ON Z.-W. SUN'S DISJOINT CONGRUENCE CLASSES CONJECTURE

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Abstract

Sun has conjectured that if k congruence classes are disjoint, then necessarily two of the moduli have greatest common divisor at least as large as k. We prove this conjecture for $k \leq 20$.

1. Introduction

In May of 2003, Prof. Sun[1] proposed a conjecture on the number theory listserver[2].

Conjecture 1 (Disjoint Congruence Classes Conjecture). *If* $k \geq 2$ *congruence classes* $a_i \pmod{m_i}$ *are disjoint, then there exist* $i < j$ *with* $\gcd(m_i, m_j) \geq k$.

The set of congruence classes $\{1 \pmod{k}, 2 \pmod{k}, \ldots, k \pmod{k}\}$ demonstrate that, if true, the DCCC is best possible.

The contrapositive of the $k = 2$ case is a familiar special case of the Chinese remainder theorem: if two congruence classes have relatively prime moduli, then they intersect. Prof. Graham[3], who first brought this problem to my attention, pointed out that the $k = 3$ case follows easily from the pigeonhole principle, as we now explain.

Suppose that $a_1 \bmod m_1$, $a_2 \bmod m_2$, $a_3 \bmod m_3$ is a counter-example: they are disjoint and $\gcd(m_i, m_j) < 3$ for all $i < j$. If $\gcd(m_i, m_j) = 1$, then by the Chinese remainder theorem $a_i \pmod{m_i}$ and $a_j \pmod{m_j}$ intersect. Thus, $\gcd(m_i, m_j) = 2$ for all $i < j$; in

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particular, all m_i are even. By the pigeonhole principle, two of the a_i's must have the same parity, say $a_1 \equiv a_2 \pmod 2$. Since $\gcd(m_1, m_2) = 2$, obviously $\gcd(\frac{m_1}{2}, \frac{m_2}{2}) = 1$. If a_1 and a_2 are both even, then by the Chinese remainder theorem we can find x in the intersection of

$$\frac{a_1}{2} \pmod{\frac{m_1}{2}} \quad \text{and} \quad \frac{a_2}{2} \pmod{\frac{m_2}{2}}$$

and consequently $2x$ is in the intersection of $a_1 \pmod{m_1}$ and $a_2 \pmod{m_2}$. If a_1 and a_2 are both odd, then we can find x in the intersection of

$$\frac{a_1 - 1}{2} \pmod{\frac{m_1}{2}} \quad \text{and} \quad \frac{a_2 - 1}{2} \pmod{\frac{m_2}{2}}.$$

But then $2x + 1$ is in the intersection of $a_1 \pmod{m_1}$ and $a_2 \pmod{m_2}$.

The following criterion is at the heart of the last paragraph.

Proposition 2. *The congruence classes* $a_i \pmod{m_i}$ *and* $a_j \pmod{m_j}$ *are disjoint if and only if* $\gcd(m_i, m_j) \nmid a_i - a_j$.

Prof. Graham also noted that the $k = 4$ case was similar but with more considerations, and considered it likely that $k = 5$ was similarly tractable. We now state our main theorem.

Theorem 3. *The DCCC holds for* $k \leq 20$. *Moreover, a counterexample to the DCCC with minimal* k *does not have* $k \in \{24, 30\}$.

We close the introduction with a stronger conjecture of Prof. Sun [3].

Conjecture 4. *Suppose that* $A_1, \ldots, A_k$ *are disjoint left-cosets of* $H_1, \ldots, H_k$ *in the group* G. *Then* $\gcd([G : H_i], [G : H_j]) \geq k$ *for some* $i < j$.

2. A Disjointness Criterion

The following criterion is stated in [1] without proof.

Lemma 5. *If* $\displaystyle\sum_{i=1}^{\ell} \frac{1}{\gcd(m_i, M)} > 1$, *where* M *is any multiple of* $\mathrm{LCM}\{\gcd(m_i, m_j) \colon 1 \leq i < j \leq \ell\}$, *then the congruence classes* $a_i \bmod m_i$ *(with* $1 \leq i \leq k$*) are not disjoint.*

The usefulness of this criterion lies in the fact that it makes no reference to the a_i.

Can there be seven disjoint congruence classes with moduli 20, 15, 12, 6, 6, 6, 6? Lemma 5 does not exclude it directly, but it does exclude the possibility that there are six disjoint congruence classes with moduli 15, 12, 6, 6, 6, 6. It is plausible that if every subset of

$\{m_1, \ldots, m_k\}$ passes the above test, then it is possible to choose $a_1, \ldots, a_k$ so that a_1 mod $m_1, \ldots, a_k$ mod m_k are disjoint, and Huhn & Megyesi [1] conjectured this. However, Z.-W. Sun [2] notes that the moduli 10, 15, 36, 42, 66 pass the above test (as does every subset of the moduli), but these are not the moduli of disjoint congruence classes.

Proof. The class a_i mod m_i intersects each of the $M/\gcd(m_i, M)$ classes $a_i + jm_i \pmod M$ (for $0 \le j < M/\gcd(m_i, M)$); that is, the class a_i mod m_i is actually $M/\gcd(m_i, M)$ classes modulo M. Since by hypothesis $\sum_{i=1}^{\ell} M/\gcd(m_i, M) > M$, the pigeonhole principle implies that two of the modulo-M congruence classes must intersect. In other words, there are integers $\alpha, \beta, \gamma, i, j$ such that

$$a_i + \alpha m_i = a_j + \beta m_j + \gamma M.$$

Since $\gcd(m_i, m_j) \mid M$, there are integers δ, ϵ such that $\delta m_i + \epsilon m_j = M$. Thus

$$a_i + (\alpha - \delta\gamma) m_i = a_j + (\beta + \epsilon\gamma) m_j,$$

and we see that a_i mod m_i and a_j mod m_j intersect.

3. Without Loss of Generality

We suppose that a minimal counterexample exists, and use that to determine a sequence $m_1, \ldots, m_k$ (as described in the lemma below) that has particular properties (in particular, we have an explicit upper bound on m_i). While there are infinitely many sets of k congruence classes, there are only finitely many such m_i sequences, and in fact we show that for $k \le 20$ there are none. We note that the existence of such an m_i sequence would not disprove Sun's Disjoint Congruence Classes Conjecture, but that nonexistence would imply his conjecture.

Set $L_k := \mathrm{LCM}\{1, 2, \ldots, k-1\}$.

k	3	4	5	6	7	8	9	10	11	12	13	14	15
L_k	2	6	12	60	60	420	840	2520	2520	27720	27720	360360	360360

Lemma 6. *Suppose that $k \ge 2$ is the least integer such that there are k disjoint congruence classes $a_i \pmod{m_i}$ with $\gcd(m_i, m_j) < k$ for $i < j$ (i.e., a counterexample to the DCCC). Further, suppose that $a_i \pmod{m_i}$ $(1 \le i \le k)$ has minimal $\sum_i m_i$. Set*

$$N := \mathrm{LCM}\{\gcd(m_i, m_j) \colon 1 \le i < j \le k\}.$$

Then $k \ge 4$, and for all i

1. *$m_i \mid N$, and $N \mid L_k$, and $N = \mathrm{LCM}\{m_1, m_2, \ldots, m_k\}$;*

2. *$1 < \gcd(m_i, m_j) < k$ for all $j \neq i$;*

3. *If a prime power q divides m_i, then there is $j \neq i$ with $q \mid m_j$.*

4. *m_i is not a prime power;*

5. *At least three of the m's are multiples of $k-1$;*

6. *If only three of the m's are multiples of $k-1$, then two others are multiples of $k-2$;*

7. *For every $\{h_1, \ldots, h_\ell\} \subseteq \{m_1, \ldots, m_k\}$, and every multiple M of $\mathrm{LCM}\{\gcd(h_i, h_j) \colon 1 \leq i < j \leq \ell\}$,*

$$\sum_{i=1}^{\ell} \frac{1}{\gcd(h_i, M)} \leq 1.$$

8. *If $7 \leq k \leq 30$, and $p \geq k/2$ is a prime that divides some m, then it divides exactly two of the m's.*

Proof. We noted in the introduction the reasons why $k \not\leq 3$.

In this paragraph, we prove the three statements in item 1. Suppose that p^r is a prime power that divides m_i but not N. By the definition of N, we see that $p^r \nmid m_j$ for $j \neq i$. Thus $\gcd(m_i, m_j) = \gcd(m_i/p, m_j)$, and by Proposition 2, the classes $a_i \pmod{m_i/p}$ and $a_j \pmod{m_j}$ are disjoint. Thus, we can replace $a_i \pmod{m_i}$ in our counterexample to the DCCC with $a_i \pmod{m_i/p}$, obtaining a counterexample with smaller $\sum m$. We began with minimal $\sum m$, so we conclude that there is no prime power dividing m_i but not N. Since N is defined to be the least common multiple of a subset of $\{1, 2, \ldots, k-1\}$, it is clear that $N \mid L_k$. Moreover, since N is the least common multiple of divisors of the m_i, and the m_i divide N, it is also now immediate that $N = \mathrm{LCM}\{m_1, \ldots, m_k\}$.

That $\gcd(m_i, m_j) < k$ is by hypothesis; that $\gcd(m_i, m_j) > 1$ follows from the chinese remainder theorem. This proves item 2.

Any prime power that divides some m_i also divides N since $N = \mathrm{LCM}\{m_1, \ldots, m_k\}$. But any prime power that divides $N = \mathrm{LCM}\{\gcd(m_i, m_j) \colon i \neq j\}$ must divide two of the m's. This proves item 3.

Items 4, 5, and 6 are based on the minimality of k. Suppose that $m_1 = p^r$, a prime power. Since $\gcd(m_1, m_j) > 1$, we know that $p \mid m_j$ for every j. By item 3, there is m_2 that is also a multiple of p^r. Since $\gcd(m_1, m_2) < k$, we see that $p < k$, whence $\lceil k/p \rceil \geq 2$. By Proposition 2, $\gcd(m_i, m_j) \nmid a_j - a_i$ for every $1 \leq i < j \leq k$. And by the pigeonhole principle, there are at least $\lceil k/p \rceil$ of the a's that are congruent to one another modulo p, say (after renumbering)

$$a_1 \equiv a_2 \equiv \cdots \equiv a_{\lceil k/p \rceil} \pmod{p}.$$

Consequently,

$$\gcd\left(\frac{m_i}{p}, \frac{m_j}{p}\right) \nmid \frac{a_j - a_i}{p}$$

for every $1 \le i < j \le \lceil k/p \rceil$. This implies, again by Proposition 2, that

$$0 \pmod{\frac{m_1}{p}}, \quad \frac{a_2 - a_1}{p} \pmod{\frac{m_2}{p}}, \quad \dots, \quad \frac{a_{\lceil k/p \rceil} - a_1}{p} \pmod{\frac{m_{\lceil k/p \rceil}}{p}}$$

are also disjoint. Since k is minimal, there must be $1 \le i < j \le \lceil k/p \rceil$ with

$$\gcd\left(\frac{m_i}{p}, \frac{m_j}{p}\right) \ge \left\lceil \frac{k}{p} \right\rceil,$$

whence $\gcd(m_i, m_j) \ge \lceil k/p \rceil p \ge k$, contradicting the existence of a counterexample with k classes. This proves item 4.

Suppose that only m_1 and m_2 are multiples of $k-1$. Then $a_i \pmod{m_i}$ (with $2 \le i \le k$) is a collection of $k-1$ disjoint congruence classes whose moduli have gcds strictly less than $k-1$. This contradicts the minimality of k, and proves item 5.

Suppose that only m_1, m_2, and m_3 are multiples of $k-1$. Then $a_i \pmod{m_i}$ (with $3 \le i \le k$) are $k-2$ disjoint congruence classes whose moduli have gcds strictly less than k (because we started with a counterexample). Moreover, the gcds are strictly less than $k-1$ because none of $m_4, \dots, m_k$ are multiples of $k-1$. If there are not two of $m_3, \dots, m_k$ that are multiples of $k-2$, then we have even more: a counterexample with $k-2$ sequences. By the minimality of k, then, two of $m_3, \dots, m_k$ must be multiples of $k-2$. If there are exactly two, then it could be that m_3 is one of them. To see that is not the case, we renumber m_2 and m_3 to conclude that both m_2 and m_3 are multiples of $k-2$. But then both $k-1$ and $k-2$ divide $\gcd(m_2, m_3)$, so that it must be at least k. This proves item 6, and that $k \ge 5$.

Item 7 is a restatement of Lemma 5 for subsets.

Now suppose that p and k are as hypothesized in item 8, and suppose that $m_1, \dots, m_\ell$ are multiples of p, while $m_{\ell+1}, \dots, m_k$ are not multiples of p. Note that $\ell \ge 2$ by item 3, and by item 2 necessarily $p < k$. Suppose that there are r primes less than k; since $k \le 30$ we know that $r \le 10$. Assume, by way of contradiction, that $\ell \ge 3$.

Set

$$\mathcal{P}_i := \{q \colon q \text{ prime}, q \mid m_i\} \setminus \{p\},$$

and recall that by Lemma 6 (item 4) the cardinality of $\mathcal{P}_i$ is at least 1 for every $i \le \ell$, and at least 2 for $i > \ell$. Moreover, since $\gcd(m_i, m_j) < k \le 2p$ the $\mathcal{P}_i$ (for $1 \le i \le \ell$) are disjoint.

Now define for $\ell < i \le k$ the ℓ-tuple

$$\omega_i := \Big(\min\{\mathcal{P}_1 \cap \mathcal{P}_i\}, \min\{\mathcal{P}_2 \cap \mathcal{P}_i\}, \dots, \min\{\mathcal{P}_\ell \cap \mathcal{P}_i\} \Big),$$

which is an element of $\mathcal{P}_1 \times \cdots \times \mathcal{P}_\ell$. If $\omega_i = \omega_j$ (with $\ell < i < j \le k$), then $\gcd(m_i, m_j)$ is at least the product of the primes in ω_i; since $\ell \ge 3$ this product must be at least $2{\cdot}3{\cdot}5 = 30 \ge k$. By the pigeonhole principle, this certainly happens if

$$k - \ell > \left|\mathcal{P}_1 \times \cdots \times \mathcal{P}_\ell\right| = \prod_{i=1}^{\ell} |\mathcal{P}_i|.$$

Since $|\mathcal{P}_i| \ge 1$ and $\sum_{i=1}^{\ell} |\mathcal{P}_i| \le r - 1$ (because the $\mathcal{P}_i$ are disjoint for $i \le \ell$, and there are r primes less than k including p), we can easily bound the size of $\prod_{i=1}^{\ell} |\mathcal{P}_i|$ in terms of r and ℓ:

r \ ℓ: $\max\{\prod_{i=1}^{\ell} \lvert\mathcal{P}_i\rvert\}$	3	4	5	6	7	8	9
4	1	0	0	0	0	0	0
5	2	1	0	0	0	0	0
6	4	2	1	0	0	0	0
7	8	4	2	1	0	0	0
8	12	8	4	2	1	0	0
9	18	16	8	4	2	1	0
10	27	24	16	8	4	2	1

This table shows that for $4 \le r \le 10$, necessarily $k - \ell > |\text{Range}(\omega)|$, and so two m's have gcd at least $30 \ge k$, with the exception of $r = 10$, $k = 30$, $\ell = 3$.

In the case of $r = 10$, $k = 30$, $\ell = 3$, each of the 27 possible values of ω must actually occur: $|\mathcal{P}_1| = |\mathcal{P}_2| = |\mathcal{P}_3| = 3$. Thus two of the four primes 17, 19, 23, 29 are in separate $\mathcal{P}_i$'s ($1 \le i \le 3$), and so must both be in three of the ω_i's ($4 \le i \le 30$). But then the two corresponding m's will have gcd at least $17 \cdot 19 > 30$.

4. Casework

4.1. The Cases $3 \le k \le 6$

These cases can be handled in a variety of ways. We handle them here using only $1 < m_i \mid L_k$, $1 < \gcd(m_i, m_j) < k$, and that m_i cannot be a prime power.

k = 3. The divisors of $L_3 = 2$ are 1 and 2. That $m_i = 1$ is impossible since $\gcd(m_i, m_j) > 1$, and that $m_i = 2$ is impossible since m_i cannot be a prime power.

k = 4. The divisors of $L_4 = 6$ are 1, 2, 3, and 6. Since m_i cannot be a prime power (or 1), every $m_i = 6$. But $\gcd(m_i, m_j) < k < 6$, so this too is impossible.

k = 5. The divisors of $L_5 = 12$ are 1, 2, 3, 4, 6, and 12. Since m_i cannot be a prime power (or 1), every m_i is either 6 or 12. But $\gcd(m_i, m_j) < k < 6$, so this too is impossible.

k = 6. The only allowed divisors of $L_6 = 60$ are $6, 10, 12, 15, 20, 30$, and 60. Since $\gcd(m_i, m_j) < 6$, the m_i must be distinct. By the pigeonhole principle, two of the m's must be multiples of 10, whence $\gcd(m_i, m_j) \geq 10$.

4.2. The Cases $6 \leq k \leq 10$

While looking through the cases below, I find it helpful to imagine the complete graph on k vertices with vertices labeled with $m_1, m_2, \dots, m_k$, and edges labeled with $\gcd(m_i, m_j)$.

With $k = 7$, the assignments $m_1 = 20$, $m_2 = 15$, $m_3 = 12$, $m_4 = m_5 = m_6 = m_7 = 6$ satisfy all of the conditions given in Lemma 6 except item 7, and also passes the test of Lemma 5. Thus, for $k \geq 7$ the arguments are necessarily more involved.

k = 7. We have $N \mid L_7 = 60$. Suppose that none of the m's are multiples of 5, so that we can take $M := 12$, and at most one $\gcd(m_i, M)$ is 12, with other six $\gcd(m_i, M)$ being ≤ 6. In this case,

$$\sum_{i=1}^{7} \frac{1}{\gcd(m_i, 12)} \geq \frac{1}{12} + 6\,\frac{1}{6} > 1.$$

Now suppose that some m is a multiple of 5, and by item 3, at least two of the m's are multiples of 5. We know that no m is exactly 5 (a prime), so that the two (or more) of the m's that are multiples of 5 are in $\{10, 15, 20, 30, 60\}$. Since $\gcd(m_i, m_j) \leq 6$ the only possibilities are "10 and 15" or "15 and 20". Delete the congruence class that gives m being either 10 or 20, and the six classes remaining fail to have the condition given in item 7 with $M = 12$:

$$\sum_{6 \text{ classes}} \frac{1}{\gcd(m_i, 12)} \geq \frac{1}{\gcd(15, 12)} + \frac{1}{\gcd(12, 12)} + 4\,\frac{1}{\gcd(6, 12)} > 1.$$

k = 8. Here, items 5 and 8 are not consistent. For clarity, we reprove here this special case of item 8, proved in full above.

By item 5, we may assume that m_1, m_2, m_3 are multiples of 7 and divisors of $L_8 = 2^2{\cdot}3{\cdot}5{\cdot}7$. No m is exactly 7, so the remaining possibilities for (m_1, m_2, m_3) are $(2 \cdot 7, 3 \cdot 7, 5 \cdot 7)$ and $(4 \cdot 7, 3 \cdot 7, 5 \cdot 7)$. In particular, there are not *four* m's that are multiples of 7. Item 6 now implies that there *are* two m's, say m_4 and m_5, that are multiples of $k - 2 = 6$, and one of those is not a multiple of 5 (or else $\gcd(m_4, m_5) = 30$). Thus, either $\gcd(m_4, m_3)$ or $\gcd(m_5, m_3)$ is 1.

k = 9. At least three of the m's are multiples of 8 and divisors of $L_9 = 2^3 \cdot 3 \cdot 5 \cdot 7$. Since $m_i \neq 8$ (a prime power), the three multiples of 8 are $3 \cdot 8, 5 \cdot 8, 7 \cdot 8$, and there isn't a fourth.

Since there isn't a fourth multiple of 8, there must be two other m's that are multiples of $k-2=7$, one of which is relatively prime to 5, and both are odd since $\gcd(m_i, 7\cdot 8) \le 8$. Thus there is one that is relatively prime to $5\cdot 8$, a contradiction.

$\mathbf{k=10}$. At least three of the m's are multiples of 9 and divisors of $L_{10} = 2^3\cdot 3^2\cdot 5\cdot 7$. Since $m_i \ne 9$ (a prime power) the three multiples of 9 are $(m_1, m_2, m_3) = (2^r\cdot 9, 5\cdot 9, 7\cdot 9)$ (for some $r \ge 1$), and there isn't a fourth m that is a multiple of 9. Since there isn't a fourth multiple of 9, there must be two other m's, say m_4 and m_5, that are multiples of 8, one of which is relatively prime to 5, and so a multiple of 3 since $\gcd(m_i, m_5) > 1$. Say m_4 is the multiple of $3\cdot 8$, and m_5 is not a multiple of 3 since $\gcd(m_4, m_5) < 10$. Since $\gcd(m_2, m_5) > 1$ and $\gcd(m_3, m_5) > 1$, we have $m_5 = 8\cdot 5\cdot 7$. Since $\gcd(m_1, m_4) < 10$ and $\gcd(m_4, m_5) = 8$, we have $m_4 = 3\cdot 8$. And since $m_1 \ne 9$ and $\gcd(m_1, m_4) < 10$, we find that $m_1 = 2^r\cdot 9 = 2\cdot 9$.

We have $m_1 = 2\cdot 3^2, m_2 = 5\cdot 3^2, m_3 = 7\cdot 3^2, m_4 = 2^3\cdot 3, m_5 = 2^3\cdot 5\cdot 7$. Suppose that there is another m that is a multiple of 7. Since $\gcd(m, m_1) > 1$, either $2 \mid m$ or $3 \mid m$. But then either $\gcd(m, m_5) = 14 \ge k$ or $\gcd(m, m_3) = 21 \ge k$. Likewise, no other m is a multiple of 5. Thus the other m's are divisors of $2^3\cdot 3^2$, but not multiples of 8 or 9 or prime powers: the possibilities are 4, 6, 12. No other m can be a multiple of 12 because $m_4 = 24$, so $m_6, \cdots, m_{10}$ are at most 6. Deleting m_5, we take $M = 2^3\cdot 3^2$ and

$$\sum_{i\ne 5} \frac{1}{\gcd(m_i, M)} \ge \frac{1}{18} + \frac{1}{9} + \frac{1}{9} + \frac{1}{24} + \frac{1}{8} + 5\,\frac{1}{6} > 1.$$

4.3. The Cases $\mathbf{k} \in \{\mathbf{8}, \mathbf{12}, \mathbf{14}, \mathbf{18}, \mathbf{20}, \mathbf{24}, \mathbf{30}\}$

For $k \in \{8, 12, 14, 18, 20, 24, 30\}$, the prime $k-1$ must divide at least 3 of the m's by item 5. By item 8, however, only 0 or 2 of the m's can be multiples of a prime as large as $k/2$.

4.4. The Cases $\mathbf{k} \le \mathbf{19}$

Let `G` be a (possibly empty) list of positive integers and let `potentials` be a (possibly empty) list of positive integers. Define a function `Grow[G,potentials]` that will return `True` if it is possible to augment `G` with elements of `potentials` to get a set of size k (a global variable) such that each pair of elements has gcd strictly between 1 and k, and every subset of it passes the test of Lemma 5. Otherwise it will return `False`. We give in Figure 1 a *Mathematica* program that accomplishes this.

```
GCDList[G_] := Union[Flatten[Table[GCD[G[[i]], G[[j]]],
                                   {i, 1, Length[G]-1},
                                   {j, i+1, Length[G]}    ]]];
LemmaTest[G_]:= Block[
                      {gcdlist = GCDList[G],
                       M },
                      M = LCM @@ gcdlist;
                      ( (Last[gcdlist] < k) &&
                        (First[gcdlist] > 1) &&
                        (Plus @@ (1/ (GCD[M,#]& /@ G) ) <= 1)
                      )                                    ];
Grow[G_, potentials_] :=
    If[Length[G] == k,
       {passed, counter} = {True, Length[Subsets[G, {3, k}]]};
       While[passed && counter > 0,
             passed = LemmaTest @@ Subsets[G, {3, k}, {counter}];
             counter--                                     ];
       passed,
       Or @@ Table[Grow[Append[ G, potentials[[i]] ],
                       Select[potentials,
                              ( (# >= potentials[[i]]) &&
                                LemmaTest[Join[G, {potentials[[i]], #}]]
                              )&                           ]],
                   {i, Length[potentials]}                 ]];
```

Figure 1: *Mathematica* code to look for counter-examples.

We can prove that there is no counter-example with $k \leq 19$ sequences by executing the following *Mathematica* loop

```
Or @@ Table[L = LCM @@ Range[k-1];
            candidates = Select[Divisors[L],
                                (Length[FactorInteger[#]] > 1)&];
            Grow[ {}, candidates ],
            {k, 2, 19}                                     ]
```

and observing that the output is `False`. This required slightly less than a week to run on the authors humble desktop PC.

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