SNAKY IS A 41-DIMENSIONAL WINNER

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Abstract

Every polyomino is a winner in the weak achievement game on a rectangular infinite board with large enough dimension. In particular, Snaky is a winner on the 41-dimensional board. An infinite polyomino is a winner on an infinite dimensional board but a loser on a finite dimensional board.

1. Introduction

Achievement games for polyominoes have been introduced by Frank Harary [Gar, Ha1, Ha2, HH, Ha3]. They are generalizations of the well known game Tic-Tac-Toe, where the target shape can be some predetermined set of polyominoes. The type of the board can vary as well. It can be a tiling of the plane by triangles [HH, BH3] or hexagons [BH2]. The game board can be a Platonic solid [BH1] or the hyperbolic plane [Bod]. A comprehensive investigation of these possibilities can be found in [Bod]. The game of n-dimensional Tic-Tac-Toe is studied in [HJ] using Ramsey theory. The research of polyomino achievement on 3-dimensional board was started in [HW] and continued in [SD].

In this paper we study polyomino achievement games on n-dimensional rectangular boards. We show that every polyomino is a winner on a board whose dimension is large enough. In particular we show that Snaky, the only undecided 2-dimensional poliomino [Ha3], is a winner on the 41-dimensional board. Finally we show that the theory of achievement games of infinite polyominoes is very simple. It only depends on the dimension of the board.

The main tool in our investigations is a theorem of Beck [Bec] about hypergraph games. This theorem uses weight functions and is a generalization of a result of Erdős and Selfridge [ES].

2. Preliminaries

A *board* is a set of cells. The cells of an *n-dimensional rectangular board* is a board whose cells are the translations of an *n*-dimensional unit cube $[0, 1]^n$ by vectors of $\mathbf{Z}^n$.

An *n-dimensional polyomino* or an *n-polyomino* is a finite set of cells of the *n*-dimensional rectangular board such that both the polyomino and its complement are connected through $(n - 1)$-dimensional faces of the cells. A polyomino is also called an *animal*.

If $n < m$ then the *n*-dimensional rectangular board embeds into the *m*-dimensional rectangular board. So an *n*-polyomino can also be considered as an *m*-polyomino. The *dimension of a polyomino* is the minimum dimension of the board on which the polyomino is realizable.

We only consider polyominoes up to congruence, that is, the location and orientation of the polyomino on the board is not important.

In a *polyomino achievement game* two players alternately mark previously unmarked cells of the board using their own colors. The player who marks a set of cells congruent to a given polyomino wins the game. In a *weak achievement* game the second player only tries to prevent the first player from achieving the given polyomino. In a weak achievement game the first player is called the *maker* and the second player is called the *breaker*.

A polyomino is called a *(weak) winner* if the first player can always win the (weak) achievement game with the given polyomino. Otherwise the polyomino is called a *loser*.

Let $\mathcal{H} = (V, E)$ be a hypergraph, that is, $E \subseteq 2^V$. In a *hypergraph positional game* $(\mathcal{H}, p, q)$, two players alternately mark previously unmarked vertices of $\mathcal{H}$. The first player marks p and the second player marks q vertices per move. In an *achievement* game the player who marks all vertices of an edge of $\mathcal{H}$ wins the game. In a *weak achievement* game the second player only tries to prevent the first player to mark the vertices of an edge.

A polyomino achievement game is equivalent to a corresponding hypergraph game where the vertices of $\mathcal{H}$ is the set of cells of the playing board and the edges of $\mathcal{H}$ is the set of target polyominoes in all possible positions on the board.

Beck [Bec] showed that using a strategy based on weight functions the maker can win the weak achievement game if

$$\sum_{A \in E} \left(1 + \frac{p}{q}\right)^{-|A|} > p^2 q^2 (p + q)^{-3} d_2(\mathcal{H})|V|,$$

where $d_2(\mathcal{H})$ denotes the maximum number of edges of $\mathcal{H}$ containing two given vertices. In our case $p = 1 = q$ so the condition becomes

(1) $$\sum_{A \in E} 2^{-|A|} > 2^{-3} d_2(\mathcal{H})|V|.$$

3. Every polyomino is a winner

Since Beck's theorem only works if the hypergraph is finite, we are going to consider a finite D-dimensional board $B_{D,L}$ of size L. If an animal is a winner on this finite board, then it is also a winner on the infinite D-dimensional board B_D. First we consider the polyomino $P_{d,l}$ which is a d-dimensional cube of size l containing l^d cells.

Proposition 3.1. *If $L \geq l$ then there is a D such that $P_{d,l}$ is a winner on $B_{D,L}$.*

Proof. Let $\mathcal{H} = (V, E)$ be the hypergraph corresponding to the $P_{d,l}$ polyomino achievement game on the board $B_{D,L}$. We are going to verify that (1) holds for a large enough D.

We have $|A| = l^d$ for all $A \in E$. To determine $|E|$, note that we can choose $\binom{D}{d}$ ways the d dimensions of the board in which $P_{d,l}$ is embedded. In each of these dimensions we can place l consecutive cells $L - l + 1$ ways. In the rest of the dimensions we can choose a coordinate L ways. So the left hand side of (1) becomes

$$\sum_{A \in E} 2^{-|A|} = |E| 2^{-l^d} = \binom{D}{d}(L - l + 1)^d L^{D-d} 2^{-l^d} .$$

Now we determine $d_2(\mathcal{H})$. The number of edges containing two given cells will be a maximum if the two cells are next to each other, that is, all of the coordinates of these cells are the same except in one dimension where the coordinates differ by one. In this one dimension we can place l consecutive cells of an edge $l - 1$ ways so that they contain the two given cells. The remaining $d - 1$ dimensions of $P_{d,l}$ can be chosen $\binom{D}{d-1}$ ways and in each of these dimensions $P_{d,l}$ can be placed l ways. The following Figure 1 shows the $(3 - 1)3^{2-1} = 6$ possible placements of $P_{2,3}$ after picking the other dimension.
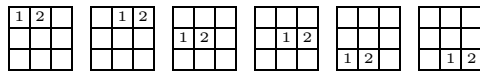

Figure 1.

So

$$d_2(\mathcal{H}) \leq (l-1)\binom{D-1}{d-1}l^{d-1}$$

and equality holds if L is large enough to contain each of the counted edges. So the maker wins if $L \geq l$ and

$$\binom{D}{d}(L - l + 1)^d L^{D-d} 2^{-l^d} > 2^{-3}(l-1)\binom{D-1}{d-1}l^{d-1}L^D$$

which holds if

(2) $$D > 2^{l^d-3}d(l-1)l^{d-1}\left(\frac{L}{L-l+1}\right)^d .$$

$\square$

Theorem 3.2. *For a polyomino P there is a D such that P is a winner on a rectangular board with dimension larger than D.*

Proof. There exist d and l such that P is a subset of $P_{d,l}$. If we can find a board on which $P_{d,l}$ is a winner then P is a winner on this same board since a subset of a winner is also a winner. By the previous proposition $P_{d,l}$ is a winner on the board $B_{D,L}$ if $L \geq l$ and D satisfies (2). Taking the limit of (2) for $L \to \infty$ shows that $P_{d,l}$ is a winner on the infinite board B_D if

$$(3) \qquad\qquad D > 2^{l^d - 3} d(l-1) l^{d-1}.$$

$\square$

By Theorem 3.2 it is guaranteed that the 2×2 square $P_{2,2}$ is a winner on B_D if $D > 9$. Actually $P_{2,2}$ is a loser on the 2-dimensional board [HS3] and a winner on the 3-dimensional board [SD].

4. Snaky

On a 2-dimensional board every polyomino is known to be a winner or a loser except Snaky (see Figure 2).
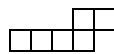

Figure 2.

Snaky is a paving winner [HS3], that is, there is no winning strategy for the breaker based on pavings by pairs of cells. Still, the best known handicap strategy [HHS, HS1] for Snaky requires two extra marks for the maker before the game starts.

Using (3) to find the dimensions of the boards on which Snaky is a winner gives the condition $D > 5 \cdot 2^{25}$. We can get a much lower value for the dimension if we recalculate condition (2) specifically for Snaky.

Proposition 4.1. *Snaky is a winner on the 41-dimensional rectangular board.*

Proof. Let $\mathcal{H} = (V, E)$ be the hypergraph corresponding to the Snaky achievement game on the board $B_{D,L}$. We have $|A| = 6$ for all $A \in E$. Now we determine $|E|$. We can choose the two dimensions in which Snaky can be embedded $D(D-1)$ ways. In these two dimensions we can shift a 5×2 rectangle containing Snaky $(L-4)(L-1)$ ways. We can place Snaky into this 5×2 rectangle 4 ways. Each of the remaining $D-2$ coordinates can be chosen L ways. So the

left hand side of (1) becomes

$$\sum_{A \in E} 2^{-|A|} = |E|2^{-6} = 4D(D-1)(L-4)(L-1)L^{D-2}2^{-6}.$$

Now we determine $d_2(\mathcal{H})$. The number of edges containing two given cells will be a maximum if the cells are next to each other. There are 10 ways to pick these two cells from Snaky. Snaky is 2-dimensional so we can pick the other dimension in which it is embedded $D-1$ ways. In this other dimension we can orient Snaky 2 ways. In Figure 3 the 20 possible placements of Snaky are shown after picking the other dimension.
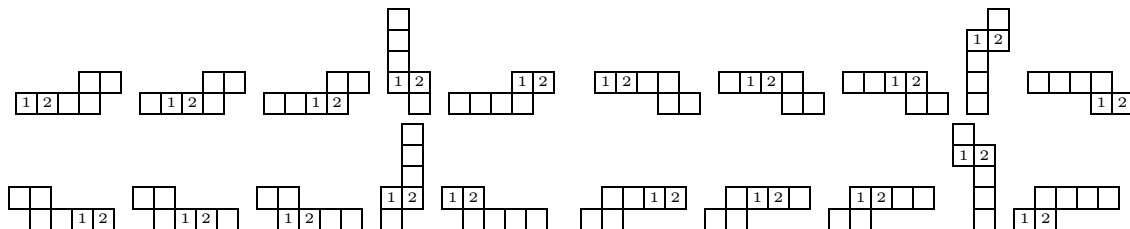

Figure 3.

So

$$d_2(\mathcal{H}) \leq 20(D-1)$$

and equality holds if L is large enough to contain each of the counted edges. So the maker wins if $L \geq l$ and

$$D(D-1)(L-4)(L-1)L^{D-2}2^{-4} > 10 \cdot 2^{-2}(D-1)L^D$$

or equivalently if

$$D > \frac{40L^2}{(L-4)(L-1)}.$$

Taking the limit for $L \to \infty$ shows that the maker wins on the infinite board B_D if $D > 40$. □

5. Infinite polyominoes

In this section we investigate achievement games on infinite polyominoes. We say that an infinite polyomino P is a *winner* if every finite subpolyomino of P is a winner. It only depends on the dimension of the board whether an infinite polyomino is a winner or a loser.

Corollary 5.1. *An infinite polyomino on a rectangular board is a winner if the dimension of the board is infinite and the polyomino is a loser if the dimension is finite.*

Proof. By [SD], there are only finitely many winners on a finite dimensional board. So an infinite polyomino cannot be a winner on a finite dimensional board. On the other hand, Theorem 3.2 shows that on an infinite dimensional board every finite polyomino is a winner and therefore every infinite polyomino is a winner as well. □

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