AN INFINITE FAMILY OF DUAL SEQUENCE IDENTITIES

Robin Chapman
Department of Mathematics, University of Exeter, Exeter, EX4 4QE, UK
rjc@maths.ex.ac.uk

Received: 7/27/05, Accepted: 11/13/05, Published: 12/9/05

Abstract

We subsume three identities of Sun [2] into an infinite family of identities, and consider some special cases.

1. Introduction

Let $(a_n)_{n=0}^{\infty}$ be a sequence of complex numbers. We call the sequence $(a_n^*)_{n=0}^{\infty}$ defined by

$$a_n^* = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i$$

the *dual sequence* of (a_n). Repeating this operation gives rise to the original sequence: $a_n^{**} = a_n$ for all n [1, 192–193].

A sequence (a_n) satisfying $a_n^* = a_n$ is called *self-dual*. An important example of a self-dual sequence is $((-1)^n B_n)$ where the Bernoulli numbers B_n satisfy

$$\sum_{n=0}^{\infty} B_n \frac{t^n}{n!} = \frac{t}{\exp(t) - 1}.$$

Sun [2] considered the polynomials

$$A_n(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i x^{n-i} \quad \text{and} \quad A_n^*(x) = \sum_{i=0}^{n} \binom{n}{i} (-1)^i a_i^* x^{n-i}. \tag{1}$$

When $a_k = (-1)^k B_k$ then $A_n(x) = B_n(x)$, the classical Bernoulli polynomial. Sun's main theorem [2, Theorem 1.1] gives three identities involving the polynomials A_n and A_n^*. Using these identities he unifies and generalizes various identities concerning Bernoulli numbers and polynomials in the recent literature. The main theorem here gives an infinite family of identities subsuming Sun's identities.

2. The main theorem

We begin with preliminaries concerning exponential generating functions. Throughout we let $(a_n)_{n=0}^{\infty}$ be a sequence and $(a_n^*)_{n=0}^{\infty}$ be its dual. Also we let A_n and A_n^* be the polynomials defined by (1).

We define
$$\alpha(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} \quad \text{and} \quad \alpha^*(t) = \sum_{n=0}^{\infty} a_n^* \frac{t^n}{n!}$$
as the exponential generating functions of the sequences (a_n) and (a_n^*). Then
$$\alpha^*(t) = \sum_{n=0}^{\infty}\sum_{i=0}^{n} \binom{n}{i}(-1)^i a_i \frac{t^n}{n!} = \sum_{i=0}^{\infty} a_i \frac{(-t)^i}{i!} \sum_{n=i}^{\infty} \frac{t^{n-i}}{(n-i)!} = \exp(t)\alpha(-t).$$
As a consequence,
$$\exp(t)\alpha^*(-t) = \exp(t)\exp(-t)\alpha(t) = \alpha(t)$$
which recovers the fact that $a_n^{**} = a_n$.

It is well-known that the Bernoulli polynomials $B_n(x)$ are symmetric or anti-symmetric about $1/2$ according to the parity of n, to wit
$$B_n(1-x) = (-1)^n B_n(x).$$
The following lemma generalizes this to arbitrary sequences, relating the polynomials A_n and A_n^*.

Lemma 1 (Duality principle) *For each n, $A_n^*(x) = (-1)^n A_n(1-x)$.*

Proof. First
$$\begin{aligned}\sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\binom{n}{i}(-1)^i a_i x^{n-i}\frac{t^n}{n!} \\ &= \sum_{i=0}^{\infty} a_i \frac{(-t)^i}{i!}\sum_{n=i}^{\infty}\frac{(xt)^{n-i}}{(n-i)!} = \exp(xt)\alpha(-t).\end{aligned}$$
Similarly
$$\sum_{n=0}^{\infty} A_n^*(x)\frac{t^n}{n!} = \exp(xt)\alpha^*(-t) = \exp((x-1)t)\alpha(t) = \sum_{n=0}^{\infty} A_n(1-x)\frac{(-t)^n}{n!}.$$
Hence $A_n^*(x) = (-1)^n A_n(1-x)$ for all n. □

Sun's main theorem uses three variables x, y and z satisfying $x+y+z=1$ and expresses three sums featuring polynomials $A_m(y)$ and $A_n^*(z)$ with coefficients in x as expressions depending only on x. Our main theorem generalizes the first two of these identities. We later see how to generalize the third.

Theorem 2 *Let $x+y+z=1$. Then for all integers k, l, $r \geq 0$,*

$$\begin{aligned}
&(-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \frac{(l+j)! A_{l+j+r}(y)}{(l+j+r)!} \\
&- (-1)^{j+r} \sum_{j=0}^{l} \binom{l}{j} x^{l-j} \frac{(k+j)! A^*_{k+j+r}(z)}{(k+j+r)!} \\
= \; & k! l! (-x)^{k+l+1} \sum_{i=0}^{r-1} \frac{(-1)^i a_i}{i!(k+l+r-i)!} \\
&\times \sum_{j=0}^{r-1-i} \binom{k+r-i-1-j}{k} \binom{l+j}{l} y^{r-i-1-j} (1-z)^j.
\end{aligned} \tag{2}$$

Proof. We first recast (2) in a more symmetric form. We replace z by $1-z$ and use the duality principle. Then $x = 1-y-z$ must be replaced by $z-y$. After some rearrangement we find that (2) is equivalent to

$$\begin{aligned}
&\sum_{j=0}^{k} \binom{k}{j} (z-y)^{k-j} \frac{(l+j)! A_{l+j+r}(y)}{(l+j+r)!} \\
&- \sum_{j=0}^{l} \binom{l}{j} (y-z)^{l-j} \frac{(k+j)! A_{k+j+r}(z)}{(k+j+r)!} \\
= \; & (-1)^k k! l! (y-z)^{k+l+1} \sum_{i=0}^{r-1} \frac{(-1)^i a_i}{i!(k+l+r-i)!} \\
&\times \sum_{j=0}^{r-1-i} \binom{k+r-i-1-j}{k} \binom{l+j}{l} y^{r-i-1-j} z^j.
\end{aligned} \tag{3}$$

Define

$$\mathcal{A}_r(k,l;y,z) = \sum_{j=0}^{k} \binom{k}{j} (z-y)^{k-j} (l+j)! \frac{A_{l+j+r}(y)}{(l+j+r)!}$$

so that (3) is concerned with $\mathcal{A}_r(k,l;y,z) - \mathcal{A}_r(l,k;z,y)$. We consider the exponential generating function of the $\mathcal{A}_r(k,l;y,z)$. We calculate

$$\begin{aligned}
&\sum_{k,l=0}^{\infty} \mathcal{A}_r(k,l;y,z) \frac{t^k u^l}{k! l!} \\
= \; & \sum_{k,l=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} (z-y)^{k-j} (l+j)! \frac{A_{l+j+r}(y)}{(l+j+r)!} \frac{t^k u^l}{k! l!} \\
= \; & \sum_{j,l=0}^{\infty} \binom{l+j}{j} t^j u^l \frac{A_{l+j+r}(y)}{(l+j+r)!} \sum_{k=j}^{\infty} \frac{(z-y)^{k-j} t^{k-j}}{(k-j)!} \\
= \; & \exp((z-y)t) \sum_{p=0}^{\infty} (t+u)^p \frac{A_{p+r}(y)}{(p+r)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\exp((z-y)t)}{(t+u)^r}\left[\exp(y(t+u))\alpha(-t-u) - \sum_{s=0}^{r-1} A_s(y)\frac{(t+u)^s}{s!}\right] \\
&= \frac{\exp(zt+yu)\alpha(-t-u)}{(t+u)^r} - \frac{\exp((z-y)t)}{(t+u)^r}\sum_{s=0}^{r-1} A_s(y)\frac{(t+u)^s}{s!}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{k,l=0}^{\infty}[\mathcal{A}_r(k,l;y,z) - \mathcal{A}_r(l,k;z,y)]\frac{t^k u^l}{k!l!} \\
&= \frac{1}{(t+u)^r}\sum_{s=0}^{r-1}[\exp((y-z)u)A_s(z) - \exp((z-y)t)A_s(y)]\frac{(t+u)^s}{s!}.
\end{aligned}
$$

As

$$
\sum_{s=0}^{r-1} A_s(x)\frac{v^s}{s!} = \sum_{s=0}^{r-1}\sum_{i=0}^{s}(-1)^i\frac{a_i}{i!}\frac{x^{s-i}}{(s-i)!}v^s = \sum_{i=0}^{r-1}(-1)^i a_i\frac{v^i}{i!}\sum_{s=i}^{r-1}\frac{(xv)^{s-i}}{(s-i)!}
$$

then

$$
\begin{aligned}
&\sum_{k,l=0}^{\infty}[\mathcal{A}_r(k,l;y,z) - \mathcal{A}_r(l,k;z,y)]\frac{t^k u^l}{k!l!} \\
&= \sum_{i=0}^{r-1}\frac{(-1)^i a_i}{i!(t+u)^{r-i}}\sum_{s=i}^{r-1}\frac{(t+u)^{s-i}}{(s-i)!}[z^{s-i}\exp((y-z)u) - y^{s-i}\exp((z-y)t)] \\
&= \sum_{i=0}^{r-1}\frac{(-1)^i a_i}{i!(t+u)^{r-i}}\sum_{s=0}^{r-1-i}\frac{(t+u)^s}{s!}[z^s\exp((y-z)u) - y^s\exp((z-y)t)] \\
&= \sum_{i=0}^{r-1}\frac{(-1)^i a_i}{i!}\mathcal{B}_{r-i}(t,u,y,z)
\end{aligned}
$$

where

$$
\mathcal{B}_q(t,u,y,z) = \frac{1}{(t+u)^q}\sum_{s=0}^{q-1}\frac{(t+u)^s}{s!}[z^s\exp((y-z)u) - y^s\exp((z-y)t)].
$$

We claim that

$$
\begin{aligned}
&\mathcal{B}_q(t,u,y,z) \\
&= \sum_{k,l=0}^{\infty}\frac{(y-z)^{k+l+1}}{(k+l+r)!}(-t)^k u^l\sum_{j=0}^{q-1}\binom{k+q-1-j}{q-1-j}\binom{l+j}{j}y^{q-1-j}z^j \\
&= \sum_{m=0}^{\infty}\frac{(y-z)^{m+1}}{(m+q)!}F_{m,q}(t,u,y,z)
\end{aligned}
\tag{4}
$$

where

$$
F_{m,q}(t,u,y,z) = \sum_{k=0}^{m}\sum_{j=0}^{q-1}\binom{k+q-1-j}{q-1-j}\binom{m-k+j}{j}(-t)^k u^{m-k}y^{q-1-j}z^j.
$$

We remark that

$$F_{m,q}(t,u,y,z) = \sum_{j=0}^{q-1} G_{m,q-1-j,j}(t,u) y^{q-1-j} z^j$$

where

$$G_{m,i,j}(t,u) = \sum_{k=0}^{m} \binom{k+i}{i}\binom{m-k+j}{j}(-t)^k u^{m-k}.$$

We need to calculate $(t+u)G_{m,i,j}(t,u)$. In doing this we adopt the convention that $\binom{x}{i}$ is defined for arbitrary x by insisting that it be a polynomial of degree i. Adopting this convention ensures that $\binom{-1}{i} = \binom{i+1}{i} = 0$. We must take particular care with the cases where $i = 0$ or $j = 0$. If $i > 0$ and $j > 0$ then

$$\begin{aligned}
& (t+u)G_{m,i,j}(t,u) \\
= & \sum_{k=0}^{m+1} \binom{k+i}{i}\binom{m-k+j}{j}(-t)^k u^{m+1-k} \\
& - \sum_{k=-1}^{m} \binom{k+i}{i}\binom{m-k+j}{j}(-t)^{k+1} u^{m-k} \\
= & \sum_{k=0}^{m+1} \left[\binom{k+i}{i}\binom{m-k+j}{j} - \binom{k-1+i}{i}\binom{m-k+1+j}{j}\right] \\
& \times (-t)^k u^{m+1-k} \\
= & \sum_{k=0}^{m+1} \left[\binom{k+i-1}{i-1}\binom{m-k+1+j}{j} - \binom{k+i}{i}\binom{m-k+1+j-1}{j-1}\right] \\
& \times (-t)^k u^{m+1-k} \\
= & G_{m+1,i-1,j}(t,u) - G_{m+1,i,j-1}(t,u).
\end{aligned}$$

If $i > 0$ then

$$\begin{aligned}
(t+u)G_{m,i,0} &= \sum_{k=0}^{m} \binom{k+i}{i}(-t)^k u^{m+1-k} - \sum_{k=-1}^{m} \binom{k+i}{i}(-t)^{k+1} u^{m-k} \\
&= \sum_{k=0}^{m} \binom{k+i}{i}(-t)^k u^{m+1-k} - \sum_{k=0}^{m+1} \binom{k-1+i}{i}(-t)^k u^{m+1-k} \\
&= \sum_{k=0}^{m+1} \binom{k+i-1}{i-1}(-t)^k u^{m+1-k} - \binom{m+1+i}{i}(-t)^{m+1} \\
&\quad - \binom{m+i}{i}(-t)^{m+1} \\
&= G_{m+1,i-1,0}(t,u) - \binom{m+1+i}{i}(-t)^{m+1}.
\end{aligned}$$

Similarly, if $j > 0$ then

$$(t+u)G_{m,0,j}(t,u) = \binom{m+1+j}{j}u^{m+1} - G_{m+1,0,j-1}(t,u).$$

Finally,

$$(t+u)G_{m,0,0}(t,u) = (t+u)\sum_{k=0}^{m}(-t)^k u^{m+1-k} = u^{m+1} - (-t)^{m+1}.$$

It follows that

$$(t+u)F_{m,1}(t,u,y,z) = (t+u)G_{m,0,0}(t,u) = u^{m+1} - (-t)^{m+1},$$

$$\begin{aligned} & (t+u)F_{m,2}(t,u,y,z) \\ = \; & (t+u)(G_{m,0,1}(t,u)z + G_{m,1,0}(t,u)y) \\ = \; & (m+2)u^{m+1}z + G_{m,0,0}(t,u)(y-z) - (m+2)(-t)^{m+1}y, \end{aligned}$$

and for $r \geq 3$,

$$\begin{aligned} & (t+u)F_{m,r}(t,u,y,z) \\ = \; & \binom{m+r}{r-1}u^{m+1}z^{r-1} - G_{m+1,0,r-2}(t,u)z^{r-1} \\ & + G_{m+1,r-2,0}(t,u)y^{r-1} - \binom{m+r}{r-1}(-t)^{m+1}y^{r-1} \\ & + \sum_{i=1}^{r-2}[G_{m+1,i-1,r-1-i}(t,u) - G_{m+1i,r-2-i}(t,u)]y^i z^{r-1-i} \\ = \; & (y-z)F_{m+1,r-1}(t,u,y,z) + \binom{m+r}{r-1}(u^{m+1}z^{r-1} - (-t)^{m+1}y^{r-1}). \end{aligned}$$

Therefore

$$(t+u)^r F_{m,r}(t,u,y,z) \;=\; \sum_{l=1}^{r}\binom{m+r}{r-l}(y-z)^{r-l}(u^{m+r}z^{r-l} - (-t)^{m+r}y^{r-l}).$$

This establishes (4).

Consequently

$$\begin{aligned} & \mathcal{A}_r(k,l;y,z) - \mathcal{A}_r(l,k;z,y) \\ = \; & (-1)^k k!l!(y-z)^{k+l+1}\sum_{i=0}^{r-i-1}\frac{(-1)^i a_i}{i!(k+l+r-i)!} \\ & \times \sum_{j=0}^{r-i-1}\binom{k+r-1-i-j}{r-1-i-j}\binom{m-k+j}{j}y^{r-i-1-j}z^j \end{aligned}$$

as required. □

3. Examples and applications

The identities (1.4) and (1.5) in [1] are respectively the $r = 1$ and $r = 0$ cases of Theorem 2.

The next special case of Theorem 2, with $r = 2$ yields, after some rearrangement,

$$\begin{aligned} & (-1)^k \sum_{j=0}^{k} \binom{k}{j} x^{k-j} \frac{A_{l+j+2}(y)}{(l+j+1)(l+j+2)} \\ & - (-1)^j \sum_{j=0}^{l} \binom{l}{j} x^{l-j} \frac{A^*_{k+j+2}(z)}{(k+j+1)(k+j+2)} \\ = \ & k!l!(-x)^{k+l+1} \left(\frac{a_0}{(k+l+2)!} \left[(k+1)y + (l+1)(1-z)\right] - \frac{a_1}{(k+l+1)!} \right) \\ = \ & \frac{(-x)^{k+l+1}}{\binom{k+l}{k}} \left(\frac{a_0 \left[(k+1)y + (l+1)(1-z)\right]}{(k+l+1)(k+l+2)} - \frac{a_1}{k+l+1} \right). \end{aligned}$$

Taking $l = k$ and $z = y$ in (2) gives the following corollary.

Corollary 3 *For all integers k, $r \geq 0$,*

$$\begin{aligned} & \sum_{j=0}^{k} \binom{k}{j} (1-2y)^{k-j} \frac{(k+j)![A_{k+j+r}(y) - (-1)^r A^*_{k+j+r}(y)]}{(k+j+r)!} \\ = \ & (-1)^k k!^2 (2y-1)^{2k+1} \sum_{i=0}^{r-1} \frac{(-1)^i a_i}{i!(2k+r-i)!} \\ & \times \sum_{j=0}^{r-1-i} \binom{k+r-i-1-j}{k} \binom{k+j}{k} y^{r-i-1-j} z^j. \end{aligned}$$

In addition if r is odd and (a_n) is self-dual, or if r is even and $a^*_n = -a_n$ for all n then

$$\begin{aligned} & 2 \sum_{j=0}^{k} \binom{k}{j} (1-2y)^{k-j} \frac{(k+j)! A_{k+j+r}(y)}{(k+j+r)!} \\ = \ & (-1)^k k!^2 (2y-1)^{2k+1} \sum_{i=0}^{r-1} \frac{(-1)^i a_i}{i!(2k+r-i)!} \\ & \times \sum_{j=0}^{r-1-i} \binom{k+r-i-1-j}{k} \binom{k+j}{k} y^{r-i-1-j} z^j. \end{aligned}$$

This corollary naturally gives rise to Bernoulli polynomial identities by taking $a_n = (-1)^n B_n$. For example, when r is odd we have

$$2 \sum_{j=0}^{k} \binom{k}{j} (1-2y)^{k-j} \frac{(k+j)! B_{k+j+r}(y)}{(k+j+r)!}$$

$$= (-1)^k k!^2 (2y-1)^{2k+1} \sum_{i=0}^{r-1} \frac{B_i}{i!(2k+r-i)!} \times \sum_{j=0}^{r-1-i} \binom{k+r-i-1-j}{k}\binom{k+j}{k} y^{r-i-1-j} z^j.$$

The following theorem which subsumes the second and third identities [2, (1.5), (1.6)] in Sun's main theorem follows from Theorem 2.

Theorem 4 *Let $x+y+z=1$. Then for all integers k, l, $r \geq 0$,*

$$(-1)^k \sum_{j=0}^{k} \binom{k}{j}\binom{l+j}{r} x^{k-j} A_{l+j-r}(y) = (-1)^{l+r} \sum_{j=0}^{l} \binom{l}{j}\binom{k+j}{r} x^{l-j} A^*_{k+j-r}(z). \quad (5)$$

Proof. In this identity we interpret $A_m(x)/m!$ as zero whenever $m < 0$.

Replacing z by $1-z$ and using the duality principle we see that (5) is equivalent to

$$\sum_{j=0}^{k} \binom{k}{j}\binom{l+j}{r} (z-y)^{k-j} A_{l+j-r}(y) = \sum_{j=0}^{l} \binom{l}{j}\binom{k+j}{r} (y-z)^{l-j} A_{k+j-r}(z) \quad (6)$$

The $r=0$ case of (6) is the same as the $r=0$ case of (3). It is immediate from the definition of $A_m(x)$ that $A'_m(x) = mA_{m-1}(x)$. The case $r>0$ of (6) follows from the $r=0$ case by applying the partial differential operator

$$\frac{1}{r!}\left(\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^r,$$

which annihilates all powers of $(y-z)$, to both sides. □

References

[1] R.L. Graham, D.E. Knuth & O. Patashnik, *Concrete Mathematics*, Addison-Wesley, 1989.

[2] Z.-W. Sun, 'Combinatorial identities in dual sequences', *European J. Combin.* **24** (2003) 709–718.