + j$ stones in the pile, then there are two moves remaining and the next player to play will lose. Thus this is a $\mathcal{P}$ position and $[n, S]$ is an $\mathcal{N}$ position.

If the game $[n, S]$ is an $\mathcal{N}$ position, then, after the first move, there must either be two or zero moves remaining. If there are no moves remaining, then the number of stones in the pile must be less than the smallest available value. Thus n is less than the sum of two values in S and so $n < i + k$. If there are two moves remaining, then the number of stones in the pile must be at least the sum of the remaining two values not yet chosen. Thus, $n \leq i+j+k$. □

For a game with four elements in S, $S = \{i, j, k, l\}$, we will show that if i is a winning move, then j is also a winning move. Therefore, in our descriptions of $\mathcal{N}$ and $\mathcal{P}$ positions, we can omit any that depend on playing i as a winning move.

Theorem 26 In four choice long global nim, if i is a winning move, then j is a winning move.

Proof. If i is a winning move, then taking i must win in either one or three moves. If i wins in one move, then $n-i < j$. Clearly, if this is true, then any other choice is also a winning move. If i wins in three moves, then after i is chosen there must be two remaining moves whose sum is less than $n - i$ and three remaining moves whose sum is greater than $n - i$. In other words, $(\forall x \neq i)(x \leq n-i \Rightarrow (\exists y \neq x, i)(x+y \leq n-i \text{ and } (\forall z \neq x, y, i)(n-i < x+y+z)))$.

From this statement, we can derive three statements all depending on choosing i first

$$(j \leq n - i \Rightarrow (j + k \leq n - i < j + k + l \text{ or } j + l \leq n - i < j + k + l)) \text{ and}$$

$$(k \leq n - i \Rightarrow (k + j \leq n - i < j + k + l \text{ or } k + l \leq n - i < j + k + l)) \text{ and}$$

$$(l \leq n - i \Rightarrow (l + j \leq n - i < j + k + l \text{ or } l + k \leq n - i < j + k + l)).$$

In each *or* statement above, the second statement implies the first: for example, $j + l \leq n - i < j + k + l$ implies $j + k \leq n - i < j + k + l$. Thus, the first statements are removable. We then convert the implications to or's. This leaves us with the statements:

$$(j > n - i \text{ or } j + k \leq n - i < j + k + l) \text{ and}$$

$$(k > n - i \text{ or } k + j \leq n - i < j + k + l) \text{ and}$$
$$(l > n - i \text{ or } l + j \leq n - i < j + k + l).$$

If $j > n-i$ is true, then certainly $k > n-i$ and $l > n-i$ are true. Following this pattern, we can now write out all of these inequalities in disjunctive normal form:

$$(j > n - i) \text{ or } (j + k \leq n - i < j + k + l \text{ and } l > n - i) \text{ or } l + j \leq n - i < j + k + l.$$

We can rewrite this statement, subtracting j and adding i whenever possible:

$$(i > n - j) \text{ or } (i + k \leq n - j < i + k + l \text{ and } l > n - j) \text{ or } l + i \leq n - j < i + k + l.$$

We can see that in each of these inequalities j is a winning move. □

Theorem 27 In four choice long global nim, let the possible moves be i, j, k, l in a pilesize n, where $i < j < k < l < n$. Then the $\mathcal{N}$ positions are those where

$$(i + j > n) \text{ or } (i + j + k \leq n < i + j + k + l \text{ and } i + j + l \leq n < i + j + k + l) \text{ or}$$

$$(i + j + k \leq n < i + j + k + l \text{ and } k + l > n) \text{ or}$$
$$(i + j + l \leq n < i + j + k + l \text{ and } k + l > n).$$

Proof. We first find the appropriate inequalities for when j is a winning move. This was done in Theorem 26. Those statements are:

$$(i > n - j) \text{ or } (i + k \leq n - j < i + k + l \text{ and } l > n - j) \text{ or } l + i \leq n - j < i + k + l.$$

The derivations for the moves where k or l is a winning move are similar. This gives us three long statements again.

$$(i > n - j) \text{ or } (i + k \leq n - j < i + k + l \text{ and } l > n - j) \text{ or } (i + l \leq n - j < i + k + l) \text{ or}$$

$$(i > n - k) \text{ or } (i + j \leq n - k < i + j + l \text{ and } l > n - k) \text{ or } (i + l \leq n - k < i + j + l) \text{ or}$$
$$(i > n - l) \text{ or } (i + j \leq n - l < i + j + k \text{ and } k > n - l) \text{ or } (i + k \leq n - l < i + j + k).$$

Since $i > n - j$ implies $i > n - k$, which implies $i > n - l$, we can disregard the latter two statements. We now consider the remaining six statements:

$$(i+j+k \leq n < i+j+k+l \text{ and } l > n-j) \text{ or } (i+j+l \leq n < i+j+k+l); \tag{1}$$

$$(i+j+k \leq n < i+j+k+l \text{ and } l > n-k) \text{ or } (i+k+l \leq n < i+j+k+l); \tag{2}$$

$$(i+j+l \leq n < i+j+k+l \text{ and } k > n-l) \text{ or } (i+k+l \leq n < i+j+k+l). \tag{3}$$

Note that (3) $\Rightarrow$ (2), so we can disregard (3). The first part of (1) implies (2), but the second part of (2) implies (1). We remove the appropriate parts, leaving us with the below statements.

$$(i+j+k \leq n < i+j+k+l \text{ and } l > n-k) \tag{4}$$

$$(i+j+l \leq n < i+j+k+l). \tag{5}$$

Finally, we note that in (4), $l+k > n \Rightarrow i+j+k+l > n$. Reducing again, we end up with three statements. These are the $\mathcal{N}$ positions in four choice long global nim:

$$(i+l > n) \text{ or } (i+j+k \leq n < k+l) \text{ or } (i+j+l \leq n < i+j+k+l).$$

□

Note that the statements show how to play the game. If the first statement is true, take l from the pile. Otherwise, take j and you are guaranteed to have a game three moves long, but no longer.

We also derived the inequalities for a game with five elements in S. The proof is similar to the proof for Theorem 27. We also showed that in five choice long global nim, if i is a winning move, then at least j or m is also a winning move. These are the $\mathcal{N}$ positions for five choice long global nim.

$$\begin{gathered}
i+m > n \text{ or} \\
(i+j+l \leq n < i+j+k+l) \text{ and } (m+j > n) \text{ or} \\
(i+j+m \leq n < i+j+k+l) \text{ or} \\
(i+j+k \leq n < l+k) \text{ or} \\
(i+k+l \leq n < i+j+k+l) \text{ and } (m+l > n) \text{ or} \\
(k+l+m \leq n < i+j+l+m) \text{ or} \\
(i+j+m \leq n < k+m) \text{ or} \\
(i+k+m \leq n < i+j+k+m) \text{ and } (l+m > n) \text{ or} \\
(i+l+m \leq n < i+j+k+m) \text{ or} \\
(i+j+k+l+m \leq n).
\end{gathered}$$

Theorem 28 If n is odd, any game of the form $[(n, n, \ldots, n), \{1, \ldots, n\}]$ is an $\mathcal{N}$ position, and any game of the form $[(n, n, \ldots, n), \{1, \ldots, n-1\}]$ is a $\mathcal{P}$ position.

Proof. Let n be an odd number. Take the game $[(n, n, \ldots, n), \{1, \ldots, n-1\}]$. Since n is odd, n has an even number of summands. Therefore, the first player will make a move in a pile, and the second player will move in that same pile to take the rest of the stones. This will leave a game with one less pile, and two summands of n will be added to the set of unavailable moves. Because of the even number of summands, it will always be possible for the previous player to take the remainder of the pile. Thus, the game $[(n, n, \ldots, n), \{1, \ldots, n-1\}]$ is a $\mathcal{P}$ position.

Take the game $[(n, n, \ldots, n), \{1, \ldots, n\}]$. The next player will take an entire pile, leaving the game $[(0, n, \ldots, n), \{1, \ldots, n-1\}]$. This is a $\mathcal{P}$ position, and so $[(n, n, \ldots, n), \{1, \ldots, n\}]$ is an $\mathcal{N}$ position. □

Proposition 29 If $b < a$, then $[(a, a+b), (\{1, \ldots, a+b\} \setminus \{1, \ldots, a-1\}) \cup \{b\}]$ is a $\mathcal{P}$-position.

Proof. If a stones are taken from the first pile, then taking $a + b$ stones from the second pile is the winning move. If a stones are taken from the second pile, then taking b stones from the second pile is the winning move. If b stones are taken from either pile, then taking a from the other pile is the winning move. If x stones are taken from the second pile (where $a < x \leq a+b$), then taking a stones from the first pile is the winning move (since $a+b-x < b$ so no moves can be made on the second pile). □

Proposition 30 If $a < b \leq 2a + 2$, then $[(a, a + b), \{1, \ldots, a + b\} \setminus \{1, \ldots, a - 1\}]$ is a $\mathcal{P}$-position.

Proof. If a stones are taken from the first pile, then taking $a + b$ stones from the second pile is the winning move. If a stones are taken from the second pile, then taking b stones from the second pile is the winning move. If x stones are taken from the second pile where $x > b$, then taking a stones from the first pile is the winning move (since $a + b - x < a$). So assume x stones are taken from the second pile where $a < x \leq b$. We claim that taking a stones from the second pile is the winning move. This leaves $b - x$ stones in the second pile. If $b - x \leq a$, then these stones cannot be taken. However, if $b \leq 2a + 1$, then $b - x < b - a \leq a + 1$, so $b - x < a$. If $b = 2a + 2$, then $b - x \leq a$ unless $x = a + 1$. But if $x = a + 1$ and we respond to taking $a + 1$ stones from the second pile by taking a stones from the second pile, the second pile only contains $a + 1$ stones, which cannot be taken. □

5. Grundy Numbers for LGN

We now move to characterizing the Grundy numbers for long global nim. A first guess for the Grundy value for the game $[n, S]$ might be the length of the longest series of terms in S whose sum is less than or equal to n. However, this guess often is too large. For example, consider the game $[8, \{1, 3, 4\}]$. Once any move is made, the game becomes a second player win, leaving a Grundy value of 1 for the game, even though there is a series of length 3 summing to 8. To start characterizing the Grundy numbers, we return to g-series.

Definition 31 A g-series for $[n, S]$ of length p is a series $s_1 + s_2 + \cdots + s_p$ of distinct terms from S whose sum is less than or equal to n where $*[n - s_1, S - \{s_1\}] = p - 1$.

Proposition 32 If $s_1 + s_2 + \cdots + s_p$ is a g-series for $[n-j, S-\{j\}]$ and $*[n-j, S-\{j\}] = p$, then $j + s_1 + s_2 + \cdots + s_p$ is a g-series for $[n, S]$.

Proof. Clearly the series has a sum less than or equal to n and consists of distinct terms from S. Moreover $*[n - j, S - \{s_1\}] = (p + 1) - 1 = p$ by construction. $\square$

Theorem 33 $*[n, S] \geq m$ if and only if there exist g-series for $[n, S]$ of length $1, 2, \ldots, m$.

Proof. Note that the theorem is trivially true for $n = 0$ and $n = 1$. Assume for all S, all m, and for all $k < n$ that $*[k, S] = m$ if and only if there exist g-series for $[k, S]$ of length $1, 2, \ldots, m$.

Assume $*[n, S] \geq m$. Then for all $p < m$, there exists $k \in S$ such that $*[n-k, S-\{k\}] = p$. Fix a given p and its corresponding k. By induction, there exists a g-series for $[n-k, S-\{k\}]$ of length p — denote the series by $s_1 + s_2 + \ldots + s_p$. Then the series $k + s_1 + s_2 + \ldots + s_p$ is a g-series for $[n, S]$ of length $p + 1$. Thus there exist g-series for $[n, S]$ of lengths $1, \ldots, m$.

Assume there exist g-series for $[n, S]$ of length $1, 2, \ldots, m$. Given $1 \leq p \leq m$, let $s_1 + s_2 + \cdots + s_p$ be the corresponding g-series. Thus $*[n - s_1, S - \{s_1\}] = p - 1$. Thus the set of Grundy number of options of $[n, S]$ contains $0, \ldots, m - 1$ and so $*[n, S] \geq m$. $\square$

While we now have some machinery for discussing Grundy numbers, to date we have not been able to prove any major facts about the values. A good first step would be to prove the following conjecture, which is true for $n \leq 50$.

Conjecture 34 If $1 + 2 + \cdots + k$ is the largest triangular number less than or equal to n, then the Grundy number of $[n, \{1, 2, \ldots, n\}]$ is k.

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