ON THE FROBENIUS NUMBER OF FIBONACCI NUMERICAL SEMIGROUPS

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Abstract

In this note we investigate the Frobenius number of *Fibonacci numerical semigroups*, that is, numerical semigroups generated by a set of Fibonacci numbers.

1. Introduction

Let $s_1, s_2, \ldots, s_n$ be positive integers such that their greatest common divisor is one. Let $S =< s_1, \ldots, s_n >$ be the numerical semigroup[1] generated by $s_1, \ldots, s_n$. A *Fibonacci numerical semigroup* is a numerical semigroup generated by a set of Fibonacci numbers $F_{i_1}, \ldots, F_{i_r}$, for some integers $3 \leq i_1 < \cdots < i_r$ where $\gcd(F_{i_1}, \ldots, F_{i_r}) = 1$.

The so-called *Frobenius number*, denoted by $g(s_1, \ldots, s_n)$, is defined as the largest integer not belonging to S, that is, the largest integer that is not representable as a nonnegative integer combination of $s_1, \ldots, s_n$. It is well known that $g(s_1, s_2) = s_1 s_2 - s_1 - s_2$. In general, finding $g(S)$ is a difficult problem and so formulas and upper bounds for particular sequences are of interest. For instance, it is known [3] $g(S)$ when S is an arithmetical sequence

$$g(a, a+d, \ldots, a+kd) = a\left(\left\lfloor \frac{a-2}{k} \right\rfloor\right) + d(a-1) \tag{1}$$

[1] Recall that a *semigroup* $(S, *)$ consists of a nonempty set S and an associative binary operation $*$ on S. If, in addition, there exists an element, which is usually denoted by 0, in S such that $a + 0 = 0 + a = a$ for all $a \in S$, we say that $(S, *)$ is a *monoid*. A *numerical semigroup* is a submonoid of $\mathbb{N}$ such that the greatest common divisor of its elements is equal to one.

We refer the reader to [2] where a complete account on the Frobenius problem can be found.

In this note, we investigate the value of $g(F_i, F_j, F_l)$ for some triples $3 \leq i < j < l$ (we always assume that $\gcd(F_i, F_j, F_l) = 1$; recall that $\gcd(F_i, F_{i+l}) = 1$ if $i \not| l$).

We first notice that $g(F_i, F_{i+1}, F_l) = g(F_i, F_{i+1})$ for any integer $l \geq i+2$. Indeed, since $F_l = F_{i+m} = F_m F_{i+1} + F_{m-1} F_i$ is a nonnegative integer combination of F_i and F_{i+1} then the semigroups $< F_i, F_{i+1}, F_l >$ and $< F_i, F_{i+1} >$ generate the same set of elements and thus they have the same Frobenius number.

Let us consider then $g(F_i, F_{i+2}, F_l)$ with $l \geq i+3$. We notice that the case when $l = i+3$ is a consequence of equation (1) since the triple $\{F_i, F_{i+2}, F_{i+3}\} = \{F_i, F_i + F_{i+1}, F_i + 2F_{i+1}\}$ form an arithmetical sequence. However, it can be checked that $\{F_i, F_{i+2}, F_{i+k}\}$ do not form an arithmetical sequence when $k \geq 3$ and the calculation of $g(F_i, F_{i+2}, F_{i+k})$ is more complicated.

We state our main result.

Theorem 1. *Let $i, k \geq 3$ be integers and let $r = \lfloor \frac{F_i - 1}{F_k} \rfloor$. Then,*

$$g(F_i, F_{i+2}, F_{i+k}) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1) & \text{if } r = 0 \text{ or } r \geq 1 \text{ and} \\ & F_{k-2}F_i < (F_i - rF_k)F_{i+2}, \\ (rF_k - 1)F_{i+2} - F_i((r-1)F_{k-2} + 1) & \text{otherwise.} \end{cases}$$

Let $N(a_1, \ldots, a_n)$ be the number of positive integers with no representation by a nonnegative integer combination of $a_1, \ldots, a_n$. Theorem 1 yields to the following result.

Corollary 2. *Let $i, k \geq 3$ be integers and let $r = \lfloor \frac{F_i - 1}{F_k} \rfloor$. Then,*

$$N(F_i, F_{i+2}, F_{i+k}) = \frac{(F_i - 1)(F_{i+2} - 1) - rF_{k-2}(2F_i - F_k(1 + r))}{2}.$$

2. Fibonacci semigroups

In order to prove Theorem 1 we need the following result due to Brauer and Shockley [1].

Lemma 3. *Let $1 < a_1 < \cdots < a_n$ be integers with $\gcd(a_1, \ldots, a_n) = 1$. Then,*

$$g(a_1, \ldots, a_n) = \max_{l \in \{1, 2, \ldots, a_n - 1\}} \{t_l\} - a_1,$$

where t_l is the smallest positive integer congruent to l modulo a_1, that is representable as a nonnegative integer combination of $a_2, \ldots, a_n$.

Proof. Let L be a positive integer. If $L \equiv 0 (\bmod a_1)$ then L is a nonnegative integer combination of a_1. If $L \equiv l (\bmod a_1)$ then L is a nonnegative integer combination of $a_1, \ldots, a_n$ if and only if $L \geq t_l$. □

Let $T^* = \{t_0^*, \ldots, t_{F_i-1}^*\}$ where t_l^* is the smallest positive integer congruent to l modulo F_i, that is representable as a nonnegative integer combination of F_{i+2} and F_{i+k}. By Lemma 3, it suffices to find t_l^* for each $l = 0, 1, \ldots, F_i - 1$. To this end, we consider all nonnegative integer combinations of F_{i+2} and F_{i+k}. We construct the following table, denoted by T_1, having as entry $t_{x,y}$ the combination of the form $xF_{i+2} + yF_{i+k}$ with integers $x, y \geq 0$, see below.

$x\backslash y$	0	1	2	$\cdots$
0	0	F_{i+k}	$2F_{i+k}$	$\cdots$
1	F_{i+2}	$F_{i+k} + F_{i+2}$	$2F_{i+k} + F_{i+2}$	$\cdots$
2	$2F_{i+2}$	$F_{i+k} + 2F_{i+2}$	$2F_{i+k} + 2F_{i+2}$	$\cdots$
3	$3F_{i+2}$	$F_{i+k} + 3F_{i+2}$	$2F_{i+k} + 3F_{i+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
$F_k - 1$	$(F_k - 1)F_{i+2}$	$F_{i+k} + (F_k - 1)F_{i+2}$	$2F_{i+k} + (F_k - 1)F_{i+2}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

We notice that

$$F_{i+k} = F_{k-2}F_{i+1} + F_{k-1}F_{i+2} = F_{k-2}(F_{i+2} - F_i) + F_{k-1}F_{i+2} = F_{i+2}F_k - F_{k-2}F_i$$

so, we obtain that

$$xF_{i+2} + yF_{i+k} = xF_{i+2} + y(F_{i+2}F_k - F_{k-2}F_i) = (x + yF_k)F_{i+2} - yF_{k-2}F_i.$$

Thus, T_1 can also be given by the following table, denoted by T_2,

$x\backslash y$	0	1	2	$\cdots$	r	$\cdots$
0	0	$F_kF_{i+2} - F_{k-2}F_i$	$2F_kF_{i+2} - 2F_{k-2}F_i$	$\cdots$	$rF_kF_{i+2} - rF_{k-2}F_i$	$\cdots$
1	F_{i+2}	$(1 + F_k)F_{i+2} - F_{k-2}F_i$	$(1 + 2F_k)F_{i+2} - 2F_{k-2}F_i$	$\cdots$	$(1 + rF_k)F_{i+2} - rF_{k-2}F_i$	$\cdots$
2	$2F_{i+2}$	$(2 + F_k)F_{i+2} - F_{k-2}F_i$	$(2 + 2F_k)F_{i+2} - 2F_{k-2}F_i$	$\cdots$	$(2 + rF_k)F_{i+2} - rF_{k-2}F_i$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
l	lF_{i+2}	$(l + F_k)F_{i+2} - F_{k-2}F_i$	$(l + 2F_k)F_{i+2} - 2F_{k-2}F_i$	$\cdots$	$(l + rF_k)F_{i+2} - rF_{k-2}F_i$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	
$F_k - 1$	$(F_k - 1)F_{i+2}$	$(2F_k - 1)F_{i+2} - F_{k-2}F_i$	$(3F_k - 1)F_{i+2} - 2F_{k-2}F_i$		$\cdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	

Let S be the set formed by the first $F_k - 1$ entries of columns zero, one, two, and so on, that is, $S = \{t_{0,0}, t_{1,0}, \ldots, t_{F_k-1,0}, t_{0,1}, t_{1,1}, \ldots, t_{F_k-1,1}, \ldots, t_{0,r}, t_{1,r}, \ldots, t_{F_k-1,r}, \ldots\}$.

Remark 4.

(a) *Let* $r = \lfloor \frac{F_i - 1}{F_k} \rfloor$ *and set* $F_i - 1 = rF_k + l$ *for some integer* $0 \le l \le F_k - 1$. *Let*

$$S' = \{t_{0,0}, t_{1,0}, \ldots, t_{F_k-1,0}, t_{0,1}, t_{1,1}, \ldots, t_{F_k-1,1}, \ldots, t_{2,r}, t_{1,r}, \ldots, t_{l,r}\},$$

Then, for each $t_{x,y} = (x + yF_k)F_{i+2} - yF_{k-2}F_i \in S'$ *we have that* $0 \le x + yF_k \le F_i - 1$. *Moreover, since* $\gcd(F_{i+2}, F_i) = 1$ *then* S' *forms a complete system of rests modulo* F_i.

(b) *The elements of* S *can be represented as* $s_x = xF_{i+2} - \lfloor \frac{x}{F_k} \rfloor F_{k-2}F_i$ *for* $x = 0, 1, \ldots$. *Indeed, it can be checked that* $S = \bigcup_{q \ge 1} S_q$ *where*

$$S_q = \{s_{qF_k}, s_{qF_k+1}, \ldots, s_{(q+1)F_k-1}\} = \{t_{0,q}, \ldots, t_{F_k-1,q}\}$$

for each integer $q = 0, 1, 2, \ldots$.

(c) *By using table* T_2 *we have that* $t_{i,j} < t_{k,l}$ *for all* $i \le k$ *and all* $j \le l$.

Lemma 5. *Let* $t_{u,v}$ *be an entry of* T_1 *such that* $t_{u,v} \notin S'$. *Then, there exists* $t_{x,y} \in S'$ *such that* $t_{u,v} \equiv t_{x,y} (\bmod F_i)$ *and* $t_{u,v} > t_{x,y}$.

Proof. We first notice that the set S can be written as follows

$$\begin{array}{llllllll} \{s_0, & \ldots & , s_{F_k-1}, & s_{F_k}, & \ldots & , s_{2F_k-1}, & \ldots & , s_{rF_k}, & \ldots & , s_{rF_k+l} = s_{F_i-1}, \\ s_{F_i}, & \ldots & , s_{F_i+F_k-1}, & s_{F_i+F_k}, & \ldots & , s_{F_i+2F_k-1}, & \ldots & , s_{F_i+rF_k}, & \ldots & , s_{2F_i-1}, \\ s_{2F_i}, & \ldots & , s_{2F_i+F_k-1}, & s_{2F_i+F_k}, & \ldots & , s_{2F_i+2F_k-1}, & \ldots & , s_{2F_i+rF_k}, & \ldots & , s_{3F_i-1}, \ldots\} \end{array}$$

where $S' = \{s_0, \ldots, s_{F_k-1}, s_{F_k}, \ldots, s_{2F_k-1}, \ldots, s_{rF_k}, \ldots, s_{F_i-1}\}$. We have two cases.

`Case A.` Suppose that $t_{u,v} \in S \setminus S'$. Then $t_{u,v}$ is of the form s_{pF_i+g} for some integers $p \ge 1$ and $0 \le g \le F_i - 1$. It is clear that,

$$s_g = gF_{i+2} - \left\lfloor \frac{g}{F_k} \right\rfloor F_iF_{k-2} \equiv (pF_i + g)F_{i+2} - \left\lfloor \frac{pF_i + g}{F_k} \right\rfloor F_iF_{k-2} = g_{pF_i+g} (\bmod F_i.)$$

We will show that $s_{pF_i+g} > s_g$. To this end, it suffices to prove that $s_{F_i+g} > s_g$ (since $s_{pF_i+g} \ge s_{F_i+g}$). Recall that $r = \lfloor \frac{F_i-1}{F_k} \rfloor$ and that $F_i - 1 = rF_k + l$ for some integer $0 \le l \le F_k - 1$. We have two subcases.

`Subcase a.` If $r = 0$ then $F_k \ge F_i$. If $F_k = F_i$ then $s_{F_i+g} = t_{g,1}$ and, by Remark 4(c), $t_{g,0} < t_{g,1}$. If $F_k > F_i$ then $s_{F_i+g} = t_{q,0}$ for some integer $q \ge F_i$ and, by Remark 4(c), $t_{g,0} < t_{q,0}$.

`Subcase b.` If $r \geq 1$, then $s_{F_i+g} > s_g$ holds if and only if

$$(F_i+g)F_{i+2} - \left\lfloor \frac{F_i+g}{F_k} \right\rfloor F_iF_{k-2} > gF_{i+2} - \left\lfloor \frac{g}{F_k} \right\rfloor F_iF_{k-2}$$

or equivalently if and only if

$$F_{i+2} > F_{k-2}\left(\left\lfloor \frac{F_i+g}{F_k} \right\rfloor - \left\lfloor \frac{g}{F_k} \right\rfloor\right).$$

Let $g = mF_k + n$ with $0 \leq n \leq F_k - 1$. Since $F_i - 1 = rF_k + l$ with $0 \leq l \leq F_k - 1$, then

$$\left\lfloor \frac{F_i - 1 + g + 1}{F_k} \right\rfloor = \left\lfloor \frac{rF_k + l + mF_k + n + 1}{F_k} \right\rfloor \leq r + m + 1$$

and thus

$$\left\lfloor \frac{F_i+g}{F_k} \right\rfloor - \left\lfloor \frac{g}{F_k} \right\rfloor \leq r + m + 1 - m = r + 1.$$

So, it is enough to show that $F_{i+2} > (r+1)F_{k-2}$ or equivalently to show that $F_i + F_{i+1} > (r+1)F_{k-2}$. Since $F_i = rF_k + l + 1$ then the latter inequality holds if and only if $rF_k + l + 1 + F_{i+1} > rF_{k-2} + F_{k-2}$, that is, if and only if

$$r(F_k - F_{k-2}) + l + 1 + F_{i+1} = r(F_{k-1}) + l + 1 + F_{i+1} > F_{k-2}$$

which is true since $r \geq 1$.

`Case B.` Suppose that $t_{u,v} \notin S$. Then we have that $0 \leq x \leq F_k - 1 < u$. If $v \geq y$ then, by Remark 4(c), $t_{x,y} < t_{x,v} < t_{u,v}$. So, we suppose that $v < y$. Since, $t_{u,v} \equiv t_{x,y} (\bmod F_i)$ then $u + vF_k \equiv x + yF_k (\bmod F_i)$ but, by Remark 4(a), $0 \leq x + yF_k \leq F_i - 1$ so $u + vF_k = d(x + yF_k)$ for some integer $d \geq 1$ and thus $u + vF_k \geq x + yF_k$. Also, since $v < y$, then $-vF_{k-2}F_i > -yF_{k-2}F_i$. So, combining the last two inequalities we have that

$$t_{u,v} = (u + vF_k)F_{i+2} - vF_{k-2}F_i > (x + yF_k)F_{i+2} - yF_{k-2}F_i = t_{x,y}.$$

□

We may now prove Theorem 1.

Proof of Theorem 1. Let $T^* = \{t^*_0, \ldots, t^*_{F_i-1}\}$ where t^*_l is the smallest positive integer congruent to l modulo F_i, that is representable as a nonnegative integer combination of F_{i+2} and F_{i+k}. Let $s_x = xF_{i+2} - \lfloor \frac{x}{F_k} \rfloor F_{k-2}F_i$ for $x = 0, 1, \ldots$. By Lemma 5, we have that for each $x = 0, \ldots, F_i - 1$, s_x is the smallest positive integer congruent to l modulo F_i, for some integer $0 \leq l \leq F_i - 1$, that is representable as a nonnegative integer combination of F_{i+2} and F_{i+k}, that is, $S' = T^*$ where $S' = \{s_0, \ldots, s_{F_k-1}, s_{F_k}, \ldots, s_{2F_k-1}, \ldots, s_{rF_k}, \ldots, s_{F_i-1}\}$. Now, by Remark 4(c), if $r \geq 1$ then

$$t_{F_k-1,i} = \max_{0 \leq x \leq F_k-1} \{t_{x,i} | t_{x,i} \in S'\} \text{ for each } i = 0, \ldots, r-1,$$

$$t_{F_k-1,r-1} = \max_{0\le i\le r-1}\{t_{F_k-1,i} | t_{F_k-1,i} \in S'\},$$

and

$$t_{l,r} = \max_{0\le x\le l}\{t_{x,r} | t_{x,r} \in S'\}.$$

Thus,

$$\max\{s|s\in S'\} = \begin{cases} t_{l,r} & \text{if } r=0,\\ \max\{t_{F_k-1,r-1}, t_{l,r}\} & \text{otherwise.}\end{cases}$$

The result follows since $t_{l,r} > t_{F_k-1,r-1}$ if and only if

$$(rF_k+l)F_{i+2} - rF_{k-2}F_i = (F_i-1)F_{i+2} - rF_{k-2}F_i > (rF_k-1)F_{i+2} - (r-1)F_{k-2}F_i$$

or equivalently, if and only if $F_{i+2}(F_i - rF_k) > F_{k-2}F_i$. □

We will use the following result due to Selmer [4] to show Corollary 2.

Lemma 6. *Let* $1 < a_1 < \cdots < a_n$ *be integers with* $\gcd(a_1,\ldots,a_n) = 1$*. If* $L = \{1,\ldots,a_1-1\}$ *then* $N(a_1,\ldots,a_n) = \dfrac{1}{a_1}\sum_{l\in L} t_l - \dfrac{a_1-1}{2}$*, where* t_l *is the smallest positive integer congruent to* l *modulo* a_1*, that is representable as a nonnegative integer combination of* $a_2,\ldots,a_n$*.*

Proof. The number of $M \equiv l \not\equiv 0 \pmod{a_1}$ with $0 < M < t_l$ is given by $\lfloor \frac{t_l}{a_1}\rfloor$. By assuming that $0 < l < a_1$, we have $\lfloor \frac{t_l}{a_1}\rfloor = \frac{t_l - l}{a_1}$. The result follows by summing over $l \in L$. □

Proof of Corollary 2. Let $r = \lfloor \frac{F_i-1}{F_k}\rfloor$ and set $F_i - 1 = rF_k + l$ for some integer $0 \le l \le F_k - 1$. By Lemma 6 and Remark 4(b), we have

$$\begin{aligned} N(F_i, F_{i+2}, F_{i+k}) &= \tfrac{1}{F_i}\sum_{s\in S'} s - \tfrac{F_i-1}{2}\\ &= \tfrac{1}{F_i}\sum_{j=0}^{F_i-1}(jF_{i+2} - F_{k-2}\lfloor \tfrac{j}{F_k}\rfloor F_i) - \tfrac{F_i-1}{2}\\ &= \tfrac{1}{F_i}\left(F_{i+2}\tfrac{(F_i-1)F_i}{2}\right) - \tfrac{1}{F_i}(F_{k-2}F_i)\sum_{j=0}^{F_i-1}\left\lfloor \tfrac{j}{F_k}\right\rfloor - \tfrac{F_i-1}{2}. \end{aligned}$$

By using the table T_1, it is easy to verify that

$$\sum_{j=0}^{F_i-1}\left\lfloor \frac{j}{F_k}\right\rfloor = 0 + F_k + 2F_k + \cdots + (r-1)F_k + r(l+1) = \frac{F_k(r-1)r}{2} + r(l+1)$$

and, since $l+1 = F_i - rF_k$, that

$$\begin{aligned} N(F_i, F_{i+2}, F_{i+k}) &= \tfrac{F_{i+2}(F_i-1)}{2} - F_{k-2}\left(\tfrac{F_k(r-1)r}{2} + r(F_i - rF_k)\right) - \tfrac{F_i-1}{2}\\ &= \tfrac{(F_i-1)(F_{i+2}-1)}{2} - F_{k-2}\left(\tfrac{F_kr^2 - F_kr + 2F_ir - 2r^2F_k}{2}\right)\\ &= \tfrac{(F_i-1)(F_{i+2}-1) - rF_{k-2}(2F_i - F_k(1+r))}{2}. \end{aligned}$$

□

We end with the following problem.

Problem. Find upper (and lower) bounds (or formulas) for $g(F_i, F_j, F_k)$ for further triples $3 \le i < j < k$.

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