# DISJOINT BEATTY SEQUENCES

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## Abstract

Consider two rational Beatty sequences $\{\lfloor p_1/q_1+\beta_1\rfloor : n \in \mathbf{Z}\}$ and $\{\lfloor p_2/q_2+\beta_2\rfloor : n \in \mathbf{Z}\}$ where $p_1, p_2, q_1, q_2$ are integers. We set $p = \gcd(p_1, p_2)$, $q = \gcd(q_1, q_2)$, $u_1 = q_1/q$, $u_2 = q_2/q$. In 1985 Morikawa showed that the sequences are disjoint for some $\beta_1$ and $\beta_2$ if and only if there exist positive integers $x$ and $y$ such that

$$xu_1 + yu_2 = p - 2u_1u_2(q - 1).$$

We give a new proof of this theorem. We also use intermediate results to obtain a generating function for a Beatty sequence and relate this to Fraenkel's conjecture about disjoint covering systems of rational Beatty sequences.

We define a **rational Beatty sequence** to be a sequence

$$S(p/q, \beta) = \{\lfloor pn/q + \beta\rfloor : n \in \mathbf{Z}\}, \tag{1}$$

where $p$ and $q$ are positive integers, $\gcd(p, q) = 1$ and $\beta$ is real. We say $p/q$ is the **modulus** of the sequence. Such sequences have an extensive literature (see [4], [10], [13]) and are the subject of a famous conjecture, postulated by Fraenkel ([4], [10], [6], [12], [14]). A **disjoint covering system of rational Beatty sequences** is a set of Beatty sequences $\{S(p_i/q_i, b_i) : i = 1, \ldots, n\}$ which partitions the integers. Fraenkel's conjecture is:

**Conjecture** If the rational Beatty sequences $\{S(p_i/q_i, b_i) : i = 1, \ldots, n\}$ form a disjoint covering system, with $n > 2$ and $p_1/q_1 < p_2/q_2 < \cdots < p_n/q_n$, then for $i = 1, \cdots, n$

$$p_i = 2^n - 1,\ q_i = 2^{n-i}.$$

We will say something more about this conjecture at the end of the paper.

Note that if we set $q = 1$ in (1) we get the arithmetic progression

$$\{np + \lfloor \beta \rfloor : n \in \mathbf{Z}\}.$$

The Chinese Remainder Theorem concerns the disjointness of pairs of such sequences. In our notation one form of it is the following.

**Chinese Remainder Theorem** There exist numbers $\beta_1$ and $\beta_2$ such that the sequences $S(p_1, \beta_1)$ and $S(p_2, \beta_2)$ are disjoint if and only if $\gcd(p_1, p_2) > 1$.

The situation is considerably more complicated when we deal with rational Beatty sequences with non-integral moduli. Ryozu Morikawa [8] has found necessary and sufficient conditions for the existence of $\beta_1$, $\beta_2$, such that $S(p_1/q_1, \beta_1)$ and $S(p_2/q_2, \beta_2)$ are disjoint. This is the "Japanese Remainder Theorem". Morikawa's proof is difficult and because it was published in a little known journal is not as well known as it should be. In this paper we present a simpler proof.

Thus we are interested in whether or not the Beatty sequences $S(p_1/q_1, \beta_1)$ and $S(p_2/q_2, \beta_2)$ intersect. We begin with some results that simplify the question.

**Lemma 1** *Given a rational Beatty sequence $S(p/q, \beta)$ there exist integers $b_1$ and $b_2$ such that*

$$\begin{aligned} S(p/q, \beta) &= S(p/q, b_1) \\ &= S(p/q, b_2/q)). \end{aligned}$$

For a proof see Lemma 1 of [12], where a slightly more general result is given.

**Remark 1** Since $\lfloor p(n+q)/q + \beta \rfloor = \lfloor pn/q + \beta \rfloor + p$ the sequence $S(p/q, \beta)$ coincides with a set of residue classes modulo $p$. It follows that:

$$\begin{aligned} S(p/q, b_1') &= S(p/q, b_1), \\ S(p/q, b_2'/q) &= S(p/q, b_2/q) \end{aligned}$$

if and only if $b_1' \equiv b_1 \bmod p$ and $b_2' \equiv b_2 \bmod p$. If $b_1$ and $b_2$ are as in Lemma 1 we have

$$S(p/q, b_1 q/q) = S(p/q, b_2/q)$$

from which

$$b_1 q \equiv b_2 \pmod{p}. \tag{2}$$

**Theorem 2** *Let $p = \gcd(p_1, p_2)$. Then*

$$S(p_1/q_1, b_1) \cap S(p_2/q_2, b_2) = \emptyset$$

*if and only if*

$$S(p/q_1, b_1) \cap S(p/q_2, b_2) = \emptyset.$$

*Proof.* The "if" part is immediate since $S(p_i/q_i, b_i) \subseteq S(p/q_i, b_i)$ for $i = 1, 2$. In the other direction let

$$p_1 = R_1 p,\ p_2 = R_2 p$$

so that $\gcd(R_1, R_2) = 1$. By Lemma 1 we may assume that $b_1$ and $b_2$ are integers. Suppose there exist integers $n_1$ and $n_2$ such that

$$\lfloor n_1 p/q_1 \rfloor + b_1 = \lfloor n_2 p/q_2 \rfloor + b_2.$$

Then for any integer $m$,

$$\lfloor (n_1 + mq_1)p/q_1 \rfloor + b_1 = \lfloor (n_2 + mq_2)p/q_2 \rfloor + b_2. \tag{3}$$

It is now sufficient to show that there exists $m$ such that

$$R_1 | (n_1 + mq_1),\ \ R_2 | (n_2 + mq_2) \tag{4}$$

for then the left and right sides of (3) are members, respectively, of $S(p_1/q_1, b_1)$ and $S(p_2/q_2, b_2)$.

Since $\gcd(q_1, R_1) = 1$. there exists $m_0$ such that $R_1$ divides $n_1 + m_0 q_1$. So for any integer $k$,

$$R_1 | n_1 + (m_0 + kR_1)q_1.$$

Now consider the sequence

$$\{n_2 + (m_0 + kR_1)q_2 : k \in \mathbf{Z}\}.$$

Since $\gcd(R_1, R_2) = 1$ and $\gcd(R_2, q_2) = 1$ there exists $k_0$ such that

$$R_2 | n_2 + m_0 q_2 + k_0 R_1 q_2,$$

so that (4) holds with $m = m_0 + k_0 R_1$. □

This theorem means that when considering conditions for the disjointness of pairs of Beatty sequences we may assume that the numerators of their moduli are equal. We shall henceforth fix $p$ to be this numerator. For this $p$ and any integer $q$ for which $\gcd(p, q) = 1$ we let $\overline{q}$ be the least non-negative residue modulo $p$ satisfying

$$q\overline{q} \equiv -1 \bmod p.$$

**Theorem 3** *The sequence $S(p/q, b)$, where $b$ is an integer, coincides with the set of residues*

$$\{\overline{q}m + b : m = 0, \ldots, q-1\} \bmod p.$$

*Proof.* For any $n$,

$$\lfloor pn/q + b \rfloor \equiv (pn - q\lfloor pn/q \rfloor)\overline{q} + b \pmod{p}.$$

The term in parentheses can take any value in the interval $[0, q-1]$. Thus for any $n$ there exists $m \in [0, q-1]$ such that

$$\lfloor pn/q + b \rfloor \equiv m\overline{q} + b \pmod{p}.$$

Similarly, for any $m \in [0, q-1]$ there exists an $n$ satisfying this relation. □

The next theorem concerns the disjointness of two Beatty sequences in which the denominator of the modulus of one divides the denominator of the modulus of the other. This will be used as a lemma for the main theorem.

**Theorem 4** *Let $u$ be a positive integer. Then the Beatty sequences $S(p/q, b_1/q)$ and $S(p/uq, b_2/uq)$ have non-empty intersection if and only if $b_2$ is congruent modulo $p$ to some integer $n$ in the interval $[(b_1 - q + 1)u, (b_1 + q)u - 1]$.*

*Proof.* Using equation (2) in Remark 1, the two Beatty sequences are $S(p/q, -b_1\overline{q})$ and $S(p/uq, -b_2\overline{uq})$. By Theorem 3 these intersect if and only if there exist integers $i$ and $j$ with $0 \le i \le q-1$, $0 \le j \le uq-1$ such that,

$$i\overline{q} - b_1\overline{q} \equiv j\overline{uq} - b_2\overline{uq} \pmod{p}. \tag{5}$$

Multiplying by $uq$ and rearranging gives

$$b_2 \equiv j + b_1 u - iu \pmod{p}.$$

That is,

$$\begin{aligned} b_2 &\in \bigcup_{i=0}^{q-1} \bigcup_{j=0}^{uq-1} \{j + b_1 u - iu\} \\ &\equiv \bigcup_{i=0}^{q-1} [(b_1 - i)u, (b_1 + q - i)u - 1] \pmod{p}. \end{aligned}$$

This is the union of $q$ blocks of $uq$ consecutive residues modulo $p$, with starting points of consecutive blocks differing by $u$. Thus it is the interval

$$[(b_1 - q + 1)u, (b_1 + q)u - 1] \pmod{p}.$$

□

**Corollary 5** *There exist $b_1$, $b_2$ such that the Beatty sequences in Theorem 4 are disjoint if and only if $p > (2q-1)u$.*

*Proof.* The interval $[(b_1 - q + 1)u, (b_1 + q)u - 1]$ contains $(2q - 1)u$ integers. By the theorem, the Beatty sequences can be made disjoint if and only if we can find $b_2$ which is not congruent modulo $p$ to any integer in the interval. This occurs if and only if $p > (2q - 1)u$. □

We now consider the general case of two Beatty sequences $S(p/q_1, \beta_1)$ and $S(p/q_2, \beta_2)$. From Remark 1 we may assume that $\beta_1 = 0$, $\beta_2 = b/q_2$. The next theorem allows us to apply Theorem 4 to this case. We let

$$q = \gcd(q_1, q_2), \;\; q_1 = u_1 q, \;\; q_2 = u_2 q. \tag{6}$$

**Theorem 6**

$$S(p/q_1, 0) = \bigcup_{i=0}^{u_1-1} S(p/q, i\overline{u}_1/q).$$

*Proof.* Treating $S(p/q_1, 0)$ as a set of residues modulo $p$, we have, by Theorem 3,

$$S(p/q_1, 0) = \bigcup_{m=0}^{q_1-1} \{m\overline{q}_1\} \pmod{p}.$$

By (6) the right hand side is

$$\bigcup_{m=0}^{u_1 q-1} \{m\overline{u_1 q}\} \equiv \bigcup_{i=0}^{u_1-1} \bigcup_{j=0}^{q-1} \{u_1 j + i\}\overline{u_1 q} \pmod{p}.$$

It is easily checked that

$$\overline{u_1 q} \equiv -\overline{u}_1 \overline{q} \pmod{p}$$

so that, using Theorem 3,

$$\begin{aligned} S(p/q, 0) &\equiv \bigcup_{i=0}^{u_1-1} \bigcup_{j=0}^{q-1} \{-\overline{u}_1\overline{q}u_1 j + i\overline{u_1 q}\} \\ &\equiv \bigcup_{i=0}^{u_1-1} \bigcup_{j=0}^{q-1} \{j\overline{q} + i\overline{u_1 q}\} \\ &\equiv \bigcup_{i=0}^{u_1-1} S(p/q, i\overline{u_1 q}) \\ &\equiv \bigcup_{i=0}^{u_1-1} S(p/q, -i\overline{u}_1\overline{q}q/q) \\ &\equiv \bigcup_{i=0}^{u_1-1} S(p/q, i\overline{u}_1/q) \pmod{p} \end{aligned}$$

as required. □

For $i = 0$ to $u_1 - 1$ we let $V_i$ be the set of residues modulo $p$ such that $b \in V_i$ implies

$$S(p/q_2, b/q_2) \cap S(p/q, i\overline{u}_1/q) \neq \emptyset.$$

**Remark 2** It follows from Theorem 6 that there exists $b$ such that $S(p/q_1, 0)$ and $S(p/q_2, b/q_2)$ are disjoint if and only if $\cup_{i=0}^{u_1-1} V_i$ is not a complete residue system modulo $p$.

If we apply Theorem 4 with $u_2$ in the role of $u$, $b$ in the role of $b_2$ and $i\overline{u}_1$ in the role of $b_1$ we obtain

$$\begin{aligned} V_i &= [(i\overline{u}_1 - q + 1)u_2, (i\overline{u}_1 + q)u_2 - 1] \\ &= [(-q+1)u_2, qu_2 - 1] + i\overline{u}_1 u_2, \end{aligned}$$

using an obvious notation. Our problem now is to decide whether the union of these intervals is a complete residue system modulo $p$. To proceed we need some more notation. Let $x$ and $y$ be integers such that

$$xu_1 + yu_2 = p - 2u_1u_2(q-1) \tag{7}$$

with

$$1 \le y \le u_1. \tag{8}$$

Since $u_1$ and $u_2$ are relatively prime by definition, (7) can always be satisfied, and (8) ensures that $x$ and $y$ are uniquely defined. Now let

$$z = u_1 - y. \tag{9}$$

Recall that $\gcd(q_1, p) = 1$, hence $\gcd(u_1, p) = 1$, hence by (7) $\gcd(y, u_1) = 1$ and by (9) $\gcd(z, u_1) = 1$.

Thus for each $i \in [0, u_1 - 1]$ there exists a unique integer $r(i)$ satisfying $r(i) \in [0, u_1 - 1]$ and $i \equiv r(i)z \bmod u_1$. For $r = 0, \dots, u_1$ let

$$M(r) = r(x + (2q-1)u_2) - \lfloor zr/u_1 \rfloor u_2.$$

We will use the functions $r(i)$ and $M(r)$ to obtain a reordering of the intervals $V_i$.

**Lemma 7** *For $i \in [0, u_1 - 1]$,*

$$M(r(i)) \equiv -i\overline{u}_1 u_2 \pmod{p}.$$

*Proof.* Fix $i$ and consider $u_1(M(r(i)) + i\overline{u_1}u_2)$. Writing $r$ for $r(i)$ and using (7) this equals

$$\begin{aligned} & u_1 r(x + (2q-1)u_2) - u_1\lfloor zr/u_1 \rfloor u_2 + u_1 i\overline{u_1}u_2 \\ \equiv\ & r(xu_1 + 2u_1u_2(q-1) + u_1u_2) - u_1u_2\lfloor zr/u_1 \rfloor - iu_2 \\ \equiv\ & r(-yu_2 + u_1u_2) - u_1u_2\lfloor zr/u_1 \rfloor - iu_2 \\ \equiv\ & u_2((u_1 - y)r - u_1\lfloor zr/u_1 \rfloor - i) \\ \equiv\ & u_2(zr - u_1\lfloor zr/u_1 \rfloor - i) \pmod{p}. \end{aligned}$$

Since $zr \equiv i \bmod u_1$ and $0 \leq i < u_1$ the term in parentheses equals zero. This, together with the observation that $\gcd(u_1, p) = 1$, implies the result. □

We can now prove our main result.

**Theorem 8** *Using the notation*

$$p = \gcd(p_1, p_2),\ q = \gcd(q_1, q_2),\ q_1 = u_1 q,\ q_2 = u_2 q,$$

*there exist $\beta_1$, $\beta_2$ such that the Beatty sequences $S(p_1/q_1, \beta_1)$ and $S(p_2/q_2, \beta_2)$ are disjoint if and only if there exist positive integers $x$ and $y$ such that,*

$$xu_1 + yu_2 = p - 2u_1u_2(q-1). \tag{10}$$

*Proof.* We can assume, as in (7) and (8) that we have $x$ and $y$ satisfying (10) with $1 \leq y \leq u_1$. We must show that disjointness is possible if and only if $x > 0$. By Remark 2 the Beatty sequences can be disjoint if and only if the union of the intervals $V_i$ is not a complete residue system modulo $p$. By Lemma 7 and the observation that as $i$ takes values from 0 to $u_1 - 1$ so, in a different order, does $r(i)$, this union is,

$$\bigcup_{i=0}^{u_1-1} \{[(-q+1)u_2, qu_2 - 1] + i\overline{u}_1 u_2\} = \bigcup_{r=0}^{u_1-1} \{[(-q+1)u_2, qu_2 - 1] - M(r)\}.$$

This is the union of $u_1$ blocks each containing $(2q-1)u_2$ consecutive residues modulo $p$. The first member of the $r$th block is $(-q+1)u_2 - M(r)$. Now

$$M(r+1) - M(r) = x + (2q-1)u_2 - u_2(\lfloor z(r+1)/u_1 \rfloor - \lfloor zr/u_1 \rfloor).$$

The term in parentheses takes the value 0 or 1, and as $r$ increases from 0 to $u_1$, $\lfloor zr/u_1 \rfloor$ increases from 0 to $z$. Thus this term will equal 1 for $z$ values of $r$ and 0 for $u_1 - z = y$ values. Now $y$ is positive by assumption so the maximum value of $M(r+1) - M(r)$ is $x + (2q-1)u_2$. The question of whether the union of the intervals $V_i$ is a complete residue system modulo $p$ is then equivalent to asking whether the block size is as big as $M(r+1) - M(r)$. That is, is $(2q-1)u_2 \geq x + (2q-1)u_2$? This is so if and only if $x \leq 0$. Thus it is possible to find $\beta_1$, $\beta_2$ such that the Beatty sequences are disjoint if and only if $x$ is strictly positive, which implies the statement of the theorem. □

We have presented an analogue of part of the Chinese Remainder Theorem. That theorem also describes the shape of the intersection $S(p_1, b_1) \cap S(p_2, b_2)$ when it is non-empty: it is an arithmetic progression with common difference equaling the lowest common multiple of $p_1$ and $p_2$. In the general case the situation is again more complicated. Fraenkel and Holzman [5] have shown that the intersection of $S(p_1/q_1, b_1)$ and $S(p_2/q_2, b_2)$ can have as many as $\min\{q_1, q_2\} + 3$ distinct differences between consecutive members.

We now return to Fraenkel's conjecture Although this concerns sets of pairwise disjoint Beatty sequences, Theorem 8 does not seem to help a great deal. However, we can use some of the earlier results to produce a generating function for the set of non-negative members of a Beatty sequence. We remark that O'Bryant [9] has recently developed a different technique for producing generating functions for Beatty sequences.

**Theorem 9** *The Beatty sequence $S(p/q, b)$, where $b$ is a non-negative integer, has generating function*

$$f(x) = g(x) + \frac{x^b}{1-x^p}\frac{1-x^{\overline{q}q}}{1-x^{\overline{q}}},$$

*where $g(x)$ is a polynomial with all coefficients equalling 1.*

*Proof.* By Theorem 3 and Remark 1 we see that the non-negative members of the Beatty sequence $S(p/q, b)$ form the set

$$\{\overline{q}m + b + np : 0 \le m < q,\ 0 \le n < \infty\} + G,$$

where $G$ is a finite set made up of those members of the sequence which are non-negative but correspond to negative values of $n$. Set $G = \{g_1, \ldots, g_M\}$. The corresponding generating function is

$$f(x) = x^b \sum_{m=0}^{q-1} x^{\overline{q}m} \sum_{n=0}^{\infty} x^{np} + g(x)$$

where $g(x)$ is the polynomial $\sum_{i=1}^{M} x^{g_i}$. Taking the closed forms of the sums gives the required formula. □

**Example** The Beatty sequence $S(7/4, 4)$ has non-negative terms 0, 2, 4, 5, 7, 9, 11, 12, 14, 16, 18, 19, 21,... We have $p = 7$, $q = 4$, $b = 4$ and $\overline{q} = 5$. Then

$$\frac{x^4}{1-x^7}\frac{1-x^{20}}{1-x^5} = x^4 + x^9 + x^{11} + x^{14} + x^{16} + x^{18} + x^{19} + x^{21} + \cdots$$

To account for the missing terms we set $g(x) = 1 + x^2 + x^5 + x^7 + x^{12}$.

**Corollary 10** *If $\{S(p_i/q_i, b_i) : i = 1, \ldots, n\}$ is a disjoint covering system then there exists a polynomial $g(x)$ whose coefficients all equal to 1 such that*

$$g(x) + \sum_{i=1}^{n} \frac{x^{b_i}}{1-x^{p_i}}\frac{1-x^{\overline{q}_i q_i}}{1-x^{\overline{q}_i}} = \frac{1}{1-x}. \tag{11}$$

*Proof.* The generating function for the non-negative integers is $1/(1-x)$. Since the Beatty sequences partition the integers we get, using the notation of the theorem,

$$\sum_{i=1}^{n} (g_i(x) + \frac{x^{b_i}}{1-x^{p_i}}\frac{1-x^{\overline{q}_i q_i}}{1-x^{\overline{q}_i}}) = \frac{1}{1-x}.$$

The result follows on writing $g(x) = \sum_{i=1}^{n} g_i(x)$. □

**Example** $\{S(\frac{7}{1},4), S(\frac{7}{2},6), S(\frac{7}{4},0)\}$ is a disjoint covering of Beatty sequences. The generating functions of the non-negative parts of the sequences are, respectively,

$$\begin{aligned} f_1(x) &= \frac{x^4}{1-x^7}, \\ f_2(x) &= x^2 + \frac{x^6}{1-x^7}\frac{1-x^6}{1-x^3}, \\ f_3(x) &= x + x^3 + x^8 + \frac{1}{1-x^7}\frac{1-x^{20}}{1-x^5}, \end{aligned}$$

so $g(x) = x + x^2 + x^3 + x^8$ and we have,

$$x + x^2 + x^3 + x^8 + \frac{x^4}{1-x^7} + \frac{x^6}{1-x^7}\frac{1-x^6}{1-x^3} + \frac{1}{1-x^7}\frac{1-x^{20}}{1-x^5} = \frac{1}{1-x}.$$

In 1950 Erdös reported the non-existence of disjoint coverings in which the quotients of the moduli all equal 1 and the numerators are distinct, with a proof obtained, independently, by Mirsky, Newman, Davenport, Rado (see [3]). This proof used a generating function like ours. Combinatorial proofs were later given by other authors ([2], [7], [11] and see [15]). We can use the corollary and their technique to give an alternative proof of a known result [1] about disjoint covering systems.

**Corollary 11** *If* $\{S(p_i/q_i, b_i) : i = 1, \ldots, n\}$ *is a disjoint covering system with* $p_1 \le p_2 \le \cdots \le p_n$ *then* $p_{n-1} = p_n$.

*Proof.* Suppose that $p_n > p_{n-1}$ and let $\zeta$ be a primitive $p_n$th root of unity. Now let $x$ approach $\zeta$ in equation (11). The absolute value of the term $\frac{x^{b_n}}{1-x^{p_n}}\frac{1-x^{\overline{q}_n q_n}}{1-x^{\overline{q}_n}}$ goes to infinity, but all other terms remain finite. This contradiction proves the result. □

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