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ON 2-ADIC ORDERS OF STIRLING NUMBERS OF THE SECOND KIND

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Abstract

We prove that for any $k = 1, \dots, 2^n$ the 2-adic order of the Stirling number $S(2^n, k)$ of the second kind is exactly $d(k) - 1$, where $d(k)$ denotes the number of 1's among the binary digits of k. This confirms a conjecture of Lengyel.

1. Introduction

For a nonzero integer m, if 2^h is the highest power of two dividing m, then we say that the 2-adic order $\rho_2(m)$ of m is h. In this paper $\rho_2(\cdot)$ is called the 2-adic valuation function.
Legendre observed that if $n \in \mathbb{N} = \{0, 1, 2, \dots\}$ then $\rho_2(n!) = n - d(n)$, where $d(n)$ is the number of 1's in the binary representation of n, in other words $d(n) = \sum_{\lambda=0}^{\infty} \varepsilon_\lambda(n)$ if $n = \sum_{\lambda=0}^{\infty} \varepsilon_\lambda(n) 2^\lambda$ with $\varepsilon_\lambda(n) \in \{0, 1\}$. Kummer proved that $\rho_2\left(\binom{n}{k}\right) = d(k) + d(n-k) - d(n)$ whenever $0 \le k \le n$.
Let $n \in \mathbb{N}$. The Stirling numbers $S(n, k)$ $(k \in \mathbb{N})$ of the second kind are given by

$$x^n = \sum_{k=0}^{\infty} S(n, k)(x)_k,$$

where $(x)_k = x(x-1)(x-2)\dots(x-k+1)$ for $k \in \mathbb{N} \setminus \{0\}$ and $(x)_0 = 1$. Actually $S(n, k)$ is the number of ways in which it is possible to partition a set with n elements into exactly k nonempty subsets. For more details and basic results on Stirling numbers of the second kind we refer the reader to [2] and [4].
In this paper we study 2-adic orders of Stirling numbers of the second kind, and establish the following theorem which was conjectured by T.Lengyel [3] and verified by him in some special cases.

Theorem 1. *Let $n, k \in \mathbb{N}$ and $1 \le k \le 2^n$. Then we have*

$$\rho_2(S(2^n, k)) = d(k) - 1.$$

In the next section we reveal some useful properties of Stirling numbers of the second kind. We are going to prove Theorem 1 in Section 3 on the basis of Section 2.

2. Auxiliary results on Stirling numbers of the second kind

The following identity relates the Stirling numbers of the second kind $S(n+m,\cdot)$ to $S(n,\cdot)$ and $S(m,\cdot)$.

Theorem 2. *Let $n, m, k \in \mathbb{N}$ such that $0 \le k \le n+m$. Then*

$$S(n+m,k) = \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}\, S(n,k-i)S(m,j).$$

Proof. Let $n, m \in \mathbb{N}$. Then

$$\begin{aligned} x^{n+m} &= x^n x^m = \sum_{r=0}^{n} S(n,r)(x)_r \sum_{j=0}^{m} S(m,j)(x)_j \\ &= \sum_{r=0}^{n} S(n,r)(x)_r \sum_{j=0}^{m} j!S(m,j)\binom{x}{j} \\ &= \sum_{r=0}^{n} S(n,r)(x)_r \sum_{j=0}^{m} j!S(m,j)\sum_{i=0}^{j}\binom{x-r}{i}\binom{r}{j-i} \\ &\qquad\qquad\qquad\qquad \text{(by the Chu-Vandermonde identity)} \\ &= \sum_{r=0}^{n} S(n,r)\sum_{j=0}^{m} S(m,j)\sum_{i=0}^{j}\frac{j!}{i!}\binom{r}{j-i}(x)_{r+i} \end{aligned}$$

Thus, for any $k = 0, 1, \ldots, n+m$ we have

$$\begin{aligned} S(n+m,k) &= \sum_{i=0}^{k}\sum_{j=i}^{k}\frac{j!}{i!}\binom{k-i}{j-i}\, S(n,k-i)S(m,j) \\ &= \sum_{i=0}^{k}\sum_{j=i}^{k}\binom{j}{i}\frac{(k-i)!}{(k-j)!}\, S(n,k-i)S(m,j). \end{aligned}$$

□

Remark: Stirling numbers of the second kind occur in a natural way while making calculations in the Witt ring (see [1] for further details). It was in this context that the previous identity arose.

Lemma 1. *Let $m, n \in \mathbb{N}$. Then*

$$d(m+n) \leq d(m) + d(n)$$

and equality holds if and only if

$$\sum_{\lambda=0}^{\infty} \varepsilon_\lambda(m)\varepsilon_\lambda(n) = 0,$$

i.e., when m and n have no non-zero binary digit in common.

Proof. If m and n have no non-zero binary digit in common then it is obvious that $d(m+n) = \sum \varepsilon_\lambda(m+n) = \sum(\varepsilon_\lambda(m) + \varepsilon_\lambda(n)) = d(m) + d(n)$. On the other hand, suppose that m and n have a non-zero binary digit in common. Let us say that λ_0 is the lowest natural number such that $\varepsilon_{\lambda_0}(m) = \varepsilon_{\lambda_0}(n) = 1$. Then it is clear that $\varepsilon_{\lambda_0}(m+n) = 0$ and 1 is added to $\varepsilon_{\lambda_0+1}(m) + \varepsilon_{\lambda_0+1}(n)$ to obtain an expression for $\varepsilon_{\lambda_0+1}(m+n)$. Anyhow, at least one non-zero binary digit is lost in $d(m+n)$. $\square$

Remark: The case $d(m+n) = d(m)+d(n)-1$ occurs if and only if $\varepsilon_{\lambda_0+1}(m) = \varepsilon_{\lambda_0+1}(n) = 0$ with λ_0 the unique natural number such that $\varepsilon_{\lambda_0}(m) = \varepsilon_{\lambda_0}(n) = 1$.

A new lower bound on the 2-adic order of Stirling numbers of the second kind can be obtained as follows.

Theorem 3. *Let $n, k \in \mathbb{N}$ and $0 \leq k \leq n$. Then*

$$\rho_2\left(S(n,k)\right) \geq d(k) - d(n).$$

Proof. We use induction on n.

For $n = 0$, $\rho_2(S(0,0)) = \rho_2(1) \geq d(0) - d(0)$.

Assume now that the above inequality is true for all $i < n$. We will prove the theorem for n. Observe that for $k = 0$ the result is obviously true.
Let $1 \leq k \leq n$. The Stirling numbers of the second kind satisfy the well-known 'vertical' recurrence relation

$$S(n,k) = \sum_{i=k-1}^{n-1} \binom{n-1}{i} S(i, k-1).$$

Combining this with the 'triangular' recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

we obtain

$$kS(n,k) = \sum_{i=k-1}^{n-1} \binom{n}{i} S(i,k-1).$$

Thus

$$\begin{aligned}
\rho_2(kS(n,k)) &= \rho_2\left(\sum_{i=k-1}^{n-1} \binom{n}{i} S(i,k-1)\right) \\
&\geq \min_{k-1\leq i\leq n-1} \{\rho_2\left(\binom{n}{i}\right) + d(k-1) - d(i)\} \\
&\qquad\qquad\qquad\qquad \text{(by the induction hypothesis)} \\
&= \min_{k-1\leq i\leq n-1} \{d(n-i) + d(k-1) - d(n)\} \\
&\qquad\qquad\qquad\qquad \text{(by the Kummer identity)} \\
&= d(k-1) - d(n) + 1.
\end{aligned}$$

So,

$$\begin{aligned}
\rho_2(S(n,k)) &\geq d(k-1) - \rho_2(k) + 1 - d(n) \\
&= d(k) - d(n).
\end{aligned}$$

□

3. Proof of Lengyel's conjecture

We use induction on n. For $n = 0$, $\rho_2(S(1,1)) = \rho_2(1) = 0 = d(1) - 1$. We assume the theorem is true for all powers 2^i where $i < n$. We will prove that the theorem holds for 2^n. By Theorem 2

$$S(2^n,k) = \sum_{i=0}^{k} \sum_{j=i}^{k} \binom{j}{i} \frac{(k-i)!}{(k-j)!}\, S(2^{n-1},k-i) S(2^{n-1},j). \tag{1}$$

We will take a closer look at the 2-adic valuation of each term in this sum (1).

$$\begin{aligned}
&\rho_2\left(\binom{j}{i}\frac{(k-i)!}{(k-j)!}\,S(2^{n-1},k-i)S(2^{n-1},j)\right)\\
&= \rho_2\left(\binom{j}{i}\right) + \rho_2((k-i)!) - \rho_2((k-j)!) + \rho_2(S(2^{n-1},k-i)) + \rho_2(S(2^{n-1},j))\\
&= \rho_2\left(\binom{j}{i}\right) + \rho_2((k-i)!) - \rho_2((k-j)!) + d(k-i) + d(j) - 2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by the induction hypothesis)}\\
&= d(i) + d(j-i) - d(j) + (k-i) - d(k-i) - (k-j) + d(k-j) + d(k-i) + d(j) - 2\\
&\qquad\qquad\qquad\qquad\qquad\qquad\text{(by the Kummer and Legendre identities)}\\
&= d(i) + d(j-i) + j - i + d(k-j) - 2.
\end{aligned}$$

The inequality of Lemma 1 implies that

$$d(i) + d(j-i) + j - i + d(k-j) - 2 \geq d(j) + j - i + d(k-j) - 2 \geq d(k) - 2 + j - i.$$

Since $j \geq i$, the 2-adic valuation of every term in the sum is at least $d(k) - 2$. To prove that the 2-adic valuation of the global sum (1) equals $d(k) - 1$ we will calculate the number of terms with 2-adic valuation $d(k) - 2$ and the number of terms with 2-adic valuation $d(k)-1$. These two results together will show that the 2-adic valuation of (1) equals $d(k)-1$.

For $k = 1$ the theorem holds since $\rho_2(S(2^n, 1)) = \rho_2(1) = 0 = d(1) - 1$, for all $n \in \mathbb{N}$. So assume $k \neq 1$.

Case 1 : $d(i) + d(j-i) + j - i + d(k-j) - 2 = d(k) - 2$**.**

Since $d(i) + d(j-i) + d(k-j) \geq d(k)$ and $j \geq i$, this situation can occur only when $j = i$ and $d(i) + d(k-i) = d(k)$. By Lemma 1 this holds only when i and $k - i$ have no non-zero binary digit in common, or equivalently, when $\varepsilon_\lambda(i) + \varepsilon_\lambda(k-i) = \varepsilon_\lambda(k)$, for all $\lambda \in \mathbb{N}$.
If $\varepsilon_\lambda(k) = 1$ (this occurs $d(k)$ times), the possible values for $\varepsilon_\lambda(i)$ are 0 and 1.
If $\varepsilon_\lambda(k) = 0$, then $\varepsilon_\lambda(i) = 0$ as well.
So, for a given k, there are $2^{d(k)}$ possibilities for $i = j$. We need to modify this number of possibilities since it includes the non-occurring situations $i = j = 0$ and $i = j = k$. This means we have $2^{d(k)} - 2$ terms in (1) with 2-adic valuation $d(k) - 2$.
In the case where $d(k) = 1$, i.e. $k = 2^m$, there are no terms satisfying the condition. When $d(k) > 1$, these $2^{d(k)} - 2$ terms contribute, in total, $M2^{d(k)-1}$ to (1).
We will show that M is odd. Let $O(i)$ be the odd part of $S(2^{n-1}, i)$. Consider the sum in this case

$$\sum_{\substack{i=1 \\ d(i)+d(k-i)=d(k)}}^{k-1} S(2^{n-1}, k-i)S(2^{n-1}, i)$$
$$= \sum_{\substack{i=1 \\ d(i)+d(k-i)=d(k)}}^{k-1} O(k-i)O(i)2^{d(k)-2}.$$

The latter expression is invariant under switching i and $k-i$ and since $i = k/2$ (in the case k even) never occurs ($d(k/2) + d(k/2) = 2d(k) \neq d(k)$) we obtain

$$\sum_{\substack{i=1 \\ d(i)+d(k-i)=d(k) \\ i<k/2}}^{k-1} O(k-i)O(i)2^{d(k)-1}.$$

This last expression consists of an odd number, $2^{d(k)-1} - 1$, of terms, so it contributes, in total, $M2^{d(k)-1}$ to (1), where M is odd.

Case 2 : $d(i) + d(j-i) + j - i + d(k-j) - 2 = d(k) - 1$**.**

Since $d(i)+d(j-i)+d(k-j) \geq d(k)$ and $j \geq i$, this situation can occur only when $j = i+1$ and $d(i) + d(k-i-1) = d(k) - 1$ or when $j = i$ and $d(i) + d(k-i) = d(k) + 1$.

Case 2.1 : $d(i) + d(k-i-1) = d(k) - 1$ **and** $j = i+1$**.**

Since $d(k-1) \leq d(i)+d(k-i-1) = d(k)-1$, k must be odd. We have $d(i)+d((k-1)-i) = d(k-1)$. As in Case 1, there are $2^{d(k-1)}$ possible values for i (the case $i = k$ doesn't occur and the case $i = 0$ and $j = 1$ is allowed). This is an even number of terms since $k \neq 1$.

Case 2.2 : $d(i) + d(k-i) = d(k) + 1$ **and** $j = i$**.**

By Lemma 1 this can occur only when there is just one value of $\lambda \in \mathbb{N}$ for which $\varepsilon_\lambda(i) = \varepsilon_\lambda(k-i) = 1$. Moreover one must have $\varepsilon_{\lambda+1}(i) = \varepsilon_{\lambda+1}(k-i) = 0$. This implies that $\varepsilon_\lambda(k) = 0$ and $\varepsilon_{\lambda+1}(k) = 1$. Following the same reasoning as in Case 1 with the remaining $d(k) - 1$ non-zero binary digits of k, we have $2^{d(k)-1}$ possibilities for i (the cases $i = 0$ and $i = k$ don't occur).
So there are $2^{d(k)-1}$ terms in (1) which come under Case 2.2 (and thus have 2-adic valuation $d(k) - 1$). When $d(k) = 1$, this number is 1, otherwise it is even.

After considering all the possible cases and counting the number of terms with 2-adic valuation $d(k) - 2$ and 2-adic valuation $d(k) - 1$, we can conclude that $\rho_2(S(2^n, k)) = d(k) - 1$.

An overview of all the cases is given in the following table.

	Case 1 coefficient of $2^{d(k)-2}$	**Case 2.1** coefficient of $2^{d(k)-1}$	**Case 2.2** coefficient of $2^{d(k)-1}$	**coefficient of $\mathbf{2^{d(k)-1}}$**
$\mathbf{d(k) = 1\ (k \neq 1)}$	0	0	odd	odd
$\mathbf{d(k) > 1}$ & **k odd**	2 x odd	even	even	odd
$\mathbf{d(k) > 1}$ & **k even**	2 x odd	0	even	odd

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