PARTITIONS OF NATURAL NUMBERS AND THEIR REPRESENTATION FUNCTIONS

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Abstract

For a given set A of nonnegative integers the representation functions $R_2(A,n)$, $R_3(A,n)$ are defined as the number of solutions of the equation $n = x+y$, $x, y \in A$ with condition $x < y$, $x \le y$, respectively. In this note we are going to determine the partitions of natural numbers into two parts such that their representation functions are the same from a certain point onwards.

1. Introduction

Throughout this paper we use the following notations: let $\mathbb{N}$ be the set of nonnegative integers. For $A \subset \mathbb{N}$ let $R_1(A,n), R_2(A,n), R_3(A,n)$ denote the number of solutions of

$$\begin{aligned} x+y &= n \qquad x,y \in A \\ x+y &= n \qquad x<y, \quad x,y \in A \\ x+y &= n \qquad x \le y, \quad x,y \in A \end{aligned}$$

respectively. A Sárközy asked whether there exist two sets A and B of nonnegative integers with infinite symmetric difference, i.e.

$$|(A \cup B) \setminus (A \cap B)| = \infty$$

and

$$R_i(A,n) = R_i(B,n) \qquad n \ge n_0$$

for $i = 1,2,3$. For i=1 the answer is negative (see [2]). For $i = 2$ G. Dombi (see [2]) and for $i = 3$ Y. G. Chen and B. Wang (see [1]) proved that the set of nonnegative integers

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can be partitioned into two subsets A and B such that $R_i(A, n) = R_i(B, n)$ for all $n \geq n_0$. In this note we determine the sets $A \subset \mathbb{N}$ for which either

$$R_2(A, n) = R_2(\mathbb{N} \backslash A, n) \qquad for \quad n \geq n_0$$

or

$$R_3(A, n) = R_3(\mathbb{N} \backslash A, n) \qquad for \quad n \geq n_0.$$

Theorem 1. *Let N be a positive integer. The equality $R_2(A, n) = R_2(\mathbb{N} \backslash A, n)$ holds for $n \geq 2N - 1$ if and only $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \in A$, $\quad 2m + 1 \in A \Leftrightarrow m \notin A$ for $m \geq N$.*

Setting out from $N = 1$ and $0 \in A$ we get Dombi's construction which is the set of nonnegative integers n where in the binary representation of n the sum of the digits is even.

Theorem 2. *Let N be a positive integer. The equality $R_3(A, n) = R_3(\mathbb{N} \backslash A, n)$ holds for $n \geq 2N - 1$ if and only if $|A \cap [0, 2N - 1]| = N$ and $2m \in A \Leftrightarrow m \notin A$, $\quad 2m + 1 \in A \Leftrightarrow m \in A$ for $m \geq N$.*

Setting out from $N = 1$ and $0 \in A$ we get Y. G. Chen and B. Wang's construction which is the set of nonnegative integers n where in the binary representation the number of the digits 0 is even.

2. Proofs

The proofs are very similar therefore we only present here the proof of Theorem 2.

Proof of Theorem 2. For $A \subset \mathbb{N}$ let

$$f(x) = \sum_{a \in A} x^a = \sum_{i=0}^{\infty} \epsilon_i x^i.$$

Then we have

$$\sum_{n=0}^{\infty} R_3(A, n) x^n = \frac{1}{2}(f(x^2) + f^2(x))$$

and

$$\sum_{n=0}^{\infty} R_3(\mathbb{N} \backslash A, n) x^n = \frac{1}{2}(\frac{1}{1 - x^2} - f(x^2) + (\frac{1}{1 - x} - f(x))^2),$$

moreover the condition $R_3(A, n) = R_3(\mathbb{N} \backslash A, n)$ for $n \geq 2N - 1$ is equivalent to the existence of a polynomial $p(x)$ of degree at most $2N - 2$ such that

$$\sum_{n=0}^{\infty} (R_3(A, n) - R_3(\mathbb{N} \backslash A, n)) x^n = p(x).$$

Then

$$\sum_{n=0}^{\infty}(R_3(A,n)-R_3(\mathbb{N}\backslash A,n))x^n =$$

$$\frac{1}{2}(f(x^2)+f^2(x)-(\frac{1}{1-x^2}-f(x^2)+(\frac{1}{1-x}-f(x))^2)) =$$

$$\frac{1}{2}(2f(x^2)-\frac{1}{1-x^2}-\frac{1}{(1-x)^2}-\frac{2f(x)}{1-x}) =$$

$$f(x^2)-\frac{1}{(1-x)^2(1+x)}+\frac{f(x)}{1-x}=p(x),$$

i.e.

$$f(x)=\frac{1}{1-x^2}-f(x^2)+f(x^2)x+p(x)(1-x).$$

First let us suppose that $R_3(A,n)=R_3(\mathbb{N}\backslash A,n)$ holds for $n\geq 2N-1$. Then there exists a polynomial $p(x)$ of degree at most $2N-2$ such that

$$f(x)=\frac{1}{1-x^2}-f(x^2)+f(x^2)x+p(x)(1-x).$$

So we have

$$p(x)(1-x)=\sum_{i=0}^{2N-1}\alpha_i x^i,$$

where $\sum_{i=0}^{2N-1}\alpha_i=0$, furthermore

$$\frac{1}{1-x^2}-f(x^2)+f(x^2)x=\sum_{i=0}^{\infty}((1-\epsilon_i)x^{2i}+\epsilon_i x^{2i+1}).$$

Hence

$$f(x)=\sum_{i=0}^{\infty}\epsilon_i x^i =$$

$$\frac{1}{1-x^2}-f(x^2)+f(x^2)x+p(x)(1-x)=$$

$$\sum_{i=0}^{N-1}((1-\epsilon_i)x^{2i}+\epsilon_i x^{2i+1})+\sum_{i=0}^{2N-1}\alpha_i x^i+\sum_{i=N}^{\infty}((1-\epsilon_i)x^{2i}+\epsilon_i x^{2i+1})=$$

$$\sum_{i=0}^{2N-1}\epsilon_i x^i+\sum_{i=2N}^{\infty}\epsilon_i x^i,$$

where

$$\sum_{i=0}^{2N-1}\epsilon_i=\sum_{i=0}^{N-1}((1-\epsilon_i)+\epsilon_i)+\sum_{i=0}^{2N-1}\alpha_i=N,$$

therefore

$$|A\cap[0,2N-1]|=N$$

and

$$\epsilon_{2m} = 1 - \epsilon_m, \quad \epsilon_{2m+1} = \epsilon_m \qquad for \quad m \geq N,$$

which means that $2m \in A$ if and only if $m \notin A$ and $2m+1 \in A$ if and only if $m \in A$ for $m \geq N$, which proves the necessary part of Theorem 2.

In the sufficient part we assume that $|A \cap [0, 2N-1]| = N$ and $2m \in A \Leftrightarrow m \notin A, \quad 2m+1 \in A \Leftrightarrow m \in A$ for $m \geq N$. This is equivalent to the assumptions that for the generating function $f(x) = \sum_{i=0}^{\infty} \epsilon_i x^i$ we have

$$\sum_{i=0}^{2N-1} \epsilon_i = N$$

and

$$\epsilon_{2m} = 1 - \epsilon_m, \quad \epsilon_{2m+1} = \epsilon_m \quad for \quad m \geq N.$$

Hence

$$f(x) = \sum_{i=0}^{\infty} \epsilon_i x^i = \sum_{i=0}^{2N-1} \epsilon_i x^i + \sum_{i=N}^{\infty} \epsilon_{2i} x^{2i} + \sum_{i=N}^{\infty} \epsilon_{2i+1} x^{2i+1} =$$

$$\sum_{i=0}^{2N-1} \epsilon_i x^i + \sum_{i=N}^{\infty} (1 - \epsilon_i) x^{2i} + \sum_{i=N}^{\infty} \epsilon_i x^{2i+1} =$$

$$\sum_{i=0}^{2N-1} \epsilon_i x^i + \sum_{i=0}^{\infty} (1 - \epsilon_i) x^{2i} - \sum_{i=0}^{N-1} (1 - \epsilon_i) x^{2i} + \sum_{i=0}^{\infty} \epsilon_i x^{2i+1} - \sum_{i=0}^{N-1} \epsilon_i x^{2i+1} =$$

$$\sum_{i=0}^{\infty} x^{2i} - \sum_{i=0}^{\infty} \epsilon_i x^{2i} + x \sum_{i=0}^{\infty} \epsilon_i x^{2i} + \sum_{i=0}^{2N-1} \epsilon_i x^i - \sum_{i=0}^{N-1} (1 - \epsilon_i) x^{2i} - \sum_{i=0}^{N-1} \epsilon_i x^{2i+1} =$$

$$\frac{1}{1 - x^2} - f(x^2) + x f(x^2) + \sum_{i=0}^{2N-1} \gamma_i x^i,$$

where

$$\sum_{i=0}^{2N-1} \gamma_i = \sum_{i=0}^{2N-1} \epsilon_i - \sum_{i=0}^{N-1} (1 - \epsilon_i) - \sum_{i=0}^{N-1} \epsilon_i = N - N = 0,$$

therefore there exists a polynomial $p(x)$ of degree at most 2N-2 such that

$$\sum_{i=0}^{2N-1} \gamma_i x^i = p(x)(1 - x).$$

Hence

$$f(x) = \frac{1}{1 - x^2} - f(x^2) + f(x^2) x + p(x)(1 - x),$$

which proves the sufficient part of Theorem 2.

References

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