COUNTING RISES, LEVELS, AND DROPS IN COMPOSITIONS

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Abstract

A composition of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of summands is called the number of parts of the composition. A palindromic composition of n is a composition of n in which the summands are the same in the given or in reverse order. In this paper we study the generating function for the number of compositions (respectively, palindromic compositions) of n with m parts in a given set $A \subseteq \mathbb{N}$ with respect to the number of rises, levels, and drops, and obtain previously known results as well as new results. We also generalize results for Carlitz compositions and for partitions.

1. Introduction

A *composition* $\sigma = \sigma_1 \sigma_2 \ldots \sigma_m$ of $n \in \mathbb{N}$ is an ordered collection of one or more positive integers whose sum is n. The number of summands, namely m, is called the number of *parts* of the composition. A *palindromic composition* of $n \in \mathbb{N}$ is a composition for which $\sigma_1 \sigma_2 \ldots \sigma_m = \sigma_m \sigma_{m-1} \ldots \sigma_1$. A *Carlitz composition* is a composition of $n \in \mathbb{N}$ in which no two consecutive parts are the same. We will derive the generating functions for the number of compositions, number of parts, and number of *rises* (a summand followed by a larger summand), *levels* (a summand followed by itself), and *drops* (a summand followed by a smaller summand) in all compositions of n whose parts are in a given set A. This unified

framework generalizes earlier work by several authors.

Alladi and Hoggatt [1] considered $A = \{1, 2\}$, and derived generating functions for the number of compositions, number of parts, and number of rises, levels and drops in compositions and palindromic compositions of n, exhibiting connections to the Fibonacci sequence. Chinn and Heubach [7] generalized to $A = \{1, k\}$ and derived all the respective generating functions. Chinn, Grimaldi and Heubach [5] considered the case $A = \mathbb{N}$, and derived generating functions for all quantities of interest. The focus of [5] was on explicit formulas for the quantities of interest, as well as combinatorial proofs for the connections among them. Carlitz [2] on the other hand approached the same topic from a generating function standpoint, and provided a connection to the Simon Newcomb problem [4]. (Note: Carlitz counts an extra rise and drop at the beginning and the end of each composition). Grimaldi [10] studied $A = \{m \,|\, m = 2k + 1, k \geq 0\}$, and derived generating functions for the number of such compositions, as well as the number of parts, but not for the number of rises, levels and drops. In addition, he studied compositions without the summand 1 [9], which was generalized by Chinn and Heubach [6] to $A = \mathbb{N} - \{k\}$. In both cases, the authors only derived generating functions for the total number of compositions and the number of parts, but not for the number of rises, levels and drops. Finally, Hoggatt and Bricknell [11] looked at compositions with parts in a general set A, and gave generating functions for the number of compositions and the number of parts. This work was generalized by Heubach and Mansour [12], which also considered Carlitz compositions and gave additional generating functions for the number of compositions with a given number of parts in a set $B \subseteq A$. Most recently, Merlini et. al. [14] obtained results on the number of compositions and palindromic compositions using Riordan arrays.

We will present a unified framework which allows us to derive previously known results as well as new results by applying the main theorem to specific sets. The main result and its proof will be stated in Section 2. In the sections that follow we will state results for compositions (Section 3), palindromic compositions (Section 4), Carlitz compositions and Carlitz palindromic compositions (Section 5), and partitions (Section 6) of n with m parts in A, respectively. In each case, we will state general results for the total number, the number of levels, rises and drops, as well as give results for specific sets A, namely $A = \mathbb{N}$, $A = \{1, 2\}$, $A = \{1, k\}$, $A = \mathbb{N} - \{k\}$, and $A = \{m \,|\, m = 2k + 1, k \geq 0\}$. In the cases of Carlitz compositions and partitions, we will restrict ourselves to the sets $A = \{1, 2\}$, $A = \{1, k\}$ and $A = \{a, b\}$.

2. Main Result

Let $\mathbb{N}$ be the set of all positive integers, and let A be any ordered (finite or infinite) set of positive integers, say $A = \{a_1, a_2, \ldots, a_k\}$, where $a_1 < a_2 < a_3 < \cdots < a_k$, with the obvious modifications in the case $|A| = \infty$. In the theorems and proofs, we will treat the two cases together if possible, and will note if the case $|A| = \infty$ requires additional steps. For ease

of notation, "ordered set" will always refer to a set whose elements are listed in increasing order.

For any ordered set $A = \{a_1, a_2, \ldots, a_k\} \subseteq \mathbb{N}$, we denote the set of all compositions (respectively palindromic compositions) of n with parts in A by C_n^A (respectively P_n^A). For any composition σ, we denote the number of parts, rises, levels, and drops by $\text{parts}(\sigma)$, $\text{rises}(\sigma)$, $\text{levels}(\sigma)$, and $\text{drops}(\sigma)$, respectively. We define the generating function for the number of compositions (respectively palindromic compositions) of n with parts in a set A specifying the number of rises, levels, and drops as

$$C_A(x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma \in C_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}$$

and

$$P_A(x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma \in P_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}.$$

The main result of this paper gives explicit expressions for these two generating functions.

Theorem 2.1 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then*

1. *The generating function for compositions is given by*

$$C_A(x; y; r, \ell, d) = \frac{1 + (1-d) \sum_{j=1}^{k} \left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y(\ell - r)}{1 - x^{a_i} y(\ell - d)} \right)}{1 - d \sum_{j=1}^{k} \left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y(\ell - r)}{1 - x^{a_i} y(\ell - d)} \right)}.$$

2. *The generating function for palindromic compositions is given by*

$$P_A(x; y; r, \ell, d) = \frac{1 + \sum_{i=1}^{k} \frac{x^{a_i} y + x^{2a_i} y^2 (\ell - d\,r)}{1 - x^{2a_i} y^2 (\ell^2 - d\,r)}}{1 - \sum_{i=1}^{k} \frac{x^{2a_i} y^2 d\,r}{1 - x^{2a_i} y^2 (\ell^2 - d\,r)}}.$$

Before giving the proof of Theorem 2.1, we will connect the generating function for compositions for $A = \mathbb{N}$ with the result in [2] to obtain the following expression, which is hard to prove directly.

Corollary 2.2 *Let $e(u) = e(u, x) = \sum_{n=0}^{\infty} \frac{u^n}{(x)_n}$ and $(x)_n = (1-x)(1-x^2)\cdots(1-x^n)$. Then*

$$\sum_{j=1}^{\infty} \frac{x^j y}{1 - x^j y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^i y(\ell - r)}{1 - x^i y(\ell - d)} = \frac{(1+d) \cdot e(xy(l - d)) - (1+r) \cdot e(xy(l - r))}{d^2 \cdot e(xy(l - d)) + (r - d(r+1)) \cdot e(xy(l - r))}.$$

Proof. Since Carlitz counts an extra rise and fall at beginning and end, respectively, we need to adjust our generating function by a factor of $r \cdot d$. Setting the two generating functions, namely Theorem 2.1 Part (1) for $A = \mathbb{N}$ and the right-hand side of Eq. (4.6) [2], equal, we get that

$$r \cdot d \cdot \frac{1 + (1-d) \sum_{j=1}^{\infty} \frac{x^j y}{1 - x^j y(\ell-d)} \prod_{i=1}^{j-1} \frac{1 - x^i y(\ell-r)}{1 - x^i y(\ell-d)}}{1 - d \sum_{j=1}^{\infty} \frac{x^j y}{1 - x^j y(\ell-d)} \prod_{i=1}^{j-1} \frac{1 - x^i y(\ell-r)}{1 - x^i y(\ell-d)}} = r \cdot d \frac{e(xy(l-d)) - e(xy(l-r))}{r \cdot e(xy(l-r)) - d \cdot e(xy(l-d))}.$$

Solving for $\sum_{j=1}^{\infty} \frac{x^j y}{1 - x^j y(\ell-d)} \prod_{i=1}^{j-1} \frac{1 - x^i y(\ell-r)}{1 - x^i y(\ell-d)}$, we obtain the desired result. □

We now turn to the proof of Theorem 2.1.

Proof. In order to find the two generating functions, we derive recursions for (palindromic) compositions that start with specific parts. Thus, we define a second set of generating functions

$$C_A(s_1 s_2 \ldots s_e | x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)}$$

and

$$P_A(s_1 s_2 \ldots s_e | x; y; r, \ell, d) = \sum_{n \geq 0} \sum_{\sigma} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} \ell^{\text{levels}(\sigma)} d^{\text{drops}(\sigma)},$$

where the sum on the right side of the equation is over all the composition $\sigma \in C_n^A$ and $\sigma \in P_n^A$, respectively, such that the (palindromic) composition σ starts with $s_1 s_2 \ldots s_e$, for $e \geq 1$. From the definitions of the generating functions, we immediately have the following:

$$C_A(x; y; r, \ell, d) = 1 + \sum_{i=1}^{k} C_A(a_i | x; y; r, \ell, d) \tag{1}$$

$$P_A(x; y; r, \ell, d) = 1 + \sum_{i=1}^{k} P_A(a_i | x; y; r, \ell, d), \tag{2}$$

where the summand 1 covers the case $n = 0$. The strategy is now to find expressions for the generating functions $C_A(a_i | x; y; r, \ell, d)$ and $P_A(a_i | x; y; r, \ell, d)$, which will allow us to prove the claim.

We will first prove part (1) of the theorem. We derive a recursion for $C_A(a_i | x; y; r, \ell, d)$ which together with Equation (1) gives a system of equations that is solved using Cramer's rule. Note that the compositions of n starting with a_i with at least two parts can be created recursively by prepending a_i to a composition of $n - a_i$ which starts with a_j for some j. This either creates a rise (if $i < j$), a level (if $i = j$), or a drop (if $i > j$), and in each case, results in one more part. Thus,

$$C_A(a_i a_j | x; y; r, \ell, d) = \begin{cases} r\, x^{a_i} y\, C_A(a_j | x; y; r, \ell, d), & i < j \\ \ell\, x^{a_i} y\, C_A(a_j | x; y; r, \ell, d), & i = j \\ d\, x^{a_i} y\, C_A(a_j | x; y; r, \ell, d), & i > j \end{cases}.$$

Summing over j and accounting for the single composition with exactly one part, namely a_i, gives

$$C_A(a_i|x;y;r,\ell,d) \;=\; x^{a_i}y + x^{a_i}y\,d\sum_{j=1}^{i-1}C_A(a_j|x;y;r,\ell,d) + x^{a_i}y\,\ell\,C_A(a_i|x;y;r,\ell,d)$$

$$+\,x^{a_i}y\,r\sum_{j=i+1}^{k}C_A(a_j|x;y;r,\ell,d) \qquad (3)$$

for $i = 1, 2, \ldots, k$. Let $t_0 = C_A(x;y;r,\ell,d)$, $t_i = C_A(a_i|x;y;r,\ell,d)$ and $b_i = x^{a_i}y$, for $i = 1, 2, \ldots, k$. Then Equations (1) and (3) result in this system of $k+1$ equations in $k+1$ variables:

$$\begin{pmatrix} 1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ 0 & 1-b_1\ell & -b_1r & -b_1r & \cdots & -b_1r & -b_1r \\ 0 & -b_2d & 1-b_2\ell & -b_2r & \cdots & -b_2r & -b_2r \\ 0 & -b_3d & -b_3d & 1-b_3\ell & \cdots & -b_3r & -b_3r \\ \vdots & & \vdots & & & \vdots & \\ 0 & -b_{k-1}d & -b_{k-1}d & -b_{k-1} & \cdots & 1-b_{k-1}\ell & -b_{k-1}r \\ 0 & -b_kd & -b_kd & -b_kd & \cdots & -b_kd & 1-b_k\ell \end{pmatrix} \cdot \begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_4 \\ \vdots \\ t_{k-1} \\ t_k \end{pmatrix} = \begin{pmatrix} 1 \\ b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_{k-1} \\ b_k \end{pmatrix}$$

By Cramer's rule, $t_0 = \frac{\det(N_k)}{\det(M_k)}$, where M_k is the $(k+1) \times (k+1)$ matrix of the system of equations and N_k is the $(k+1) \times (k+1)$ matrix which results from replacing the first column in M_k by the vector of the right-hand side of the system, i.e.,

$$N_k = \begin{pmatrix} 1 & -1 & -1 & \cdots & -1 & -1 & -1 \\ b_1 & 1-b_1\ell & -b_1r & -b_1r & \cdots & -b_1r & -b_1r \\ b_2 & -b_2d & 1-b_2\ell & -b_2r & \cdots & -b_2r & -b_2r \\ b_3 & -b_3d & -b_3d & 1-b_3\ell & \cdots & -b_3r & -b_3r \\ \vdots & & & \vdots & & & \vdots \\ b_{k-1} & -b_{k-1}d & -b_{k-1}d & -b_{k-1} & \cdots & 1-b_{k-1}\ell & -b_{k-1}r \\ b_k & -b_kd & -b_kd & -b_kd & \cdots & -b_kd & 1-b_k\ell \end{pmatrix}.$$

We now derive formulas for $\det(N_k)$ and $\det(M_k)$. Expanding down the first column of M_k, we get that

$$\det(M_k) = \begin{vmatrix} 1-b_1\ell & -b_1r & -b_1r & \cdots & -b_1r & -b_1r \\ -b_2d & 1-b_2\ell & -b_2r & \cdots & -b_2r & -b_2r \\ -b_3d & -b_3d & 1-b_3\ell & \cdots & -b_3r & -b_3r \\ \vdots & & \vdots & & \vdots & \\ -b_{k-1}d & -b_{k-1}d & -b_{k-1} & \cdots & 1-b_{k-1}\ell & -b_{k-1}r \\ -b_kd & -b_kd & -b_kd & \cdots & -b_kd & 1-b_k\ell \end{vmatrix}.$$

Subtracting the $(k-1)^{\text{st}}$ column from the kth column of the above matrix, then expanding down the resulting column gives

$$\det(M_k) = (1 - b_k(\ell - d))\det(M_{k-1}) - b_kd(1 - b_{k-1}(\ell - r))\det(E(b_1, b_2, ..., b_{k-2})), \qquad (4)$$

where

$$E(b_1, b_2, ..., b_{k-2}) = \begin{pmatrix} 1 - b_1\ell & -b_1 r & -b_1 r & \cdots & -b_1 r \\ -b_2 d & 1 - b_2\ell & -b_2 r & \cdots & -b_2 r \\ -b_3 d & -b_3 d & 1 - b_3\ell & \cdots & -b_3 r \\ \vdots & & \vdots & & \vdots \\ -b_{k-2}d & -b_{k-2}d & -b_{k-2}d & \cdots & -b_{k-2}r \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix}.$$

Adding $(b_1 r)$ times the last row to the first row in the matrix $E(b_1, b_2, ..., b_{k-2})$, then expanding across the resulting first row gives

$$\det(E(b_1, b_2, ..., b_{k-2})) = (1 - b_1(\ell - r)) \det(E(b_2, ..., b_{k-2})),$$

and, since $\det(E(b_{k-2})) = (1 - b_{k-2}(\ell - r))$,

$$\det(E(b_1, b_2, ..., b_{k-2})) = \prod_{j=1}^{k-2}(1 - b_j(\ell - r)). \tag{5}$$

Equations (4) and (5) result in

$$\det(M_k) = (1 - b_k(\ell - d)) \det(M_{k-1}) - b_k d \prod_{j=1}^{k-1}(1 - b_j(\ell - r)).$$

Thus, if we define $\det(M_0) = 1$ and use the fact that $\det(M_1) = 1 - b_1\ell = 1 - b_1(\ell - d) - b_1 d$, then we can show by induction on k that for all $k \geq 1$,

$$\det(M_k) = \prod_{j=1}^{k}(1 - b_j(\ell - d)) - d \sum_{j=1}^{k} b_j \prod_{i=1}^{j-1}(1 - b_i(\ell - r)) \prod_{i=j+1}^{k}(1 - b_j(\ell - d)). \tag{6}$$

Similarly, by subtracting $(b_k d)$ times the last row from the kth row in the matrix N_k and then expanding across the resulting k^{th} row we get

$$\det(N_k) = (1 - b_k(\ell - d)) \det(N_{k-1}) + b_k(1 - d) \det(D(b_1, b_2, ..., b_{k-1})),$$

where $D(b_1, b_2, ..., b_{k-1})$ agrees with $E(b_1, b_2, ..., b_{k-1})$ except for the signs of the last row. Thus, $\det(D(b_1, b_2, ..., b_{k-1})) = -\det(E(b_1, b_2, ..., b_{k-1}))$, which yields

$$\det(N_k) = (1 - b_k(\ell - d)) \det(N_{k-1}) - b_k(1 - d) \prod_{j=1}^{k-1}(1 - b_j(\ell - r)).$$

With $\det(N_0) = 1$ and $\det(N_1) = 1 - b_1\ell + b_1 = 1 - b_1(\ell - d) + (1 - d)b_1$, we can show by induction on k that for all $k \geq 1$,

$$\det(N_k) = \prod_{j=1}^{k}(1 - b_j(\ell - d)) + (1 - d) \sum_{j=1}^{k} b_j \prod_{i=1}^{j-1}(1 - b_i(\ell - r)) \prod_{i=j+1}^{k}(1 - b_j(\ell - d)). \tag{7}$$

Substituting Equations (6) and (7) and $b_i = x^{a_i}y$ into $\frac{\det(N_k)}{\det(M_k)}$ completes the proof of Theorem 2.1(1). Note that if $|A| = \infty$, then the result follows by taking limits as $k \to \infty$.

We now prove part(2), which uses the same general idea of first deriving a recursion for $P_A(a_i|x; y; r, \ell, d)$ and then solving the resulting system of equations. Palindromes of n that start with a_i (for a fixed i) either have one part, a_i, or two parts and one level, $a_i a_i$, or three or more parts. Those of three or more parts can be created by adding the part a_i both at the beginning and the end of a palindromic composition of $n - 2a_i$ that starts with a_j. If $i = j$, then two additional levels are created; if $i \neq j$, then a rise and a drop are created, and in both cases we have two additional parts. Thus, the generating function is as follows:

$$
\begin{aligned}
P_A(a_i|x; y; r, \ell, d) &= x^{a_i}y + x^{2a_i}y^2\ell + x^{2a_i}y^2\ell^2 P_A(a_i|x; y; r, \ell, d) \\
&\quad + x^{2a_i}y^2 d\, r \sum_{j \neq i, j=1}^{k} P_A(a_j|x; y; r, \ell, d) \\
&= x^{a_i}y + x^{2a_i}y^2\ell + x^{2a_i}y^2(\ell^2 - r\,d)P_A(a_i|x; y; r, \ell, d) \\
&\quad + x^{2a_i}y^2 d\, r (P_A(x; y; r, l, d) - 1),
\end{aligned}
$$

where the last equality follows from Equation (2). We can now solve for $P_A(a_i|x; y; r, \ell, d)$, and substitute the result into Equation (2), which gives that

$$
P_A(x; y; r, \ell, d) = \frac{1 + \sum\limits_{i=1}^{k} \frac{x^{a_i}y + x^{2a_i}y^2(\ell - dr)}{1 - x^{2a_i}y^2(\ell^2 - dr)}}{1 - \sum\limits_{i=1}^{k} \frac{x^{2a_i}y^2 dr}{1 - x^{2a_i}y^2(\ell^2 - dr)}}.
$$

The reason that we get a less tedious proof for palindromic compositions comes from the fact that we only need to distinguish between the cases $i = j$ and $i \neq j$ when deriving $P_A(a_i|x; y; r, \ell, d)$. This allows us to solve for $P_A(a_i|x; y; r, \ell, d)$ in terms of $P_A(x; y; r, \ell, d)$, and allows for direct substituion. $\square$

We now apply Theorem 2.1 to specific sets A to obtain previous and new results.

3. Results for compositions with parts in A

In this section we study the number of compositions of n as well as the number of rises, levels, and drops in the compositions of n with parts in A, both for general and specific sets A.

Applying Theorem 2.1(1) for $r = \ell = d = 1$, we get that the generating function for the number of compositions of n with m parts in A is given by

$$
\frac{1}{1 - y\sum_{j=1}^{k} x^{a_j}}. \tag{8}
$$

Therefore, the generating function for the number of compositions of n with m parts in $\mathbb{N}$ is given by

$$\sum_{n\geq 0}\sum_{\sigma\in C_n^{\mathbb{N}}} x^n y^{\text{parts}(\sigma)} = \frac{1}{1-y\sum_{j=1}^{\infty}x^j} = \frac{1}{1-\frac{yx}{1-x}} = \sum_{m\geq 0}\frac{x^m}{(1-x)^m}y^m.$$

Furthermore, setting $y=1$ in Equation (8) gives the generating function for the number of compositions of n with parts in A (see [12], Theorem 2.4):

$$\frac{1}{1-\sum_{j=1}^k x^{a_j}}.$$

In particular, for $A=\mathbb{N}$, the generating function for the number of compositions of n with parts in $\mathbb{N}$ is given by (see [5], Theorem 6)

$$\frac{1-x}{1-2x}.$$

Additional examples for specific choices of A are given in [12]. We now state results concerning the number of rises and drops.

3.1 The number of rises and drops

Note that the number of rises equals the number of drops in all compositions of n: for each non-palindromic composition there exists a composition in reverse order, thus the rises match the drops, and for palindromic compositions, symmetry matches up rises and drops within the composition. Thus, we will derive results only for rises, and the results for drops follow by interchanging the roles of r and d in the proofs.

In order to obtain the relevant generating functions, we use a common technique, namely taking partial derivatives. Setting $\ell=d=1$ in Theorem 2.1(1) gives

$$C_A(x;y;r,1,1) = \frac{1}{1-\sum_{j=1}^k\left(x^{a_j}y\prod_{i=1}^{j-1}(1-x^{a_i}y(1-r))\right)}. \tag{9}$$

Using Equation (9) together with the fact that for $f_i(r)\neq 0$

$$\frac{\partial}{\partial r}\prod_{i=1}^m f_i(r) = \left(\prod_{i=1}^m f_i(r)\right)\sum_{i=1}^m\frac{\frac{\partial}{\partial r}f_i(r)}{f_i(r)}, \tag{10}$$

we get that

$$\sum_{n\geq 0}\sum_{\sigma\in C_n^A}\text{rises}(\sigma)x^n y^{\text{parts}(\sigma)} = \frac{\partial}{\partial r}C_A(x;y;r,1,1)\bigg|_{r=1} = \frac{y^2\sum_{k\geq j>i\geq 1}x^{a_i+a_j}}{\left(1-y\sum_{j=1}^k x^{a_j}\right)^2}.$$

Hence, expressing this function as a power series about $y=0$, we get the following result.

Corollary 3.1 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then*

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} rises(\sigma) x^n y^{parts(\sigma)} = \left(\sum_{k \geq j > i \geq 1} x^{a_i + a_j} \right) \sum_{m \geq 0} (m+1) \left(\sum_{j=1}^k x^{a_j} \right)^m y^{m+2}$$

and

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} drops(\sigma) x^n y^{parts(\sigma)} = \left(\sum_{k \geq j > i \geq 1} x^{a_i + a_j} \right) \sum_{m \geq 0} (m+1) \left(\sum_{j=1}^k x^{a_j} \right)^m y^{m+2}.$$

For example, letting $A = \mathbb{N}$ and looking at the coefficient of y^m in Corollary 3.1 we get that the generating function for the number of rises (drops) in the compositions of n with a fixed number of parts, $m \geq 2$, in $\mathbb{N}$ is given by

$$\sum_{j > i \geq 1} x^{i+j} (m-1) \left(\sum_{j \geq 1} x^j \right)^{m-2}$$

$$= \sum_{i \geq 1} \sum_{j \geq i+1} x^{i+j} (m-1) \left(\frac{x}{1-x} \right)^{m-2} = \sum_{i \geq 1} x^i \sum_{j \geq 1} x^{2j} \frac{(m-1)x^{m-2}}{(1-x)^{m-2}}$$

$$= \frac{x^3}{(1-x)(1-x^2)} \cdot \frac{(m-1)x^{m-2}}{(1-x)^{m-2}} = \frac{(m-1)x^{m+1}}{(1+x)(1-x)^m}.$$

Furthermore, setting $y = 1$ and $A = \mathbb{N}$ in Corollary 3.1 allows us to compute the generating function for the number of rises (drops) in all compositions of n with parts in $\mathbb{N}$ (see [5], Theorem 6) in a similar way:

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} rises(\sigma) x^n = \sum_{j > i \geq 1} x^{i+j} \sum_{m \geq 0} (m+1) \left(\frac{x}{1-x} \right)^m = \frac{x^3}{(1-x)(1-x^2)} \cdot \frac{1}{(1 - \frac{x}{1-x})^2}$$

$$= \frac{x^3}{(1+x)(1-2x)^2}.$$

For $A = \{1, k\}$ and $y = 1$, Corollary 3.1 gives the generating function for the number of rises (drops) in all compositions of n with parts in $\{1, k\}$ as (see [7], Theorem 4)

$$\frac{x^{k+1}}{(1 - x - x^k)^2}.$$

For $A = \{m \mid m = 2k + 1, k \geq 0\}$ and $y = 1$, and using that $\sum_{0 \leq i < j} x^{(2i+1)+(2j+1)} = \sum_{i \geq 0} (x^2)^i \sum_{j \geq 1} (x^4)^j$, Corollary 3.1 yields a new result, namely that the generating function for the number of rises (drops) in compositions of n with odd parts is given by

$$\frac{x^{k+1}}{(1 - x - x^k)^2}.$$

We also obtain new results for $A = \mathbb{N} - \{k\}$ (studied in [6]) and $A = \mathbb{N} - \{1\}$ (studied in [9]). For $A = \mathbb{N} - \{k\}$, we define $g(x, y; k) = \sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{rises}(\sigma) x^n y^{\text{parts}(\sigma)}$. Then Corollary 3.1 gives

$$g(x, y; k) = \left(\frac{x^3}{(1-x)(1-x^2)} - \frac{x^{k+1}(1-x^{k-1}) + x^{2k+1}}{1-x} \right) \sum_{m \geq 0} (m+1) \left(\frac{x}{1-x} - x^k \right)^m y^{m+2}.$$

For $k = 1$, i.e., $A = \mathbb{N} - \{1\}$ this expression reduces to

$$g(x, y; 1) = \sum_{m \geq 2} (m-1) \frac{x^{2m+1}}{(1+x)(1-x)^m} y^m.$$

Thus, the generating function for the number of rises (drops) in the compositions of n without 1s with a fixed number of parts ($m \geq 2$) is given by

$$(m-1) \frac{x^{2m+1}}{(1+x)(1-x)^m} = \sum_{n \geq 0} x^{n+2m-1}(m-1) \sum_{j=0}^{n} (-1)^{n-j} \binom{j+m-1}{m-1}.$$

We now look at results concerning levels.

3.2 The number of levels

Proceeding similarly to the case of rises, we set $r = d = 1$ in Theorem 2.1(1) to get

$$C_A(x; y; 1, \ell, 1) = \frac{1}{1 - \sum_{j=1}^{k} \frac{x^{a_j} y}{1 - x^{a_j} y(\ell-1)}}.$$

Therefore,

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{levels}(\sigma) x^n y^{\text{parts}(\sigma)} = \frac{\partial}{\partial \ell} C_A(x; y; 1, \ell, 1) \Big|_{\ell=1} = \frac{y^2 \sum_{j=1}^{k} x^{2a_j}}{\left(1 - y \sum_{j=1}^{k} x^{a_j}\right)^2}.$$

Expressing the above function as a power series about $y = 0$, we get the following result.

Corollary 3.2 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then*

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} levels(\sigma) x^n y^{parts(\sigma)} = \left(\sum_{j=1}^{k} x^{2a_j} \right) \sum_{m \geq 0} (m+1) \left(\sum_{j=1}^{k} x^{a_j} \right)^m y^{m+2}.$$

Using computations similar to those for rises and drops, by looking at the coefficient of y^m, we get from Corollary 3.2 that the generating function for the number of levels in all compositions of n with a fixed number of parts m in $\mathbb{N}$ is given by

$$\frac{(m-1)x^m}{(1+x)(1-x)^{m-1}}.$$

In addition, by setting $y = 1$ and $A = \mathbb{N}$ in Corollary 3.2 we obtain that the generating function for the number of levels in the compositions of n with parts in $\mathbb{N}$ (see [5], Theorem 6) is given by

$$\frac{x^2(1-x)}{(1+x)(1-2x)^2}.$$

Applying Corollary 3.2 for $A = \{1, 2\}$ and $y = 1$, we get the result given in Theorem 1.1 [1] for the generating function for the number of levels in all compositions with only 1's and 2's:

$$\frac{x^2 + x^4}{(1-(x+x^2))^2},$$

and more generally, for $A = \{1, k\}$ and $y = 1$, we obtain the generalization stated in Theorem 4 [7]:

$$\frac{x^2 + x^{2k}}{(1-(x^k + x^{2k}))^2}.$$

If we apply Corollary 3.2 to $A = \{m \,|\, m = 2k + 1, k \geq 0\}$, then we get a new result, namely that the generating function for the number of levels in the compositions of n with odd summands is given by

$$\frac{x^2(1-x^2)}{(1+x^2)(1-x-x^2)^2}.$$

Finally, we look at $A = \mathbb{N} - \{k\}$ and define $g(x, y; k) = \sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{levels}(\sigma) x^n y^{\text{parts}(\sigma)}$. Then Corollary 3.2 gives

$$g(x, y; k) = \left(\frac{x^2}{1-x^2} - x^{2k}\right) \sum_{m \geq 0} (m+1) \left(\frac{x}{1-x} - x^k\right)^m y^{m+2}.$$

If we set $y = 1$, then we get a new result, namely that the generating function for the number of levels in the compositions of n without k is given by

$$\frac{(1-x)x^2(1-x^{2(k-1)} + x^{2k})}{(1+x)(1-2x+x^k-x^{k+1})^2}.$$

4. Results for palindromic compositions with parts in A

Applying Theorem 2.1(2) for $r = \ell = d = 1$ we get that the generating function for the number of palindromic compositions of n with m parts in A is given by

$$\frac{1 + y \sum_{i=1}^{k} x^{a_i}}{1 - y^2 \sum_{i=1}^{k} x^{2a_i}}.$$

Setting $y = 1$ we get that the number of palindromic compositions of n with parts in A is given by (see [12], Theorem 3.2)

$$\frac{1 + \sum_{i=1}^{k} x^{a_i}}{1 - \sum_{i=1}^{k} x^{2a_i}}.$$

Using $A = \mathbb{N}$ we get that the generating function for the number of palindromic compositions of n with parts in $\mathbb{N}$ is given by (see [5], Theorem 6)

$$\frac{1 + x}{1 - 2x^2}.$$

We now turn our attention to the number of rises and drops.

4.1 The number of rises and drops

As before, the number of rises equals the number of drops. Theorem 2.1(2) for $\ell = 1$ and $d = 1$ gives

$$P_A(x; y; r, 1, 1) = \frac{1 + \sum\limits_{i=1}^{k} \frac{x^{a_i}y + x^{2a_i}y^2(1-r)}{1 - x^{2a_i}y^2(1-r)}}{1 - \sum\limits_{i=1}^{k} \frac{x^{2a_i}y^2 r}{1 - x^{2a_i}y^2(1-r)}}.$$

Therefore, by finding $\frac{\partial}{\partial r} P_A(x; y; r, 1, 1)$ and setting $r = 1$ we obtain the following result.

Corollary 4.1 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function $g_A(x; y) = \sum_{n \geq 0} \sum_{\sigma \in P_n^A} rises(\sigma) x^n y^{parts(\sigma)} = \sum_{n \geq 0} \sum_{\sigma \in P_n^A} drops(\sigma) x^n y^{parts(\sigma)}$ is given by*

$$\frac{y^2 \left(1 + y \sum_{i=1}^{k} x^{a_i}\right) \left(\sum_{i=1}^{k} x^{2a_i}(1 - x^{2a_i}y^2)\right) - y^2 \left(1 - y^2 \sum_{i=1}^{k} x^{2a_i}\right) \sum_{i=1}^{k} x^{2a_i}(1 + x^{a_i}y)}{\left(1 - y^2 \sum_{i=1}^{k} x^{2a_i}\right)^2}.$$

For example, if $A = \mathbb{N}$, then Corollary 4.1 gives that

$$g_{\mathbb{N}}(x; y) = \frac{y^2 \left(1 + \frac{yx}{1-x}\right) \left(\frac{x^2}{1-x^2} - \frac{x^4y^2}{1-x^4}\right) - y^2 \left(1 - \frac{y^2x^2}{1-x^2}\right) \left(\frac{x^2}{1-x^2} + \frac{x^3y}{1-x^3}\right)}{\left(1 - \frac{y^2x^2}{1-x^2}\right)^2}. \tag{11}$$

Thus, we can derive the generating function for the number of rises (drops) in the compositions of n with a given number of parts, m, in $\mathbb{N}$, by looking at the coefficient of y^m in $g_{\mathbb{N}}(x; y)$. To do so, we expand the numerator of $g_{\mathbb{N}}(x; y)$ and collect terms according to powers of y:

$$\frac{x^4 y^3}{(1-x)^2(1+x)} \cdot \left(\frac{2x + 1}{(x^2 + x + 1)} + \frac{2x^2}{(x+1)(x^2+1)} y - \frac{x^2}{(x^2+x+1)(x^2+1)} y^2\right).$$

Furthermore,

$$\frac{1}{\left(1 - \frac{y^2 x^2}{1-x^2}\right)^2} = \sum_{m \geq 0} \frac{(m+1)x^{2m}}{(1-x^2)^m} y^{2m},$$

so altogether,

$$g_{\mathbb{N}}(x; y) = \sum_{m \geq 0} \frac{(m+1)x^{2m+4}}{(1-x)^2(1+x)(1-x^2)^m} y^{2m+3} \cdot$$

$$\left(\frac{2x+1}{(x^2+x+1)} + \frac{2x^2}{(x+1)(x^2+1)}y - \frac{x^2}{(x^2+x+1)(x^2+1)}y^2\right).$$

We now have to distinguish between two cases, namely, m even and m odd. In the first case, only the summand with factor y (resulting in terms with factor $y^{2m+3}y$) needs to be taken into account, whereas in the second case, the summands with factors y^0 and y^2 (resulting in terms with factors y^{2m+3} and $y^{2m+3}y^2$) need to be considered. Thus, the generating function for the number of rises (drops) in the compositions of n with a given number of parts, m, in $\mathbb{N}$ is given by

$$\frac{(2m'-2)x^{2m'+2}}{(1+x^2)(1-x^2)^{m'}} \qquad \text{for } m = 2m',$$

and

$$\frac{x^{2m'}(1-x)(1+(2m'-2)x+(2m'-3)x^2+(2m'-2)x^3)}{(1+x^2)(1+x+x^2)(1-x^2)^{m'}} \qquad \text{for } m = 2m'-1.$$

Furthermore, setting $y = 1$ in Equation (11) and simplifying yields that the generating function for the number of rises (drops) in the compositions of n with parts in $\mathbb{N}$ (see [5], Theorem 6) is given by

$$g_{\mathbb{N}}(x; 1) = \frac{x^4(4x^4 + 4x^3 + 4x^2 + 3x + 1)}{(1+x^2)(1+x+x^2)(1-2x^2)^2}.$$

We now apply Corollary 4.1 for $A = \{1, k\}$ and get that

$$g_{\{1,k\}}(x; y) = \frac{x^{k+1}y^3(x + x^k + 2x^{k+1}y - y^2(x^3 + x^{3k} - x^{k+2} - x^{2k+1}))}{(1 - y^2(x^2 + x^{2k}))^2}.$$

In particular, when setting $y = 1$ in the above expression we get that the generating function for the number of rises (drops) in the palindromic compositions of n with any number of parts in $A = \{1, k\}$ is given by (see [7], Theorem 5)

$$g_{\{1,k\}}(x; 1) = \frac{x^{k+1}(x - x^3 + x^k - x^{3k} + 2x^{k+1} + x^{k+2} + x^{2k+1})}{(1 - x^2 - x^{2k})^2}.$$

If we let $A = \{m \mid m = 2k+1, k \geq 0\}$ in Corollary 4.1, then we get that the generating function $g_A(x; y)$ is given by

$$\frac{y^2\left(1 + \frac{xy}{1-x^2}\right)\left(\frac{x^2}{1-x^4} - \frac{y^2x^4}{1-x^8}\right) - y^2\left(1 - \frac{x^2y^2}{1-x^4}\right)\left(\frac{x^2}{1-x^4} + \frac{x^3y}{1-x^6}\right)}{\left(1 - \frac{x^2y^2}{1-x^4}\right)^2}.$$

Furthermore, if we set $y = 1$ in the above expression, then we get that the generating function for the number of rises (drops) in the palindromic compositions of n with any number of odd parts is given by

$$g_A(x; 1) = \frac{x^5(1 + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 3x^6 + 2x^7 + 2x^8)}{(1 + x^4)(1 - x^2 - x^4)^2(1 + x^2 + x^4)},$$

which extends the work of Grimaldi [10].

Applying Corollary 4.1 to $A = \mathbb{N} - \{k\}$ gives that

$$g_{\mathbb{N}-\{k\}}(x; y) = \frac{y^2 \left(1 + \frac{yx}{1-x} - yx^k\right) \left(\frac{x^2}{1-x^2} - x^{2k} - \frac{y^2 x^4}{1-x^4} + y^2 x^{4k}\right)}{\left(1 - \frac{y^2 x^2}{1-x^2} + y^2 x^{2k}\right)^2}$$
$$- \frac{y^2 \left(1 - \frac{y^2 x^2}{1-x^2} + y^2 x^{2k}\right) \left(\frac{x^2}{1-x^2} - x^{2k} + \frac{yx^3}{1-x^3} - yx^{3k}\right)}{\left(1 - \frac{y^2 x^2}{1-x^2} + y^2 x^{2k}\right)^2}.$$

In particular, when setting $y = 1$ in the above expression we get that the generating function for the number of rises (drops) in the palindromic compositions of n with any number of parts in $A = \mathbb{N} - \{k\}$ is given by

$$g_{\mathbb{N}-\{k\}}(x; 1) = \frac{x^4(1 + 3x + 4x^2 + 4x^3 + 4x^4) + x^{2k+1}(x^4 - 1)(1 + 4x + 5x^2 + 4x^3)}{(1 + x^2)(1 + x + x^2)(1 - 2x^2 + x^{2k} - x^{2(k+1)})^2}$$
$$+ \frac{(x^2 - 1)(x^{k+2} + x^{3k}(1 + x^2)(3x^2 - 2) + x^{4k}(1 + x)(x - 2))}{(1 + x^2)(1 - 2x^2 + x^{2k} - x^{2(k+1)})^2}.$$

This extends the work of Chinn and Heubach [6]. Likewise, we can extend the work of Grimaldi [9] by setting $k = 1$ to get that

$$g_{\mathbb{N}-\{1\}}(x; 1) = \frac{(x^5 + 3x^4 + 5x^3 + 3x^2 + 3x + 1)x^7}{(1 - x^2 - x^4)^2(1 + x + x^2)(1 + x^2)}.$$

We now look at the number of levels in palindromic compositions.

4.2 The number of levels

Theorem 2.1(2) for $r = d = 1$ gives

$$P_A(x; y; 1, \ell, 1) = \frac{1 + \sum_{i=1}^{k} \frac{x^{a_i} y + x^{2a_i} y^2(\ell-1)}{1 - x^{2a_i} y^2(\ell^2-1)}}{1 - \sum_{i=1}^{k} \frac{x^{2a_i} y^2}{1 - x^{2a_i} y^2(\ell^2-1)}}.$$

Therefore, finding $\frac{\partial}{\partial \ell} P_A(x; y; 1, \ell, 1)$ and setting $\ell = 1$ yields the following result.

Corollary 4.2 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function $g_A(x; y) = \sum_{n \geq 0} \sum_{\sigma \in P_n^A} levels(\sigma) x^n y^{parts(\sigma)}$ is given by*

$$\frac{y^2 \left(1 - y^2 \sum_{i=1}^k x^{2a_i}\right) \sum_{i=1}^k x^{2a_i}(1 + 2x^{a_i}y) + 2y^4 \left(1 + y \sum_{i=1}^k x^{a_i}\right) \sum_{i=1}^k x^{4a_i}}{\left(1 - y^2 \sum_{i=1}^k x^{2a_i}\right)^2}.$$

For example, applying Corollary 4.2 for $A = \mathbb{N}$ gives that the generating function $g_\mathbb{N}(x; y)$ for the number of levels in all palindromic compositions of n with m parts in $\mathbb{N}$ is given by

$$\frac{x^2 y^2 \left(2x^4(x + 1)y^3 + x^2(1 - 3x^2)(1 + x + x^2)y^2 + (1 - x^4)(2x(1 + x)y + 1 + x + x^2)\right)}{(1 + x^2)(1 + x + x^2)(1 - x^2 - x^2 y^2)^2}. \quad (12)$$

Since

$$\frac{1}{(1 - x^2 - x^2 y^2)^2} = \frac{1}{(1 - x^2)^2 (1 - \frac{x^2 y^2}{1 - x^2})^2} = \frac{1}{(1 - x^2)^2} \sum_{m \geq 0} (m + 1) \frac{x^{2m}}{(1 - x^2)^m} y^{2m},$$

we can compute the generating function $l_m(x)$ for the number of levels in palindromic compositions of n with a given number of parts, m, by looking at the coefficient of y^m in expression (12):

$$l_m(x) = \begin{cases} \frac{x^2}{1 - x^2} & \text{for} \quad m = 2 \\ \frac{(2m' - 1 - (2m' - 3)x^2)x^{2m'}}{(1 + x^2)(1 - x^2)^{m'}} & \text{for} \quad m = 2m', \ m' \geq 2 \\ \frac{2(1 + x)(m' + (m' - 1)x + m'x^2)x^{2m' + 1}}{(1 + x^2)(1 + x + x^2)(1 - x^2)^{m'}} & \text{for} \quad m = 2m' + 1, \ m' \geq 1 \end{cases}.$$

In addition, setting $y = 1$ in (12) gives that the generating function for the number of levels in the palindromic compositions of n with parts in $\mathbb{N}$ (see [5], Theorem 6) is given by

$$g_\mathbb{N}(x; 1) = \frac{x^2(1 + 3x + 4x^2 + x^3 - x^4 - 4x^5 - 6x^6)}{(1 + x^2)(1 + x + x^2)(1 - 2x^2)^2}.$$

If we let $A = \{1, k\}$ in Corollary 4.2, then $g_{\{1,k\}}(x; y)$ is given by

$$\frac{y^2(x^2 + x^{2k}) + 2y^3(x^3 + x^{3k}) + y^4(x^4 + x^{4k} - 2x^{2(k+1)}) + 2y^5(x^{k+4} - x^{2k+3} - x^{3k+2} + x^{4k+1})}{(1 - y^2 x^2 - y^2 x^{2k})^2}.$$

Setting $y = 1$ in the above expression yields that the generating function for the number of levels in the palindromic compositions of n with any number of parts in $\{1, k\}$ is given by

$$g_{\{1,k\}}(x; 1) = \frac{x^2 + x^{2k} + x^3 + x^{3k} + x^4 + x^{4k} + 2(x^{k+4} - x^{2(k+1)} - x^{2k+3} - x^{3k+2} + x^{4k+1})}{(1 - y^2 x^2 - y^2 x^{2k})^2}.$$

This result was not explicitly stated in [7], but can be easily computed from the generating functions for other quantities given in [7].

We look next at $A = \{m \mid m = 2k + 1, k \geq 0\}$. Applying Corollary 4.2 for this case, we get that

$$g_A(x; y) = \frac{y^2 \left(1 - \frac{x^2 y^2}{1 - x^4}\right)\left(\frac{x^2}{1 - x^4} + 2\frac{x^3 y}{1 - x^6}\right) + 2\frac{x^4 y^4}{1 - x^8}\left(1 + \frac{x\,y}{1 - x^2}\right)}{\left(1 - \frac{x^2 y^2}{1 - x^4}\right)^2}.$$

Furthermore, if we set $y = 1$ in the above expression, then we get that the generating function for the number of levels in the palindromic compositions of n with any number of odd parts is given by

$$g_A(x; 1) = \frac{x^2(1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 - 2x^6 + 2x^7 - 4x^8 - 2x^9 - 4x^{10} - 2x^{11} - x^{12})}{(1 + x^4)(1 - x^2 - x^4)^2(1 + x^2 + x^4)},$$

which extends the work of Grimaldi [10].

Finally, applying Corollary 4.2 for $A = \mathbb{N} - \{k\}$ gives that the generating function $g_{\mathbb{N}-\{k\}}(x; y)$ is given by

$$\frac{y^2\left(1 - \frac{y^2 x^2}{1 - x^2} + y^2 x^{2k}\right)\left(\frac{x^2}{1 - x^2} - x^{2k} + \frac{2yx^3}{1 - x^3} - 2yx^{3k}\right) + 2y^4\left(1 + \frac{yx}{1 - x} - yx^k\right)\left(\frac{x^4}{1 - x^4} - x^{4k}\right)}{\left(1 - \frac{y^2 x^2}{1 - x^2} + y^2 x^{2k}\right)^2}.$$

In particular, when setting $y = 1$ in the above expression we get that the generating function for the number of levels in the palindromic compositions of n that do not contain k is given by

$$\frac{x^2(1 + 3x + 4x^2 + x^3 - x^4 - 4x^5 - 6x^6) + x^{2k}(x^4 - 1)(1 + x - 2x^2 - 5x^3 - 5x^4)}{(1 + x^2)(1 + x + x^2)(1 - 2x^2 + x^{2k} - x^{2(k+1)})^2}$$
$$+ \frac{(x^2 - 1)(2x^{k+4} + 2x^{3k}(1 + x^2)(1 - 2x^2) + x^{4k}(1 + x)(3 - x)(1 + x^2))}{(1 + x^2)(1 - 2x^2 + x^{2k} - x^{2(k+1)})^2}.$$

This extends the work of Chinn and Heubach [6]. Likewise, we can extend the work of Grimaldi [9] by setting $k = 1$ to get that

$$g_{\mathbb{N}-\{1\}}(x; 1) = \frac{(1 + x + 3x^2 + 2x^3 - 5x^6 - 3x^7 - x^8)x^4}{(1 - x^2 - x^4)^2(1 + x^2)(1 + x + x^2)}.$$

In the next section, we look at another special type of compositions.

5. Results for Carlitz compositions with parts in A

A *Carlitz composition*[1] of n, introduced in [3], is a composition of n in which no adjacent parts are the same. In other words, a Carlitz composition σ is a composition with levels$(\sigma) = 0$.

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[1]Originally called *waves* by L. Carlitz in [3], and also known as *Smirnov sequences* (see [8], page 68). Named *Carlitz compositions* by Knopfmacher and Prodinger in [13] in honor of L. Carlitz

Carlitz [2] specialized the general results for compositions with parts in $A = \mathbb{N}$ to Carlitz compositions, but did not consider *Carlitz palindromic compositions*. Knopfmacher and Prodinger[13] stated results for the total number of Carlitz compositions without reference to the function $e(u, x)$ defined in Corollary 2.2.

In this section, we will apply Theorem 2.1 to state general results for Carlitz compositions and Carlitz palindromic compositions, and also look at a few specific sets A. We denote the set of Carlitz compositions and Carlitz palindromic compositions of n with parts in A, respectively, by E_n^A and F_n^A. Note that $F_n^A = E_n^A \cap P_n^A$.

5.1 The number of Carlitz compositions

Let E_n^A denote the set of Carlitz compositions of n with parts in A, and $E_A(x; y; r, d)$ the generating function for the number of Carlitz compositions of n with m parts in A with respect to the number of rises and drops, i.e.,

$$E_A(x; y; r, d) = \sum_{n \geq 0} \sum_{\sigma \in E_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)} d^{\text{drops}(\sigma)}.$$

Since $E_A(x; y; r, d) = C_A(x; y; r, 0, d)$, Theorem 2.1(1) for $\ell = 0$ gives the following result.

Corollary 5.1 *Let* $A = \{a_1, \ldots, a_k\}$ *be any ordered subset of* $\mathbb{N}$. *Then*

$$E_A(x; y; r, d) = 1 + \frac{\displaystyle\sum_{j=1}^{k} \left(\frac{x^{a_j} y}{1 + x^{a_j} y \, d} \prod_{i=1}^{j-1} \frac{1 + x^{a_i} y \, r}{1 + x^{a_i} y \, d} \right)}{1 - \displaystyle\sum_{j=1}^{k} \left(\frac{x^{a_j} y \, d}{1 + x^{a_j} y \, d} \prod_{i=1}^{j-1} \frac{1 + x^{a_i} y \, r}{1 + x^{a_i} y \, d} \right)}.$$

Note that equating $E_A(x; y; r, d)$ with the result of Theorem 3 [2] gives Corollary 2.2 for $\ell = 0$. We now look at specific sets A. Setting $r = d = 1$ in Corollary 5.1 we obtain the generating function for the number of Carlitz compositions with m parts in A (for the case $A = \mathbb{N}$, see [13]) as

$$E_A(x; y; 1, 1) = \frac{1}{1 - \sum_{j=1}^{k} \frac{x^{a_j} y}{1 + x^{a_j} y}}.$$

Applying Corollary 5.1 for $A = \{a, b\}$ and $r = d = 1$ yields the generating function for the number of Carlitz compositions of n with m parts in $\{a, b\}$ is given by

$$\frac{(1 + x^a y)(1 + x^b y)}{1 - x^{a+b} y^2} = 1 + (x^a + x^b)y + \sum_{m \geq 1} x^{m(a+b)} (2y^{2m} + (x^a + x^b)y^{2m+1}).$$

In particular, setting $y = 1$ in the expression above yields that the generating function for the number of Carlitz compositions of n with parts in $\{a, b\}$ is given by

$$\frac{(1 + x^a)(1 + x^b)}{1 - x^{a+b}}.$$

Remark: In the case $A = \{a, b\}$, the requirement that no adjacent parts are to be the same restricts the compositions to those with alternating a's and b's. This results in the following possibilities:

$$
\begin{array}{ccc}
n & & \text{Carlitz compositions of } n \\
n'(a + b) & & abab\dots ab,\ baba\dots ba \\
n'(a + b) + a & & abab\dots aba \\
n'(a + b) + b & & babab\dots ab
\end{array}
\tag{13}
$$

Thus, the number of Carlitz compositions of $n > 0$ is 2 if $n \equiv 0 \,(\mathrm{mod}(a + b))$, 1 if $n \equiv a \,(\mathrm{mod}(a + b))$ or $\equiv b \,(\mathrm{mod}(a + b))$, and 0 otherwise.

Next we look at the number of rises and drops in Carlitz compositions.

5.2 The number of rises and drops

As before, the number of rises equals the number of drops. Using Corollary 5.1 to find an explicit expression for $\frac{\partial}{\partial r} E_A(x; y; r, 1)\big|_{r=1}$ gives the following result.

Corollary 5.2 *Let $A = \{a_1, \dots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating functions $\sum_{n \geq 0} \sum_{\sigma \in E_n^A} rises(\sigma) x^n y^{parts(\sigma)}$ and $\sum_{n \geq 0} \sum_{\sigma \in E_n^A} drops(\sigma) x^n y^{parts(\sigma)}$ are given by*

$$\sum_{j=1}^{k} \left(\frac{x^{a_j} y}{1 + x^{a_j} y} \sum_{i=1}^{j-1} \frac{x^{a_i} y}{1 + x^{a_i} y} \right) \bigg/ \left(1 - \sum_{j=1}^{k} \frac{x^{a_j} y}{1 + x^{a_j} y} \right)^2 .$$

Setting $A = \mathbb{N}$ and $y = 1$ in Corollary 5.2 yields that the generating function for the number of rises (drops) in the Carlitz compositions of n with parts in $\mathbb{N}$ is given by

$$\frac{\sum_{j \geq 1} \left(\frac{x^j}{1 + x^j} \sum_{i=1}^{j-1} \frac{x^i}{1 + x^i} \right)}{\left(1 - \sum_{j \geq 1} \frac{x^j}{1 + x^j} \right)^2}.$$

Applying Corollary 5.2 for $A = \{a, b\}$ gives that

$$\sum_{n \geq 0} \sum_{\sigma \in E_n^A} \mathrm{rises}(\sigma) x^n y^{\mathrm{parts}(\sigma)} = \frac{x^{a+b} y^2 (1 + x^a y)(1 + x^b y)}{(1 - x^{a+b} y^2)^2}$$

$$= \sum_{m \geq 1} x^{m(a+b)} \left((2m - 1) y^{2m} + m(x^a + x^b) y^{2m+1} \right),$$

where the second equation follows after collecting even and odd powers of y.

In particular, setting $y = 1$ in the expression above yields that the generating function for the number of rises (drops) in the Carlitz compositions of n with parts in $\{a, b\}$ is given by

$$\frac{x^{a+b}(1 + x^a)(1 + x^b)}{(1 - x^{a+b})^2}.$$

Thus, the number of rises (drops) in Carlitz compositions of $n \geq (a + b)$ with parts in $\{a, b\}$ is given by

$$\begin{cases} 2n' - 1 & \text{if } n = (a + b)n' \quad \text{for } n' \geq 1 \\ n' & \text{if } n = (a + b)n' + a \text{ or } n = (a + b)n' + b \end{cases}.$$

This follows immediately from (13) since there is a rise for every occurrence of "ab". If $n = (a+b)n'$ and the composition starts with a, then there are n' rises. For the composition that starts with b, there is one less rise, for a total of $2n' - 1$ rises. If n is not a multiple of $a + b$, then the composition starts with r, where $n = (a + b)n' + r$. In either case, there are exactly n' rises, as there are n' occurrences of "ab" in the composition.

We now derive results for Carlitz palindromic compositions.

5.3 The number of Carlitz palindromic compositions

We denote the generating function for the number of Carlitz palindromic compositions of n with m parts in A with respect to the number of rises by $F_A(x; y; r)$, that is,

$$F_A(x; y; r) = \sum_{n \geq 0} \sum_{\sigma \in F_n^A} x^n y^{\text{parts}(\sigma)} r^{\text{rises}(\sigma)}.$$

Note that $F_A(x; y; r) = P_A(x; y; r, 0, 1)$. Using Theorem 2.1(2) for $\ell = 0$ and $d = 1$ gives the following result.

Corollary 5.3 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then*

$$F_A(x; y; r) = 1 + \frac{\sum_{i=1}^{k} \frac{x^{a_i} y}{1 + x^{2a_i} y^2 r}}{1 - \sum_{i=1}^{k} \frac{x^{2a_i} y^2 r}{1 + x^{2a_i} y^2 r}}.$$

Applying Corollary 5.3 for $A = \{a, b\}$ and $y = r = 1$ yields that the generating function for the number of Carlitz palindromic compositions of n with parts in $\{a, b\}$ is given by

$$\frac{1 + x^a + x^b - x^{a+b}}{1 - x^{a+b}}.$$

Thus, the number of Carlitz palindromic compositions of n with parts in $\{a, b\}$ is 1 if $n = (a+b)n' + a$ or $n = (a+b)n' + b$ for some $n' \geq 0$, and 0 otherwise. This follows immediately from (13), since the Carlitz compositions for $n = (a+b)n'$ are not symmetric.

5.4 The number of rises and drops

We now study the number of rises (drops) in all Carlitz palindromic compositions of n with m parts in A. Using Corollary 5.3 to compute $\frac{\partial}{\partial r}F_A(x; y; r)\big|_{r=1}$ gives the following result.

Corollary 5.4 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function for the number of rises in all Carlitz palindromic compositions of n with m parts in A is given by*

$$\frac{\partial}{\partial r}F_A(x; y; r)\bigg|_{r=1} = \frac{\sum_{i=1}^{k}\frac{x^{3a_i}y^3}{(1+x^{2a_i}y^2)^2}\left(\sum_{i=1}^{k}\frac{x^{2a_i}y^2}{1+x^{2a_i}y^2} - 1\right) + \sum_{i=1}^{k}\frac{x^{a_i}y}{1+x^{2a_i}y^2}\sum_{i=1}^{k}\frac{x^{2a_i}y^2}{(1+x^{2a_i}y^2)^2}}{\left(1 - \sum_{i=1}^{k}\frac{x^{2a_i}y^2}{1+x^{2a_i}y^2}\right)^2}.$$

Applying Corollary 5.4 for $A = \{a, b\}$ gives that

$$\sum_{n\geq 0}\sum_{\sigma \in F_n^A} \text{rises}(\sigma)x^n y^{\text{parts}(\sigma)} = \frac{x^{a+b}y^3(x^a + x^b)}{(1 - x^{a+b}y^2)^2} = (x^a + x^b)\sum_{m\geq 1} m\, x^{m(a+b)}y^{2m+1}.$$

In particular, setting $y = 1$ in the expression above yields that the generating function for the number of rises (drops) in all Carlitz palindromic compositions of n with parts in $\{a, b\}$ is given by

$$\frac{x^{a+b}(x^a + x^b)}{(1 - x^{a+b})^2}.$$

Thus, the number of rises (drops) in the Carlitz palindromic compositions of $n \geq a+b$ with parts in $\{a, b\}$ is given by

$$n' \text{ if } n = (a+b)n' + a \text{ or } n = (a+b)n' + b \text{ for } n' \geq 1 \quad \text{and} \quad 0 \text{ otherwise.}$$

This follows immediately from (13), as the Carlitz compositions for $n = (a+b)n' + a$ and $n = (a+b)n' + b$ are symmetric.

Last, but not least, we apply Theorem 2.1 to obtain results for partitions.

6. Partitions with parts in A

A *partition* σ of n is a composition of n with $\text{rises}(\sigma) = 0$. Let G_n^A be the set of all partitions of n with parts in A.

6.1 The number of partitions

We denote the generating function for the number of partitions of n with m parts in A with respect to the number of levels and drops by

$$G_A(x; y; \ell, d) = \sum_{n \geq 0} \sum_{\sigma \in G_n^A} x^n y^{\mathrm{parts}(\sigma)} \ell^{\mathrm{levels}\sigma} d^{\mathrm{drops}(\sigma)}.$$

Note that $G_A(x; y; \ell, d) = C_A(x; y; 0, l, d)$. Using Theorem 2.1(1) for $r = 0$ we get the following result.

Corollary 6.1 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function $G_A(x; y; \ell, d)$ is given by*

$$1 + \frac{\sum_{j=1}^{k} \left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y\ell}{1 - x^{a_i} y(\ell - d)} \right)}{1 - d \sum_{j=1}^{k} \left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - d)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y\ell}{1 - x^{a_i} y(\ell - d)} \right)}.$$

For example, if we apply Corollary 6.1 for $A = \mathbb{N}$ and $\ell = d = 1$ and use the identity

$$\sum_{j=1}^{k} x^{a_j} \prod_{i=1}^{j-1} (1 - x^{a_i} \alpha) = \frac{1}{\alpha} \left(1 - \prod_{j=1}^{k} (1 - x^{a_j} \alpha) \right), \tag{14}$$

then we get that the generating function for the number of partitions of n with m parts in $A = \mathbb{N}$ is given by

$$F_{\mathbb{N}}(x; y; 1, 1) = \prod_{j \geq 1} (1 - x^j y)^{-1}.$$

Note that the identity in (14) follows from the fact that

$$1 - \alpha \sum_{j=1}^{k} x^{a_j} \prod_{i=1}^{j-1} (1 - x^{a_i} \alpha) = \left(1 - \prod_{j=1}^{k} (1 - x^{a_j} \alpha) \right),$$

which can be easily proved by induction.

Another interesting example, namely setting $\ell = 0$ and $d = 1$ in Corollary 6.1, gives that the generating function for the number of partitions of n with m parts in A in which no adjacent parts are the same, that is, the partitions with distinct parts, is given by

$$G_A(x; y; 0, 1) = \frac{1}{1 - \sum_{j=1}^{k} x^{a_j} y \prod_{i=1}^{j} (1 + x^{a_i} y)^{-1}} = \prod_{j=1}^{k} (1 + x^{a_j} y),$$

where the second equality is easily proved by induction. In particular, the generating function for the number of partitions of n with parts in $\mathbb{N}$ with distinct parts is given by $\prod_{j \geq 1} (1 + x^j)$.

6.2 The number of levels and drops

We now study the number of levels and drops in all partitions of n. Using Corollary 6.1 to compute $\frac{\partial}{\partial \ell} G_A(x; y; \ell, 1)\big|_{\ell=1}$ and $\frac{\partial}{\partial d} G_A(x; y; 1, d)\big|_{d=1}$, we get the following result.

Corollary 6.2 *Let $A = \{a_1, \ldots, a_k\}$ be any ordered subset of $\mathbb{N}$. Then the generating function $\sum_{n \geq 0} \sum_{\sigma \in G_n^A} levels(\sigma) x^n y^{parts(\sigma)}$ is given by*

$$\frac{\sum_{j=1}^k \left(x^{2a_j} y^2 \prod_{i=1}^{j-1} (1 - x^{a_i} y) \right) - \sum_{j=1}^k \left(x^{a_j} y \prod_{i=1}^{j-1} (1 - x^{a_i} y) \sum_{i=1}^{j-1} \frac{x^{2a_i} y^2}{1 - x^{a_i} y} \right)}{\prod_{j=1}^k (1 - x^{a_j} y)^2},$$

and the generating function $\sum_{n \geq 0} \sum_{\sigma \in G_n^A} drops(\sigma) x^n y^{parts(\sigma)}$ is given by

$$\frac{\left(1 - \prod_{j=1}^k (1 - x^{a_j} y) \right)^2 - y^2 \sum_{j=1}^k \left(x^{a_j} \prod_{i=1}^{j-1} (1 - x^{a_i} y) \sum_{i=1}^j x^{a_i} \right)}{\prod_{j=1}^k (1 - x^{a_j} y)^2}.$$

Proof. We give a sketch of the proof for the first generating function. Since $G_A(x; y; l, 1) = 1 + \frac{S(\ell)}{1 - S(\ell)}$, where

$$S(\ell) = \sum_{j=1}^k \left(\frac{x^{a_j} y}{1 - x^{a_j} y(\ell - 1)} \prod_{i=1}^{j-1} \frac{1 - x^{a_i} y \ell}{1 - x^{a_i} y(\ell - 1)} \right) = \sum_{j=1}^k \left(g_j(\ell) \prod_{i=1}^{j-1} f_i(\ell) \right),$$

we get that

$$\frac{\partial}{\partial \ell} G_A(x; y; \ell, 1) = \frac{\frac{\partial}{\partial \ell} S(\ell)}{(1 - S(\ell))^2} = \frac{\sum_{j=1}^k \frac{\partial}{\partial \ell} g_j(\ell) \prod_{i=1}^{j-1} f_i(\ell) + g_j(\ell) \frac{\partial}{\partial \ell} \prod_{i=1}^{j-1} f_i(\ell)}{(1 - S(\ell))^2}.$$

Using Equation (10) gives that $\frac{\partial}{\partial \ell} \prod_{i=1}^{j-1} f_i(\ell) = - \prod_{i=1}^k f_i(\ell) \sum_{j=1}^k \frac{x^{2a_j} y^2}{(1 - x^{a_j} y \ell)(1 - x^{a_j} y(\ell - 1))}$. Computing $\frac{\partial}{\partial \ell} g_j(\ell)$, setting $\ell = 1$ in the expression for $\frac{\partial}{\partial \ell} G_A(x; y; \ell, 1)$, then using Equation (14) to simplify the denominator gives the stated result. $\square$

Applying Corollary 6.2 to $A = \{a, b\}$ gives that the generating function for the number of levels and drops, respectively, in the partitions of n with m parts in $\{a, b\}$ is given by

$$\frac{x^{2a} y^2 (1 - x^b y) + x^{2b} y^2 (1 - x^a y)}{(1 - x^a y)^2 (1 - x^b y)^2} \qquad \text{and} \qquad \frac{x^{a+b} y^2}{(1 - x^a y)(1 - x^b y)}.$$

In particular, setting $y = 1$ in the above expression yields that the generating function for the number of drops in all partitions of n with parts in $\{1, k\}$ is given by $\frac{x^{k+1}}{(1-x)(1-x^k)}$. Thus, the number of drops in the partitions of n with parts in $\{1, k\}$ is $\lfloor (n-1)/k \rfloor$. This follows from the specific structure of the partitions with parts in $\{1, k\}$. A single drop occurs in all the partitions that do not consist of either all 1's or all k's. Thus, for $n \in [n'k + 1, (n'+1)k)$, there are exactly $n' = \lfloor (n-1)/k \rfloor$ drops.

7. Concluding Remarks

We have provided a very general framework for answering questions concerning the number of compositions, number of parts, and number of rises, levels and drops in all compositions of n with parts in A. We have used this framework to investigate compositions, palindromic compositions, Carlitz compositions, Carlitz palindromic compositions and partitions of n. Our results generalize work by several authors, and we have applied our results to the specific sets studied previously, which has led to several new results. In addition, our results can be applied to any set $A \subseteq \mathbb{N}$, which will allow for further study of special cases.

In addition, the techniques used in this paper can be used to investigate products of the number of rises, levels and drops which show interesting connections to the Fibonacci sequence. For example, by computing the derivative with respect to d twice in Theorem 2.1 (2) and setting $y = r = \ell = 1$, we get that

$$\sum_{n \geq 0} \sum_{\sigma \in C_n^A} \text{drops}(\sigma)(\text{drops}(\sigma) - 1)x^n = \frac{2x^6}{(1 - x - x^2)^3} = 2x^3 \sum_{n \geq 3} \left(\sum_{a+b+c=n} F_a F_b F_c \right) x^n,$$

i.e., a convolution of three Fibonacci sequences. However, the formulas for the various products become more complicated, and not as easy to evaluate.

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