INTEGERS: ELECTRONIC JOURNAL OF COMBINATORIAL NUMBER THEORY 7 (2007), #A30

A PROOF OF THE CONTINUED FRACTION EXPANSION OF $e^{2/s}$

Takao Komatsu[1]
Graduate School of Science and Technology, Hirosaki University, Hirosaki, 036-8561, Japan
komatsu@cc.hirosaki-u.ac.jp

Received: 3/13/07, Accepted: 6/6/07, Published: 6/19/07

Abstract

We give a proof of the continued fraction expansion of $e^{2/s}$, where $s \ge 3$ is an odd integer, by expressing the error between $e^{2/s}$ and its each convergent explicitly in terms of integrals.

1. Introduction

Let $\alpha = [a_0; a_1, a_2, \dots]$ denote the simple continued fraction expansion of a real α, where

$$\alpha = a_0 + 1/\alpha_1, \qquad a_0 = \lfloor \alpha \rfloor,$$
$$\alpha_n = a_n + 1/\alpha_{n+1}, \qquad a_n = \lfloor \alpha_n \rfloor \quad (n \ge 1).$$

The n-th convergent of the continued fraction expansion is denoted by $p_n/q_n = [a_0; a_1, \dots, a_n]$, and p_n and q_n satisfy the recurrence relation:

$$p_n = a_n p_{n-1} + p_{n-2} \quad (n \ge 0), \qquad p_{-1} = 1, \qquad p_{-2} = 0,$$
$$q_n = a_n q_{n-1} + q_{n-2} \quad (n \ge 0), \qquad q_{-1} = 0, \qquad q_{-2} = 1.$$

An irrational number α is well approximated by its n-th convergent p_n/q_n. In fact, for $n \ge 0$

$$\frac{1}{q_n(q_{n+1} + q_n)} < \left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{q_n q_{n+1}}$$

([4, p. 20]). Precisely speaking, by using the algorithm mentioned above, we can express ([1, Lemma 5.4]) the error as

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n(\alpha_{n+1} q_n + q_{n-1})}.$$

[1]This research was supported in part by the Grant-in-Aid for Scientific research (C) (No. 18540006), the Japan Society for the Promotion of Science.

Osler [5] gave a remarkable proof of the simple continued fraction

$$e^{1/s} = [1; \overline{(2k-1)s-1, 1, 1}]_{k=1}^{\infty} \quad (s \geq 2)$$

by expressing this error explicitly in terms of integrals. Namely, when p_n/q_n is the n-th convergent of the continued fraction of $e^{1/s}$, he showed that for $n \geq 0$

$$\frac{p_{3n}}{q_{3n}} - e^{1/s} = -\frac{1}{q_{3n}} \int_0^1 \frac{x^n(x-1)^n}{s^{n+1} n!} e^{x/s} dx\,,$$
$$\frac{p_{3n+1}}{q_{3n+1}} - e^{1/s} = \frac{1}{q_{3n+1}} \int_0^1 \frac{x^{n+1}(x-1)^n}{s^{n+1} n!} e^{x/s} dx$$

and

$$\frac{p_{3n+2}}{q_{3n+2}} - e^{1/s} = \frac{1}{q_{3n+2}} \int_0^1 \frac{x^n(x-1)^{n+1}}{s^{n+1} n!} e^{x/s} dx\,.$$

This was the direct extension of the result given by Cohn [2] concerning e. A similar expression can be seen in [3] too.

It is known that the continued fraction expansion of $e^{2/s}$ is given by

$$e^{2/s} = \big[1; \overline{\frac{(6k-5)s-1}{2}, (12k-6)s, \frac{(6k-1)s-1}{2}, 1, 1}\,\big]_{k=1}^{\infty}\,,$$

where $s > 1$ is odd (See [6], §32, (2)). We shall give another proof of the continued fraction expansion of $e^{2/s}$ by showing similar errors explicitly. For convenience, for $n \geq 0$ put

$$A_n = \left(\frac{2}{s}\right)^{3n+1} \int_0^1 \frac{x^{3n}(x-1)^{3n}}{(3n)!} e^{2x/s} dx\,,$$
$$B_n = \frac{2^{3n+1}}{s^{3n+2}} \int_0^1 \frac{x^{3n+1}(x-1)^{3n+1}}{(3n+1)!} e^{2x/s} dx\,,$$
$$C_n = \left(\frac{2}{s}\right)^{3n+3} \int_0^1 \frac{x^{3n+2}(x-1)^{3n+2}}{(3n+2)!} e^{2x/s} dx\,,$$
$$D_n = \left(\frac{2}{s}\right)^{3n+3} \int_0^1 \frac{x^{3n+3}(x-1)^{3n+2}}{(3n+2)!} e^{2x/s} dx\,,$$

and

$$E_n = \left(\frac{2}{s}\right)^{3n+3} \int_0^1 \frac{x^{3n+2}(x-1)^{3n+3}}{(3n+2)!} e^{2x/s} dx\,.$$

Then, our main theorem is stated as follows.

Theorem 1.1. *Let p_n/q_n be the n-th convergent of the continued fraction of $e^{2/s}$. Then, for $n \geq 0$*

$$\begin{aligned}
\frac{p_{5n}}{q_{5n}} - e^{2/s} &= -\frac{1}{q_{5n}}A_n\,,\\
\frac{p_{5n+1}}{q_{5n+1}} - e^{2/s} &= -\frac{1}{q_{5n+1}}B_n\,,\\
\frac{p_{5n+2}}{q_{5n+2}} - e^{2/s} &= -\frac{1}{q_{5n+2}}C_n\,,\\
\frac{p_{5n+3}}{q_{5n+3}} - e^{2/s} &= \frac{1}{q_{5n+3}}D_n\,,
\end{aligned}$$

and

$$\frac{p_{5n+4}}{q_{5n+4}} - e^{2/s} = \frac{1}{q_{5n+4}}E_n\,.$$

2. The Continued Fraction of $e^{2/s}$

The proof of the main theorem is based upon the following identities.

Lemma 2.1.

$$A_n = -E_{n-1} - D_{n-1}\,, \tag{2.1}$$

$$B_n = \frac{(6n+1)s-1}{2}A_n - E_{n-1}\,, \tag{2.2}$$

$$C_n = (12n+6)sB_n + A_n\,, \tag{2.3}$$

$$D_n = -\frac{(6n+5)s-1}{2}C_n - B_n\,, \tag{2.4}$$

and

$$E_n = D_n - C_n\,. \tag{2.5}$$

These identities correspond with the desired relations:

$$\begin{aligned}
p_{5n} &= p_{5n-1} + p_{5n-2}, & q_{5n} &= q_{5n-1} + q_{5n-2},\\
p_{5n+1} &= \frac{(6n+1)s-1}{2}p_{5n} + p_{5n-1}, & q_{5n+1} &= \frac{(6n+1)s-1}{2}q_{5n} + q_{5n-1},\\
p_{5n+2} &= (12n+6)sp_{5n+1} + p_{5n}, & q_{5n+2} &= (12n+6)sq_{5n+1} + q_{5n},\\
p_{5n+3} &= \frac{(6n+5)s-1}{2}p_{5n+2} + p_{5n+1}, & q_{5n+3} &= \frac{(6n+5)s-1}{2}q_{5n+2} + q_{5n+1},\\
p_{5n+4} &= p_{5n+3} + p_{5n+2}, & q_{5n+4} &= q_{5n+3} + q_{5n+2}.
\end{aligned}$$

We shall prove Theorem 1.1 and Lemma 2.1 simultaneously.

Proof. When $n = 0$, the relations in Theorem 1.1 are true. Indeed,

$$A_0 = \frac{2}{s}\int_0^1 e^{2x/s}dx = e^{2/s} - 1 = q_0 e^{2/s} - p_0 \,.$$

$$\begin{aligned} B_0 &= \frac{2}{s^2}\int_0^1 x(x-1)e^{2x/s}dx = \frac{1}{2s}\big[\big(2x^2 - 2(s+1)x + s^2 + s\big)e^{2x/s}\big]_0^1 \\ &= \frac{s-1}{2}e^{2/s} - \frac{s+1}{2} = q_1 e^{2/s} - p_1 \,. \end{aligned}$$

$$\begin{aligned} C_0 &= \left(\frac{2}{s}\right)^3 \frac{1}{2}\int_0^1 x^2(x-1)^2 e^{2x/s}dx \\ &= \frac{1}{s^2}\big[\big(2x^4 - 4(s+1)x^3 + 2(3s^2+3s+1)x^2 \\ &\qquad - 2s(3s^2+3s+1)x + 3s^4 + 3s^3 + s^2\big)e^{2x/s}\big]_0^1 \\ &= (3s^2 - 3s + 1)e^{2/s} - (3s^2 + 3s + 1) = q_2 e^{2/s} - p_2 \,. \end{aligned}$$

$$\begin{aligned} D_0 &= \left(\frac{2}{s}\right)^3 \frac{1}{2}\int_0^1 x^3(x-1)^2 e^{2x/s}dx \\ &= -\frac{1}{2s^2}\big[\big(4x^5 - (10s+8)x^4 + (20s^2+16s+4)x^3 - (30s^3+24s^2+6s)x^2 \\ &\qquad + (30s^4 + 24s^3 + 6s^2)x - (15s^5 + 12s^4 + 3s^3)\big)e^{2x/s}\big]_0^1 \\ &= \frac{15s^3}{2} + 6s^2 + \frac{3s}{2} - \left(\frac{15s^3}{2} - 9s^2 + \frac{9s}{2} - 1\right)e^{2/s} = p_3 - q_3 e^{2/s} \,. \end{aligned}$$

$$\begin{aligned} E_0 &= \left(\frac{2}{s}\right)^3 \frac{1}{2}\int_0^1 x^2(x-1)^3 e^{2x/s}dx \\ &= -\frac{1}{2s^2}\big[\big(4x^5 - (10s+12)x^4 + (20s^2+24s+12)x^3 - (30s^3+36s^2+18s+4)x^2 \\ &\qquad + (30s^4 + 36s^3 + 18s^2 + 4s)x - (15s^5 + 18s^4 + 9s^3 + 2s^2)\big)e^{2x/s}\big]_0^1 \\ &= \frac{15s^3}{2} + 9s^2 + \frac{9s}{2} + 1 - \big(\frac{15s^3}{2} - 6s^2 + \frac{3s}{2}\big)e^{2/s} = p_4 - q_4 e^{2/s} \,. \end{aligned}$$

Suppose that Theorem 1.1 is true up to some integer $n - 1 (\geq 0)$. Since

$$\frac{d}{dx}\big(x^{3n}(x-1)^{3n}e^{2x/s}\big) = 3nx^{3n-1}(x-1)^{3n}e^{2x/s} + 3nx^{3n}(x-1)^{3n-1} + \frac{2}{s}x^{3n}(x-1)^{3n}e^{2x/s} \,,$$

by integrating from 0 to 1 we get (2.1). Hence,

$$\begin{aligned} p_{5n} - q_{5n}e^{2/s} &= (p_{5n-1} + p_{5n-2}) - (q_{5n-1} + q_{5n-2})e^{2/s} \\ &= E_{n-1} + D_{n-1} = -A_n \,. \end{aligned}$$

Since

$$\begin{aligned}&\frac{d}{dx}\bigl(x^{3n}(x-1)^{3n+1}e^{2x/s}\bigr)\\&=3nx^{3n-1}(x-1)^{3n+1}e^{2x/s}+(3n+1)x^{3n}(x-1)^{3n}e^{2x/s}+\frac{2}{s}x^{3n}(x-1)^{3n+1}e^{2x/s}\\&=-3nx^{3n-1}(x-1)^{3n}e^{2x/s}+\bigl(6n+1-\frac{2}{s}\bigr)x^{3n}(x-1)^{3n}e^{2x/s}+\frac{2}{s}x^{3n+1}(x-1)^{3n}e^{2x/s}\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dx}\bigl(x^{3n+1}(x-1)^{3n+1}e^{2x/s}\bigr)\\&=(3n+1)x^{3n}(x-1)^{3n+1}e^{2x/s}+(3n+1)x^{3n+1}(x-1)^{3n}e^{2x/s}+\frac{2}{s}x^{3n+1}(x-1)^{3n+1}e^{2x/s}\\&=2(3n+1)x^{3n+1}(x-1)^{3n}e^{2x/s}-(3n+1)x^{3n}(x-1)^{3n}e^{2x/s}+\frac{2}{s}x^{3n+1}(x-1)^{3n+1}e^{2x/s},\end{aligned}$$

by integrating from 0 to 1 and canceling the term of $x^{3n+1}(x-1)^{3n}e^{2x/s}$ we get

$$\begin{aligned}&\frac{s}{2}(-3n)\int_0^1 x^{3n-1}(x-1)^{3n}e^{2x/s}dx+\frac{(6n+1)s-1}{2}\int_0^1 x^{3n}(x-1)^{3n}e^{2x/s}dx\\&\qquad-\frac{1}{s(3n+1)}\int_0^1 x^{3n+1}(x-1)^{3n+1}e^{2x/s}dx=0\,.\end{aligned}$$

Thus, we have (2.2). Hence,

$$\begin{aligned}p_{5n+1}-q_{5n+1}e^{2/s}&=\left(\frac{(6n+1)s-1}{2}p_{5n}+p_{5n-1}\right)-\left(\frac{(6n+1)s-1}{2}q_{5n}+q_{5n-1}\right)e^{2/s}\\&=-\frac{(6n+1)s-1}{2}A_n+E_{n-1}=-B_n\,.\end{aligned}$$

Notice that

$$\begin{aligned}&\frac{d}{dx}\bigl(x^{3n+2}(x-1)^{3n+2}e^{2x/s}\bigr)\\&=(3n+2)x^{3n+1}(x-1)^{3n+2}e^{2x/s}+(3n+2)x^{3n+2}(x-1)^{3n+1}e^{2x/s}+\frac{2}{s}x^{3n+2}(x-1)^{3n+2}e^{2x/s}\\&=(6n+4)x^{3n+2}(x-1)^{3n+1}e^{2x/s}-(3n+2)x^{3n+1}(x-1)^{3n+1}e^{2x/s}+\frac{2}{s}x^{3n+2}(x-1)^{3n+2}e^{2x/s},\end{aligned}$$

$$\begin{aligned}&\frac{d}{dx}\bigl(x^{3n+2}(x-1)^{3n+1}e^{2x/s}\bigr)\\&=(3n+2)x^{3n+1}(x-1)^{3n+1}e^{2x/s}+(3n+1)x^{3n+2}(x-1)^{3n}e^{2x/s}+\frac{2}{s}x^{3n+2}(x-1)^{3n+1}e^{2x/s}\\&=(6n+3)x^{3n+1}(x-1)^{3n+1}e^{2x/s}+(3n+1)x^{3n+1}(x-1)^{3n}e^{2x/s}+\frac{2}{s}x^{3n+2}(x-1)^{3n+1}e^{2x/s}\end{aligned}$$

and

$$\frac{d}{dx}\left(x^{3n+1}(x-1)^{3n+1}e^{2x/s}\right)$$
$$= 2(3n+1)x^{3n+1}(x-1)^{3n}e^{2x/s} - (3n+1)x^{3n}(x-1)^{3n}e^{2x/s} + \frac{2}{s}x^{3n+1}(x-1)^{3n+1}e^{2x/s}\,.$$

Integrating these three equations from 0 to 1 and canceling the terms of $x^{3n+2}(x-1)^{3n+1}e^{2x/s}$ and $x^{3n+1}(x-1)^{3n}e^{2x/s}$, we get

$$\left(\frac{2}{s}\right)^2\int_0^1 x^{3n+2}(x-1)^{3n+2}e^{2x/s}dx$$
$$= (12n+6)(3n+2)\int_0^1 x^{3n+1}(x-1)^{3n+1}e^{2x/s}dx + (3n+2)(3n+1)\int_0^1 x^{3n}(x-1)^{3n}e^{2x/s}dx\,.$$

Thus, we have (2.3). Hence,

$$\begin{aligned} p_{5n+2} - q_{5n+2}e^{2/s} &= \big((12n+6)sp_{5n+1} + p_{5n}\big) - \big((12n+6)sq_{5n+1} + q_{5n}\big)e^{2/s} \\ &= -(12n+6)sB_n - A_n = -C_n\,. \end{aligned}$$

Notice that

$$\frac{d}{dx}\left(x^{3n+3}(x-1)^{3n+2}e^{2x/s}\right)$$
$$= (3n+3)x^{3n+2}(x-1)^{3n+2}e^{2x/s} + (3n+2)x^{3n+3}(x-1)^{3n+1}e^{2x/s} + \frac{2}{s}x^{3n+3}(x-1)^{3n+2}e^{2x/s}\,,$$

$$\frac{d}{dx}\left(x^{3n+3}(x-1)^{3n+1}e^{2x/s}\right)$$
$$= (3n+3)x^{3n+2}(x-1)^{3n+1}e^{2x/s} + (3n+1)x^{3n+3}(x-1)^{3n}e^{2x/s} + \frac{2}{s}x^{3n+3}(x-1)^{3n+1}e^{2x/s}$$
$$= \left(6n+4+\frac{2}{s}\right)x^{3n+2}(x-1)^{3n+1}e^{2x/s} + (3n+1)x^{3n+2}(x-1)^{3n}e^{2x/s} + \frac{2}{s}x^{3n+2}(x-1)^{3n+2}e^{2x/s}\,,$$

$$\frac{d}{dx}\left(x^{3n+2}(x-1)^{3n+2}e^{2x/s}\right)$$
$$= \left(6n+4-\frac{2}{s}\right)x^{3n+2}(x-1)^{3n+1}e^{2x/s} - (3n+2)x^{3n+1}(x-1)^{3n+1}e^{2x/s} + \frac{2}{s}x^{3n+3}(x-1)^{3n+1}e^{2x/s}$$

and

$$\frac{d}{dx}\left(x^{3n+2}(x-1)^{3n+1}e^{2x/s}\right)$$
$$= (3n+2)x^{3n+1}(x-1)^{3n+1}e^{2x/s} + (3n+1)x^{3n+2}(x-1)^{3n}e^{2x/s} + \frac{2}{s}x^{3n+2}(x-1)^{3n+1}e^{2x/s}\,.$$

Integrating these four equations from 0 to 1 and eliminating the terms of $x^{3n+3}(x-1)^{3n+1}e^{2x/s}$, $x^{3n+2}(x-1)^{3n+1}e^{2x/s}$ and $x^{3n+2}(x-1)^{3n}e^{2x/s}$, we get

$$\frac{4}{s}\int_0^1 x^{3n+3}(x-1)^{3n+2}e^{2x/s}dx$$
$$= -\left(12n+10-\frac{2}{s}\right)\int_0^1 x^{3n+2}(x-1)^{3n+2}e^{2x/s}dx-(3n+2)\int_0^1 x^{3n+1}(x-1)^{3n+1}e^{2x/s}dx\,.$$

Thus, we have (2.4). Hence,

$$p_{5n+3}-q_{5n+3}e^{2/s} = \left(\frac{(6n+5)s-1}{2}p_{5n+2}+p_{5n+1}\right)-\left(\frac{(6n+5)s-1}{2}q_{5n+2}+q_{5n+1}\right)e^{2/s}$$
$$= -\frac{(6n+5)s-1}{2}C_n - B_n = D_n\,.$$

Since $x^{3n+3}(x-1)^{3n+2}-x^{3n+2}(x-1)^{3n+2} = x^{3n+2}(x-1)^{3n+3}$, we get $D_n - C_n = E_n$, which is (2.5). Hence,

$$p_{5n+4}-q_{5n+4}e^{2/s} = (p_{5n+3}+p_{5n+2})-(q_{5n+3}+q_{5n+2})e^{2/s} = D_n - C_n = E_n\,.$$

□

3. The Continued Fraction of e^2

Let $\frac{p_n^*}{q_n^*}$ be the n^{th} convergent of the continued fraction of $e^2 = [7;\overline{3k-1,1,1,3k,12k+6}\,]_{k=1}^\infty$. Then, for $n \geq 0$ we have

$$\frac{p_n^*}{q_n^*} = \frac{p_{n+2}}{q_{n+2}}\,,$$

where p_n/q_n is the n-th convergent of the continued fraction of $e^{2/s}$ mentioned above. Thus, by replacing p_n/q_n by p_{n-2}^*/q_{n-2}^* ($n \geq 2$), Theorem 1.1 with Lemma 2.1 holds for $s=1$.

4. Additional Comments

Some results in our theorem can be derived directly from Osler's results. By the relation for $i = 1, 2, \ldots$

$$[\ldots,1,a_i-\frac{1}{2},1,\ldots] = [\ldots,1,a_{3i-2},4a_{3i-1}+2,a_{3i},1,\ldots]\,,$$

if we replace s by $s/2$ in the continued fraction $[1;\overline{(2k-1)s-1,1,1}]_{k=1}^\infty$, then we have

$$[1;\overline{(2k-1)s/2-1,1,1}]_{k=1}^\infty = \Big[1;\overline{ks-\frac{s+1}{2}-\frac{1}{2},1,1}\Big]_{k=1}^\infty$$
$$= \Big[1;\overline{\frac{(6k-5)s-1}{2},(12k-6)s,\frac{(6k-1)s-1}{2},1,1}\,\Big]_{k=1}^\infty\,.$$

Therefore, if we replace s by $s/2$ and n by $3n$ in the Osler's integral for $p_{3n} - q_{3n}e^{1/s}$, we get our integral for $p_{5n} - q_{5n}e^{2/s}$. If we replace s by $s/2$ and n by $3n+2$ in the Osler's integral for $p_{3n+1} - q_{3n+1}e^{1/s}$, we get our integral for $p_{5n+3} - q_{5n+3}e^{2/s}$. If we replace s by $s/2$ and n by $3n + 2$ in the Osler's integral for $p_{3n+2} - q_{3n+2}e^{1/s}$, we get our integral for $p_{5n+4} - q_{5n+4}e^{2/s}$.

5. Acknowledgement

This work was initiated when the author stayed in the Department of Mathematics and Statistics, University of Tennessee at Martin in 2006. In special, he thanks Sarah Holliday, Bill Austin, Louis Kolitsch, and Chris Caldwell. The author also thanks the anonymous referee for some helpful comments.

References

[1] E. B. Burger, *Exploring the Number Jungle: A journey into Diophantine Analysis*, STML 8, American Mathematical Society, 2000.

[2] H. Cohn, *A short proof of the simple continued fraction expansion of* e, Amer. Math. Monthly **113** (2006), 57–62.

[3] C. S. Davis, *On some simple continued fractions connected with* e, J. London Math. Soc, **20** (1945), 194–198.

[4] A. Ya. Khinchin, *Continued fractions*, Dover, New York, 1997.

[5] T. Osler, *A proof of the continued fraction expansion of* $e^{1/M}$, Amer. Math. Monthly **113** (2006), 62–66.

[6] O. Perron, *Die Lehre von den Kettenbrüchen*, Chelsea Publishing Company, New York, 1950.