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A NOTE ON BOOLEAN LATTICES AND FAREY SEQUENCES II

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Abstract

We establish monotone bijections between subsequences of the Farey sequences and the half-sequences of Farey subsequences associated with elements of the Boolean lattices.

1. Introduction

The *Farey sequence of order* n, denoted by $\mathcal{F}_n$, is the ascending sequence of irreducible fractions $\frac{h}{k} \in \mathbb{Q}$ such that $\frac{0}{1} \le \frac{h}{k} \le \frac{1}{1}$ and $1 \le k \le n$; see, e.g., [3, Chapter 27], [4, §3], [5, Chapter 4], [6, Chapter III], [11, Chapter 6], [12, Chapter 6], [13, Sequences A006842 and A006843], [14, Chapter 5].

The Farey sequence of order n contains the subsequences

$$\mathcal{F}_n^m := \left(\tfrac{h}{k} \in \mathcal{F}_n : \ h \le m\right) \ , \tag{1}$$

for all integers $m \ge 1$, and the subsequences

$$\mathcal{G}_n^m := \left(\tfrac{h}{k} \in \mathcal{F}_n : \ k-h \le n-m\right) \ , \tag{2}$$

for $m \le n-1$, that inherit many familiar properties of the Farey sequences. In particular, if $n > 1$ and $m \ge n-1$, then $\mathcal{F}_n^m = \mathcal{F}_n$; if $n > 1$ and $m \le 1$, then $\mathcal{G}_n^m = \mathcal{F}_n$.

The Farey subsequence $\mathcal{F}_n^m$ was presented in [1], and some comments were given in [10, Remark 7.10].

Let $\mathbb{B}(n)$ denote the Boolean lattice of rank $n > 1$, whose operation of meet is denoted by $\wedge$. If a is an element of $\mathbb{B}(n)$, of rank $\rho(a) =: m$ such that $0 < m < n$, then the integers n and m determine the subsequence

$$\begin{aligned}\mathcal{F}\big(\mathbb{B}(n), m\big) :&= \left(\tfrac{\rho(b\wedge a)}{\gcd(\rho(b\wedge a),\rho(b))} \Big/ \tfrac{\rho(b)}{\gcd(\rho(b\wedge a),\rho(b))} : \ b \in \mathbb{B}(n), \ \rho(b) > 0\right) \\ &= \left(\tfrac{h}{k} \in \mathcal{F}_n : \ h \le m, \ k-h \le n-m\right)\end{aligned} \tag{3}$$

of $\mathcal{F}_n$, considered in [7, 8, 9, 10].

Notice that the map

$$\mathcal{F}\big(\mathbb{B}(n),m\big) \to \mathcal{F}\big(\mathbb{B}(n),n-m\big) \ , \quad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} \ , \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{4}$$

is order-reversing and bijective, by analogy with the map

$$\mathcal{F}_n \to \mathcal{F}_n \ , \quad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} \ , \quad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{5}$$

which is order-reversing and bijective as well; here $\left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \in \mathbb{Z}^2$ is a vector presentation of the fraction $\frac{h}{k}$.

We call the ascending sets

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big) := \Big(\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(n),m\big) : \ \tfrac{h}{k} \leq \tfrac{1}{2}\Big) \ ,$$

and

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big) := \Big(\tfrac{h}{k} \in \mathcal{F}\big(\mathbb{B}(n),m\big) : \ \tfrac{h}{k} \geq \tfrac{1}{2}\Big) \ .$$

the *left* and *right halfsequences* of the sequence $\mathcal{F}\big(\mathbb{B}(n),m\big)$, respectively.

This work is a sequel to note [7] which concerns the sequences $\mathcal{F}\big(\mathbb{B}(2m),m\big)$. See [7] for more about Boolean lattices, Farey (sub)sequences and related combinatorial identities.

The key observation is Theorem 3 which in particular asserts that there is a bijection between the sequence $\mathcal{F}^m_{n-m}$ and the left halfsequence of $\mathcal{F}\big(\mathbb{B}(n),m\big)$, on the one hand; there is also a bijection between the sequence $\mathcal{F}^{n-m}_m$ and the right halfsequence of $\mathcal{F}\big(\mathbb{B}(n),m\big)$, on the other hand.

Throughout the note, n and m represent integers; n is always greater than one. The fractions of Farey (sub)sequences are indexed starting with zero. Terms 'precedes' and 'succeeds' related to pairs of fractions always mean relations of immediate consecution. When we deal with a Farey subsequence $\mathcal{F}\big(\mathbb{B}(n),m\big)$, we implicitly suppose $0 < m < n$.

2. The Farey Subsequences $\mathcal{F}^m_n$ and $\mathcal{G}^m_n$

The connection between the sequences of the form (1) and (2) comes from their definitions:

Lemma 1 *The maps*

$$\mathcal{F}^m_n \to \mathcal{G}^{n-m}_n \ , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ ,$$

and

$$\mathcal{G}^m_n \to \mathcal{F}^{n-m}_n \ , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{k} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ ,$$

are order-reversing and bijective, for any m, $0 \leq m \leq n$.

Following [2, §4.15], we let $\overline{\mu}(\cdot)$ denote the Möbius function on positive integers. For an interval of positive integers $[i,l] := \{j : \ i \le j \le l\}$ and for a positive integer h, let $\phi(h;[i,l]) := |\{j \in [i,l] : \ \gcd(h,j)=1\}|$.

We now describe several basic properties of the sequences $\mathcal{G}_n^m$ which are dual, in view of Lemma 1, to those of the sequences $\mathcal{F}_n^{n-m}$, cf. [10, Remark 7.10].

Remark 2 *Suppose* $0 \le m < n$.

(i) *In* $\mathcal{G}_n^m = (g_0 < g_1 < \cdots < g_{|\mathcal{G}_n^m|-2} < g_{|\mathcal{G}_n^m|-1})$, *we have*

$$g_0 = \tfrac{0}{1}\ , \quad g_1 = \tfrac{1}{\min\{n-m+1,n\}}\ , \quad g_{|\mathcal{G}_n^m|-2} = \tfrac{n-1}{n}\ , \quad g_{|\mathcal{G}_n^m|-1} = \tfrac{1}{1}\ .$$

(ii) (a) *The cardinality of the sequence* $\mathcal{G}_n^m$ *equals*

$$1 + \sum_{j\in[1,n]} \phi\bigl(j;[\max\{1,j+m-n\},j]\bigr)$$
$$= 1 + \sum_{j\in[1,n-m+1]} \phi\bigl(j;[1,j]\bigr) + \sum_{j\in[n-m+2,n]} \phi\bigl(j;[j+m-n,j]\bigr)$$
$$= 1 + \sum_{d\ge 1} \overline{\mu}(d)\cdot\Bigl(\bigl\lfloor \tfrac{n}{d}\bigr\rfloor - \tfrac{1}{2}\bigl\lfloor \tfrac{n-m}{d}\bigr\rfloor\Bigr)\cdot\bigl\lfloor \tfrac{n-m}{d}+1\bigr\rfloor\ .$$

(b) *If* $g_t \in \mathcal{G}_n^m - \{\frac{0}{1}\}$, *then*

$$t = \sum_{j\in[1,n]} \phi\bigl(j;[\max\{1,j+m-n\},\lfloor j\cdot g_t\rfloor]\bigr)$$
$$= \sum_{j\in[1,n-m+1]} \phi\bigl(j;[1,\lfloor j\cdot g_t\rfloor]\bigr) + \sum_{j\in[n-m+2,n]} \phi\bigl(j;[j+m-n,\lfloor j\cdot g_t\rfloor]\bigr)$$
$$= 1 + \sum_{d\ge 1} \overline{\mu}(d)\cdot\Biggl(\bigl\lfloor \tfrac{n-m}{d}\bigr\rfloor\cdot\Bigl(\bigl\lfloor \tfrac{n}{d}\bigr\rfloor - \tfrac{1}{2}\bigl\lfloor \tfrac{n-m}{d}+1\bigr\rfloor\Bigr) - \sum_{j\in[1,\lfloor n/d\rfloor]} \min\Bigl\{\bigl\lfloor \tfrac{n-m}{d}\bigr\rfloor, \lfloor j\cdot(1-g_t)\rfloor\Bigr\}\Biggr)\ .$$

(iii) *Let* $\frac{h}{k} \in \mathcal{G}_n^m$, $\frac{0}{1} < \frac{h}{k} < \frac{1}{1}$.

(a) *Let* x_0 *be the integer such that* $kx_0 \equiv -1 \pmod{h}$ *and* $m-h+1 \le x_0 \le m$. *Define integers* y_0 *and* t^* *by* $y_0 := \frac{kx_0+1}{h}$ *and* $t^* := \bigl\lfloor \min\bigl\{\frac{n-m+x_0-y_0}{k-h}, \frac{n-y_0}{k}\bigr\}\bigr\rfloor$. *The fraction* $\frac{x_0+t^*h}{y_0+t^*k}$ *precedes the fraction* $\frac{h}{k}$ *in* $\mathcal{G}_n^m$.

(b) *Let* x_0 *be the integer such that* $kx_0 \equiv 1 \pmod{h}$ *and* $m-h+1 \le x_0 \le m$. *Define integers* y_0 *and* t^* *by* $y_0 := \frac{kx_0-1}{h}$ *and* $t^* := \bigl\lfloor \min\bigl\{\frac{n-m+x_0-y_0}{k-h}, \frac{n-y_0}{k}\bigr\}\bigr\rfloor$. *The fraction* $\frac{x_0+t^*h}{y_0+t^*k}$ *succeeds the fraction* $\frac{h}{k}$ *in* $\mathcal{G}_n^m$.

(iv) (a) *If* $\frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}}$ *are two successive fractions of* $\mathcal{G}_n^m$ *then*

$$k_j h_{j+1} - h_j k_{j+1} = 1 \ .$$

(b) *If* $\frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}}$ *are three successive fractions of* $\mathcal{G}_n^m$ *then*

$$\frac{h_{j+1}}{k_{j+1}} = \frac{h_j+h_{j+2}}{\gcd(h_j+h_{j+2},k_j+k_{j+2})} \Big/ \frac{k_j+k_{j+2}}{\gcd(h_j+h_{j+2},k_j+k_{j+2})} \ .$$

(c) *If* $\frac{h_j}{k_j} < \frac{h_{j+1}}{k_{j+1}} < \frac{h_{j+2}}{k_{j+2}}$ *are three successive fractions of* $\mathcal{G}_n^m$ *then the integers* h_j*,* k_j*,* h_{j+2} *and* k_{j+2} *are related in the following way:*

$$\begin{aligned} h_j &= \left\lfloor \min\left\{ \tfrac{k_{j+2}+n}{k_{j+1}}, \tfrac{k_{j+2}-h_{j+2}+n-m}{k_{j+1}-h_{j+1}} \right\} \right\rfloor h_{j+1} - h_{j+2} \ , \\ k_j &= \left\lfloor \min\left\{ \tfrac{k_{j+2}+n}{k_{j+1}}, \tfrac{k_{j+2}-h_{j+2}+n-m}{k_{j+1}-h_{j+1}} \right\} \right\rfloor k_{j+1} - k_{j+2} \ , \\ h_{j+2} &= \left\lfloor \min\left\{ \tfrac{k_j+n}{k_{j+1}}, \tfrac{k_j-h_j+n-m}{k_{j+1}-h_{j+1}} \right\} \right\rfloor h_{j+1} - h_j \ , \\ k_{j+2} &= \left\lfloor \min\left\{ \tfrac{k_j+n}{k_{j+1}}, \tfrac{k_j-h_j+n-m}{k_{j+1}-h_{j+1}} \right\} \right\rfloor k_{j+1} - k_j \ . \end{aligned}$$

(v) *If* $\frac{1}{k} \in \mathcal{G}_n^m$*, where* $n > 1$*, for some* $k > 1$*, then the fraction* $\frac{\lfloor \min\{\frac{n-m-1}{k-1},\frac{n-1}{k}\}\rfloor}{k\lfloor \min\{\frac{n-m-1}{k-1},\frac{n-1}{k}\}\rfloor+1}$ *precedes* $\frac{1}{k}$*, and the fraction* $\frac{\lfloor \min\{\frac{n-m+1}{k-1},\frac{n+1}{k}\}\rfloor}{k\lfloor \min\{\frac{n-m+1}{k-1},\frac{n+1}{k}\}\rfloor-1}$ *succeeds* $\frac{1}{k}$ *in* $\mathcal{G}_n^m$*.*

3. The Farey Subsequence $\mathcal{F}(\mathbb{B}(n),m)$

Definition (3) implies that a Farey subsequence $\mathcal{F}(\mathbb{B}(n),m)$ can be regarded as the intersection

$$\mathcal{F}(\mathbb{B}(n),m) = \mathcal{F}_n^m \cap \mathcal{G}_n^m \ .$$

Its halfsequences can be described with the help of the following statement:

Theorem 3 *Consider a Farey subsequence* $\mathcal{F}(\mathbb{B}(n),m)$*. The maps*

$$\mathcal{F}^{\le\frac{1}{2}}(\mathbb{B}(n),m) \to \mathcal{F}_{n-m}^m \ , \qquad \tfrac{h}{k} \mapsto \tfrac{h}{k-h} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{6}$$

$$\mathcal{F}_{n-m}^m \to \mathcal{F}^{\le\frac{1}{2}}(\mathbb{B}(n),m) \ , \qquad \tfrac{h}{k} \mapsto \tfrac{h}{k+h} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{7}$$

$$\mathcal{F}^{\ge\frac{1}{2}}(\mathbb{B}(n),m) \to \mathcal{G}_m^{2m-n} \ , \qquad \tfrac{h}{k} \mapsto \tfrac{2h-k}{h} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{8}$$

and

$$\mathcal{G}_m^{2m-n} \to \mathcal{F}^{\ge\frac{1}{2}}(\mathbb{B}(n),m) \ , \qquad \tfrac{h}{k} \mapsto \tfrac{k}{2k-h} \ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 0 & 1 \\ -1 & 2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \ , \tag{9}$$

are order-preserving and bijective.

The maps

$$\begin{aligned}
\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big) &\to \mathcal{G}^{n-2m}_{n-m}\ , && \tfrac{h}{k}\mapsto\tfrac{k-2h}{k-h}\ , && \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-2&1\\-1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\\
\mathcal{G}^{n-2m}_{n-m} &\to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ , && \tfrac{h}{k}\mapsto\tfrac{k-h}{2k-h}\ , && \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\-1&2\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ ,\\
\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big) &\to \mathcal{F}^{n-m}_{m}\ , && \tfrac{h}{k}\mapsto\tfrac{k-h}{h}\ , && \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}-1&1\\1&0\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \qquad (10)
\end{aligned}$$

and

$$\mathcal{F}^{n-m}_{m}\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k}{k+h}\ ,\qquad \left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\mapsto\left[\begin{smallmatrix}0&1\\1&1\end{smallmatrix}\right]\cdot\left[\begin{smallmatrix}h\\k\end{smallmatrix}\right]\ , \qquad (11)$$

are order-reversing and bijective.

Proof. For any integer h, $1\leq h\leq m$, we have

$$\begin{aligned}
&\Big|\big\{\tfrac{h}{k}\in\mathcal{F}\big(\mathbb{B}(n),m\big):\ \tfrac{h}{k}<\tfrac{1}{2}\big\}\Big| = \phi(h;[2h+1,h+n-m])\\
&\quad=\sum_{d\in[1,h]:\ d|h}\overline{\mu}(d)\cdot\Big(\big\lfloor\tfrac{h+n-m}{d}\big\rfloor-\tfrac{2h}{d}\Big)=\sum_{d\in[1,h]:\ d|h}\overline{\mu}(d)\cdot\Big(\big\lfloor\tfrac{n-m}{d}\big\rfloor-\tfrac{h}{d}\Big)\\
&\quad=\phi(h;[h+1,n-m])=\Big|\big\{\tfrac{h}{k}\in\mathcal{F}^{m}_{n-m}:\ \tfrac{h}{k}<\tfrac{1}{1}\big\}\Big|
\end{aligned}$$

and see that the sequences $\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)$ and $\mathcal{F}^{m}_{n-m}$ are of the same cardinality, but this conclusion also implies $\big|\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\big|=|\mathcal{F}^{n-m}_{m}|$, due to bijection (4). The proof of the assertions concerning maps (6), (7), (10) and (11) is completed by checking that, on the one hand, a fraction $\frac{h_j}{k_j}$ precedes a fraction $\frac{h_{j+1}}{k_{j+1}}$ in $\mathcal{F}^{m}_{n-m}$ if and only if the fraction $\frac{h_j}{k_j+h_j}$ precedes the fraction $\frac{h_{j+1}}{k_{j+1}+h_{j+1}}$ in $\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)$; on the other hand, a fraction $\frac{h_j}{k_j}$ precedes a fraction $\frac{h_{j+1}}{k_{j+1}}$ in $\mathcal{F}^{n-m}_{m}$ if and only if $\frac{k_{j+1}}{k_{j+1}+h_{j+1}}$ precedes $\frac{k_j}{k_j+h_j}$ in $\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)$. The remaining assertions of the theorem now follow, thanks to Lemma 1. □

For example,

$$\begin{aligned}
\mathcal{F}_6 &= \big(\tfrac{0}{1}<\tfrac{1}{6}<\tfrac{1}{5}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{2}{5}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{5}{6}<\tfrac{1}{1}\big)\ ,\\
\mathcal{F}^4_6 &= \big(\tfrac{0}{1}<\tfrac{1}{6}<\tfrac{1}{5}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{2}{5}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{1}{1}\big)\ ,\\
\mathcal{G}^4_6 &= \big(\tfrac{0}{1}<\tfrac{1}{3}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{5}{6}<\tfrac{1}{1}\big)\ ,\\
\mathcal{F}\big(\mathbb{B}(6),4\big) &= \big(\tfrac{0}{1}<\tfrac{1}{3}<\tfrac{1}{2}<\tfrac{3}{5}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{4}{5}<\tfrac{1}{1}\big)\ ,\\
\mathcal{F}^4_{6-4}=\mathcal{F}_2 &= \big(\tfrac{0}{1}<\tfrac{1}{2}<\tfrac{1}{1}\big)\ ,\\
\mathcal{G}^{2\cdot4-6}_4=\mathcal{G}^2_4 &= \big(\tfrac{0}{1}<\tfrac{1}{3}<\tfrac{1}{2}<\tfrac{2}{3}<\tfrac{3}{4}<\tfrac{1}{1}\big)\ ,\\
\mathcal{G}^{6-2\cdot4}_{6-4}=\mathcal{F}_2 &= \big(\tfrac{0}{1}<\tfrac{1}{2}<\tfrac{1}{1}\big)\ ,\\
\mathcal{F}^{6-4}_4=\mathcal{F}^2_4 &= \big(\tfrac{0}{1}<\tfrac{1}{4}<\tfrac{1}{3}<\tfrac{1}{2}<\tfrac{2}{3}<\tfrac{1}{1}\big)\ .
\end{aligned}$$

Theorem 3 allows us to write down the formula

$$\big|\mathcal{F}\big(\mathbb{B}(n),m\big)\big|=|\mathcal{F}^{m}_{n-m}|+|\mathcal{F}^{n-m}_{m}|-1=\big|\mathcal{F}^{\min\{m,n-m\}}_{n-m}\big|+\big|\mathcal{F}^{\min\{m,n-m\}}_{m}\big|-1\ ,$$

for the number of elements of a sequence $\mathcal{F}\big(\mathbb{B}(n),m\big)$, cf. [10, Proposition 7.3(ii)]. Recall that $|\mathcal{F}_q^p| = 1+\sum_{d\geq 1}\overline{\mu}(d)\cdot\big(\lfloor\frac{q}{d}\rfloor - \frac{1}{2}\lfloor\frac{p}{d}\rfloor\big)\cdot\lfloor\frac{p}{d}+1\rfloor$, for any Farey subsequence $\mathcal{F}_q^p$ with $0<p\leq q$; see [10, Remark 7.10(ii)(b)]. Notice that the well-known relation $\sum_{d\geq 1}\overline{\mu}(d)\cdot\lfloor\frac{t}{d}\rfloor = 1$, for any positive integer t, leads us to one more formula: $|\mathcal{F}_q^p| = \frac{3}{2}+\sum_{d\geq 1}\overline{\mu}(d)\cdot\lfloor\frac{p}{d}\rfloor\cdot\big(\lfloor\frac{q}{d}\rfloor - \frac{1}{2}\lfloor\frac{p}{d}\rfloor\big)$.

Thus, we have

$$\big|\mathcal{F}\big(\mathbb{B}(n),m\big)\big| = \frac{3}{2}+\sum_{d\geq 1}\overline{\mu}(d)\cdot\Big\lfloor\tfrac{\min\{m,n-m\}}{d}\Big\rfloor\cdot\Big(\big\lfloor\tfrac{n-m}{d}\big\rfloor - \tfrac{1}{2}\Big\lfloor\tfrac{\min\{m,n-m\}}{d}\Big\rfloor\Big) + \frac{3}{2}+\sum_{d\geq 1}\overline{\mu}(d)\cdot\Big\lfloor\tfrac{\min\{m,n-m\}}{d}\Big\rfloor\cdot\Big(\big\lfloor\tfrac{m}{d}\big\rfloor - \tfrac{1}{2}\Big\lfloor\tfrac{\min\{m,n-m\}}{d}\Big\rfloor\Big) - 1$$

or

$$\big|\mathcal{F}\big(\mathbb{B}(n),m\big)\big| = 2+\sum_{d\geq 1}\overline{\mu}(d)\cdot\Big\lfloor\tfrac{\min\{m,n-m\}}{d}\Big\rfloor\cdot\Big(\big\lfloor\tfrac{n-m}{d}\big\rfloor + \big\lfloor\tfrac{m}{d}\big\rfloor - \Big\lfloor\tfrac{\min\{m,n-m\}}{d}\Big\rfloor\Big)\ ,$$

that is,

$$\big|\mathcal{F}\big(\mathbb{B}(n),m\big)\big| - 2 = \sum_{d\geq 1}\overline{\mu}(d)\cdot\big\lfloor\tfrac{m}{d}\big\rfloor\cdot\big\lfloor\tfrac{n-m}{d}\big\rfloor\ .$$

In particular, for any positive integer t we have

$$\sum_{d\geq 1}\overline{\mu}(d)\cdot\big\lfloor\tfrac{t}{d}\big\rfloor^2 = \big|\mathcal{F}\big(\mathbb{B}(2t),t\big)\big| - 2 = 2\,|\mathcal{F}_t| - 3\ .$$

Proposition 4 *Consider a Farey subsequence* $\mathcal{F}\big(\mathbb{B}(n),m\big)$.

(i) *Suppose* $m\geq\frac{n}{2}$. *The map*

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\to\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k-2h}{2k-3h}\ ,\qquad \big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\mapsto\big[\begin{smallmatrix}-2&1\\-3&2\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\ , \tag{12}$$

is order-reversing and bijective. The map

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k-h}{2k-3h}\ ,\qquad \big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\mapsto\big[\begin{smallmatrix}-1&1\\-3&2\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\ ,$$

is order-preserving and injective. The map

$$\mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{k-h}{k}\ ,\qquad \big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\mapsto\big[\begin{smallmatrix}-1&1\\0&1\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\ ,$$

is order-reversing and injective.

(ii) *Suppose* $m\leq\frac{n}{2}$. *The map*

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\to\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ ,\qquad \tfrac{h}{k}\mapsto\tfrac{h}{3h-k}\ ,\qquad \big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\mapsto\big[\begin{smallmatrix}1&0\\3&-1\end{smallmatrix}\big]\cdot\big[\begin{smallmatrix}h\\k\end{smallmatrix}\big]\ , \tag{13}$$

is order-reversing and bijective. The map

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{2h-k}{3h-k}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} 2 & -1 \\ 3 & -1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

is order-preserving and injective. The map

$$\mathcal{F}^{\geq\frac{1}{2}}\big(\mathbb{B}(n),m\big) \to \mathcal{F}^{\leq\frac{1}{2}}\big(\mathbb{B}(n),m\big)\ , \qquad \tfrac{h}{k} \mapsto \tfrac{k-h}{k}\ , \qquad \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} -1 & 1 \\ 0 & 1 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} h \\ k \end{smallmatrix}\right]\ ,$$

is order-reversing and injective.

Proof. To prove that map (12) is order-reversing and bijective, notice that $\mathcal{F}^{m}_{n-m} = \mathcal{F}_{n-m}$, and consider the composite map

$$\frac{h}{k} \overset{\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(n),m)\xrightarrow{(6)}\mathcal{F}_{n-m}}{\longmapsto} \frac{h}{k-h} \overset{\mathcal{F}_{n-m}\xrightarrow{(5)}\mathcal{F}_{n-m}}{\longmapsto} \frac{k-2h}{k-h} \overset{\mathcal{F}_{n-m}\xrightarrow{(7)}\mathcal{F}^{\leq\frac{1}{2}}(\mathbb{B}(n),m)}{\longmapsto} \frac{k-2h}{2k-3h}\ .$$

Similarly, under the hypothesis of assertion (ii) we have $\mathcal{G}^{2m-n}_{m} = \mathcal{F}_{m}$; consider the composite map

$$\frac{h}{k} \overset{\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n),m)\xrightarrow{(8)}\mathcal{F}_{m}}{\longmapsto} \frac{2h-k}{h} \overset{\mathcal{F}_{m}\xrightarrow{(5)}\mathcal{F}_{m}}{\longmapsto} \frac{k-h}{h} \overset{\mathcal{F}_{m}\xrightarrow{(9)}\mathcal{F}^{\geq\frac{1}{2}}(\mathbb{B}(n),m)}{\longmapsto} \frac{h}{3h-k}$$

to see that map (13) is order-reversing and bijective.

The remaining assertions follow from the observation that map (4) is the order-reversing bijection. □

We conclude the note by listing a few pairs of fractions adjacent in $\mathcal{F}\big(\mathbb{B}(n),m\big)$; see [8, 9] on such pairs within the sequences $\mathcal{F}\big(\mathbb{B}(2m),m\big)$. We find the neighbors of the images of several fractions of $\mathcal{F}\big(\mathbb{B}(n),m\big)$ under bijections from Theorem 3, and we then reflect them back to $\mathcal{F}\big(\mathbb{B}(n),m\big)$:

Remark 5 *Consider a Farey subsequence $\mathcal{F}\big(\mathbb{B}(n),m\big)$, with $n \neq 2m$.*

(i) *Suppose $m > \frac{n}{2}$.*
The fraction $\frac{n-m-1}{2(n-m)-1}$ precedes $\frac{1}{2}$, and the fraction $\frac{n-m+1}{2(n-m)+1}$ succeeds $\frac{1}{2}$.
The fraction $\frac{2\min\{n-m,\lfloor\frac{m+1}{2}\rfloor\}-1}{3\min\{n-m,\lfloor\frac{m+1}{2}\rfloor\}-1}$ precedes $\frac{2}{3}$; the fraction $\frac{2\min\{n-m,\lfloor\frac{m-1}{2}\rfloor\}+1}{3\min\{n-m,\lfloor\frac{m-1}{2}\rfloor\}+1}$ succeeds $\frac{2}{3}$.
If we additionally have $n-m>1$, then the fraction

$$\begin{cases} \frac{n-m-2}{2}\Big/\frac{3(n-m)-4}{2}, & \text{if } n-m \text{ is even,} \\ \frac{n-m-1}{2}\Big/\frac{3(n-m)-1}{2}, & \text{if } n-m \text{ is odd,} \end{cases}$$

precedes $\frac{1}{3}$; the fraction

$$\begin{cases} \frac{n-m}{2}\Big/\frac{3(n-m)-2}{2}, & \text{if } n-m \text{ is even,} \\ \frac{n-m+1}{2}\Big/\frac{3(n-m)+1}{2}, & \text{if } n-m \text{ is odd,} \end{cases}$$

succeeds $\frac{1}{3}$.

(ii) *Suppose* $m < \frac{n}{2}$.

The fraction $\frac{m}{2m+1}$ *precedes* $\frac{1}{2}$, *and the fraction* $\frac{m}{2m-1}$ *succeeds* $\frac{1}{2}$.

The fraction $\frac{\min\{m,\lfloor\frac{n-m+1}{2}\rfloor\}-1}{3\min\{m,\lfloor\frac{n-m+1}{2}\rfloor\}-2}$ *precedes* $\frac{1}{3}$; *the fraction* $\frac{\min\{m,\lfloor\frac{n-m-1}{2}\rfloor\}+1}{3\min\{m,\lfloor\frac{n-m-1}{2}\rfloor\}+2}$ *succeeds* $\frac{1}{3}$.

If we additionally have $m > 1$, *then the fraction*

$$\begin{cases} (m-1)\Big/\frac{3m-2}{2}, & \text{if } m \text{ is even,} \\ m\Big/\frac{3m+1}{2}, & \text{if } m \text{ is odd,} \end{cases}$$

precedes $\frac{2}{3}$; *the fraction*

$$\begin{cases} (m-1)\Big/\frac{3m-4}{2}, & \text{if } m \text{ is even,} \\ m\Big/\frac{3m-1}{2}, & \text{if } m \text{ is odd,} \end{cases}$$

succeeds $\frac{2}{3}$.

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