AN INTRODUCTION TO CLOBBER

Michael H. Albert
Department of Computer Science, University of Otago, New Zealand
malbert@cs.otago.nz

J. P. Grossman
Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada
jpg@mathstat.dal.ca

Richard J. Nowakowski
Department of Mathematics and Statistics, Dalhousie University, Halifax, NS, Canada
rjn@mathstat.dal.ca

David Wolfe
Department of Mathematics and Computer Science, Gustavus Adolphus College, St. Peter, MN, USA
wolfe@gustavus.edu

Received: 2/27/04, Revised: 8/25/04, Accepted: 5/9/05, Published: 9/1/05

Abstract

Clobber is an all-small game usually played on a checkerboard but the rules easily generalize to play on an arbitrary graph. We show that, on a graph, determining the value of the game is NP-hard. We also examine clobber on a $1 \times n$ and a $2 \times n$ strip.

Keywords: Clobber, combinatorial game, all-small, infinitesimal, NP-hard.

1 Introduction

Clobber is a partisan game that was first developed and investigated by the authors at Games-at-Dal 2001. It was then introduced at the 2002 Dagstuhl seminar on algorithmic and Combinatorial Game Theory. In the initial position, black and white tokens are placed on the vertices of an undirected graph, at most one per vertex. A player moves by picking up one of their tokens and "clobbering" an adjacent token of the opposite color; the clobbered token is replaced by the one that was moved. The last player to move wins. In practice, the game is played on a checkerboard (grid graph), where all squares are occupied with tokens of the same color as the square. A 6×7 board suffices to highlight the features of the game. See [10] for a report of the first clobber world championship. Recently, a 10×10 board has become more popular.

Because there is no position in which one player can move and the other cannot (both players have a move if and only if there are adjacent tokens of opposite colors), clobber is an *all-small* [1, p. 229] [2, p. 101] game. In Amazons and Go, for example, a player wins by gaining territory in which the other player cannot move. In clobber, neither player can gain such an advantage; in particular, the game values are all *infinitesimals* [1, p. 36–37] [2, p. 100].

In the next section of this paper, we give a genealogy, some examples of game values that occur and leave with some conjectures about $1 \times n$ and $2 \times n$ positions. The game values for clobber, in general, do not have nice canonical forms. Atomic weights [1, Ch. 8][2, p. 217] are a way of avoiding complicated infinitesimals. In the third section, we introduce atomic weights. They help to show that the game will not have an easy solution. The addition of just one piece can change the atomic weight, in either direction, by an arbitrary amount. In Section 4, we determine the atomic weight for an arbitrary graph that has just one white token and the rest are black. This result allows us to show that clobber is NP-hard. The final section describes a position justified by its aesthetic appeal.

For background on combinatorial games please see [1], [2], or the survey articles in [12]. The notation and game values studied here are developed, in depth, in [1] and [2].

2 Genealogy

Clobber originated from work on one-dimensional peg duotaire. Peg duotaire [14] is an impartial game; the moves are the same as in peg-solitaire but there are two players and the last player to move wins. The one-dimensional version, studied in [11], is played with a row of holes that can hold pegs. Each move consists of jumping one peg over an adjacent peg and landing in an empty hole; the peg that was jumped is removed. The last player to move is the winner. A closed set of positions results from pairing up the holes and allowing at most one peg in each pair. Such positions can be represented using a shorthand notation with $\bullet$ for a peg, $\circ$ for a hole and letting $x = \circ\bullet$ and $o = \bullet\circ$. For example,

$$\circ\bullet\circ\bullet\bullet\circ\circ\bullet\circ\bullet\bullet\circ\bullet\circ \text{ becomes } xxoxxoo$$

It is not necessary to explicitly represent $\circ\circ$ since it can be shown that in these positions an empty pair of holes separates the position into disjoint sub-games. Using this shorthand notation, the legal moves are $xo \to \_x$ and $xo \to o\_$. That is 'x' can capture 'o' to the right and 'o' can capture 'x' to the left. This naturally becomes a partisan game by allowing Left to move x and Right to move o. We can generalize the game beyond one dimension by removing the directional restrictions and allowing the capture of any adjacent and opposite piece; the resulting game is clobber.

It should be noted that another partisan generalization of peg-duotaire is *konane,* although konane predates peg-duotaire by at least 200 years. This game is played on a 10×10 checkerboard, black pieces and white pieces on their own colors except for two adjacent squares. Pieces move as in peg solitaire (over an opponent's piece on to an empty square) and jumps can be chained but the piece jumping is not allowed to change direction. See [5] for the exact rules and variants. Konane was a popular game in Hawaii before 1900, and it is not well understood.

Using the taxonomy proposed by Fraenkel [7], clobber is both a *math-game* and a *play-game*. The $1 \times n$ and $2 \times n$ positions considered in this paper are primarily of theoretical interest and, for the most part, do not generate interesting play between two players. By contrast, the $m \times n$ checkerboard, with white pieces starting on white squares and black on black squares, are, currently, mathematically intractable but they are good positions for a two player game that involves a combination of skill, intuition and knowledge of combinatorial game theory.

For clobber positions, we use $\circ$ for a white token which is played by Right, and $\bullet$ for a black token which is played by Left.

3 Values, Conjectures and Atomic Weights for $1 \times n$ and $2 \times n$ positions

3.1 $1 \times n$ Clobber

We begin by evaluating some simple positions.[1]

$$\begin{aligned}
\circ &= \bullet = 0 \\
\circ\bullet &= * \\
\circ\bullet\bullet &= \{0 \mid \circ\bullet\} = \{0 \mid *\} = \uparrow \\
\circ\bullet\bullet\bullet &= \{0 \mid \circ\bullet\bullet\} = \{0 \mid \uparrow\} = 2.\!\uparrow + *
\end{aligned}$$

If X is a pattern of black, white and empty spaces we denote a string consisting of n copies of this pattern by X^n, so $(\circ\bullet^2)^3 = \circ\bullet\bullet\circ\bullet\bullet\circ\bullet\bullet$.

Lemma 3.1

$$\circ^m\bullet^n = 0 \textit{ for } m, n \geq 2\textit{; and}$$

[1] Since clobber is usually played on the grid graph, throughout this paper, nodes which are adjacent horizontally or vertically are assumed to be connected by an edge.

$$\circ\bullet^n \quad = \quad (n-1).\!\uparrow + n.*.$$

Proof: In $\circ^m\bullet^n$ with $m, n \geq 2$ the first player always loses; after the first player captures, the second player captures back to end the game. The second assertion follows by induction and the base cases were evaluated before the statement of the Lemma. Specifically,

$$\begin{aligned} \circ\bullet^n &= \left\{0 \,\middle|\, \circ\bullet^{n-1}\right\} \\ &= \{0 \mid (n-2).\!\uparrow + (n-1).*\} \\ &= (n-1).\!\uparrow + n.* \end{aligned}$$

■

We leave this section with two conjectures.

Conjecture 3.2 $(\bullet\circ)^n$ *is a first player win for* $n \neq 3$.

Conjecture 3.3 $(\bullet\bullet\circ)^n = \lfloor (n+1)/2 \rfloor.\!\uparrow$.

The first conjecture has been verified by computer up to $n = 19$ and, except for $n = 3$, the first player has few losing moves. The second conjecture has been verified up to $n = 17$.

3.2 $2 \times n$ Clobber

Consider the $2 \times n$ position $\binom{\circ}{\bullet}^n = \overbrace{\begin{matrix}\circ\circ\circ\cdots\circ\\ \bullet\bullet\bullet\cdots\bullet\end{matrix}}^{n}$

Lemma 3.4 *If* n *is even then* $\binom{\circ}{\bullet}^n = 0$.

Proof: The second player can use one of (at least) two strategies: (1) partition the board into 2×2 squares; when the first player plays a vertical move in a square the second player responds in that square with the unique horizontal response. Or, (2) play rotationally symmetric moves maintaining the invariant that a 180 degree rotation about the midpoint inverts $\circ$ and $\bullet$. ■

For n odd, we can only present conjectures.

Conjecture 3.5 *For* n *odd* $\binom{\circ}{\bullet}^n$ *is a first player win.*

This has been verified by computer up to $n = 13$. The position quickly gives rise to many sub-positions, some of which can be dealt with here.

Claim 3.6 $^{\circ}\binom{\circ}{\bullet}^{n}{}_{\bullet} = 0$.

Proof: By symmetry, it suffices to show that $\circ$ can win if $\bullet$ moves first. White's strategy is as follows: each time $\bullet$ captures up, $\circ$ captures by moving to the right leaving two smaller positions of the same form. ■

Claim 3.7 *If n is even then* ${}_{\circ}\binom{\circ}{\bullet}^{n}_{\circ} \leq 0$.

Proof: We need to show that $\circ$ has a winning strategy if $\bullet$ (Left) moves first. Pair up the n middle columns. Whichever column $\bullet$ moves in, $\circ$ responds by clobbering down in that column's pair. This splits the position into two subgames one of which is a smaller position of the same form and the other which is rotationally anti-symmetric and therefore zero. (Here we use the fact that the number of columns is even; if n were odd then Left could destroy the symmetry by moving vertically in the middle column.) ■

3.3 Atomic Weights

In all-small games, *atomic weights* [1, Ch. 8] [2, p. 217] play an important role in the determining of the outcome of a position. In [1] they are used to analyze small positions of Hackenbush and Childish Hackenbush. In clobber, all the positions are small. We begin with a brief overview of atomic weights.

Let $= \{0, *, *2, \ldots \mid 0, *, *2, \ldots\}$, i.e. the game whose options are $*n$ for all non-negative integers n. The *atomic weight* of a game, $\mathrm{aw}(g)$, is defined recursively:

Definition 3.8 *If $g = \{a, b, c, \ldots \mid d, e, f, \ldots\}$ where a, b, c, d, e, f, … have atomic weights A, B, C, D, E, …, then the atomic weight of g is*

$$G_0 = \{A - 2, B - 2, C - 2, \ldots \mid D + 2, E + 2, F + 2, \ldots\}$$

UNLESS G_0 is an integer and either $g >$ or $g <$. In these exceptional cases, if $g >$ then $\mathrm{aw}(g)$ *is the largest integer G for which G $D + 2, E + 2, F + 2, \ldots$. Similarly, if if $g <$ then* $\mathrm{aw}(g)$ *is the least integer for which G $A - 2, B - 2, C - 2, \ldots$.*

A number of useful facts concerning atomic weights are not at all obvious from the definition. (Refer to [1, Ch. 8] for proofs.)

First, atomic weights are additive:

$$\mathrm{aw}(G + H) = \mathrm{aw}(G) + \mathrm{aw}(H)$$

(A game of clobber breaks down in independent sub-games relatively quickly so this is a very useful property.)

Atomic weights are order preserving:

$$\text{If } G \geq H \text{ then } \mathrm{aw}(G) \geq \mathrm{aw}(H)$$

Sometimes the atomic weight suffices to determine the winner. If $\mathrm{aw}(G) \geq 2$ then $G > 0$; if $\mathrm{aw}(G) \leq -2$ then $G < 0$; if $2 > \mathrm{aw}(G) > -2$ then the atomic weight does not determine the winner. In the context of a , the atomic weights do determine the winner:

$$g + \;\geq 0 \text{ if and only if } \mathrm{aw}(g) \geq 1 \quad .$$

In this last statement, can be replaced by any 'remote' nimber $*n$, where remote means that no position of g, including g itself, has value $*n$. We have not encountered a clobber position with value $*4$, however, ●●○○●○●●○○ has value $*2$ and will serve as a remote nimber for many positions.

Following from Lemma 3.1, it would be nice to assume that the atomic weight cannot change by more than 1 when an extra piece is added, but this is false. For example, Lemma 3.1 shows us that $\mathrm{aw}(\bullet^n) = 0$, $\mathrm{aw}(\circ\bullet^n) = n - 1$ and, for $n \geq 2$, $\mathrm{aw}(\circ\circ\bullet^n) = 0$.

For $n > 4$, the game $\begin{smallmatrix}\bullet&\circ&\circ&\circ^n\\ \bullet&\circ&\bullet&\end{smallmatrix}$ has atomic weight $\{0 \mid \{0 \mid 4-n\}\} = {}_{n-4}$. In addition, positions with atomic weights of $1/2$, $1/4$, $1/8$, $\uparrow$, $*$ and ${}_1$ can fit on a 2×7 board (Figure 1). The game $\circ\bullet\circ\bullet^n$ has atomic weight $\{n - 3 \mid 2\}$ showing that arbitrarily hot atomic weights can occur.

3.4 Award Winning Clobber Problem

The problem in Figure 2 was composed by Adam Duffy and Garrett Kolpin, and was the winning entry at the Clobber Problem Composition contest held at the Third International Conference on Computer and Games in Edmonton in the summer of 2002.

Their solution involves calculating the atomic weight of each component, and treating the component of value $*2$ as a remote star relative to the other positions. The components have atomic weights which total to $-1 + {}_2$, and so, by properties listed in [1, p. 236], the first player can win. Black can win by first moving on the component $\begin{smallmatrix}\bullet&\circ&\bullet&\bullet&\circ&\circ&\circ\\ \circ&\circ&\circ&\circ&&&\circ\end{smallmatrix}$ to $\begin{smallmatrix}&\bullet&\bullet&\bullet&\circ&\circ&\circ\\ \circ&\circ&\circ&\circ&&&\circ\end{smallmatrix}$. Duffy and Kolpin's detailed solution can be downloaded from [4].

Position	Value	Atomic Weight
●●○○ ●●○○	$*2$	0
○●●○●○○	$\Uparrow * \mid \downarrow *$	$1/2$
○●●○○ ●○	$\Uparrow * \mid \{*, \downarrow \mid \Downarrow *\}, \{0 \mid \}$	$1/4$
○○○●○○ ○●●○●	$\Uparrow \mid 0, \{0 \parallel \Downarrow *, \ \mid 5\downarrow\}$	$1/8$
●○●○●○ ●○ ○	$\Uparrow \mid 0, \{0 \mid *\}$	$\uparrow$
●●○●●● ○○	$\Uparrow \mid \Downarrow *$	$*$
○●○●○○ ●○ ○	$\Uparrow * \mid \downarrow, \{0 \mid *, 5\downarrow\}$	1
○○●●○● ○○ ○○○○	$\downarrow, \{\Uparrow * \mid \uparrow \parallel 0, \{*, \uparrow \mid *, \downarrow\} \mid\mid\mid * \mid \downarrow\} \parallel \Downarrow * \mid 8\downarrow *$	$-2 + 2$
○○●●○● ○●○○○○○	(extremely complex)	$-2 + \frac{1}{2}$

Figure 1: A sampling of atomic weights appearing in clobber

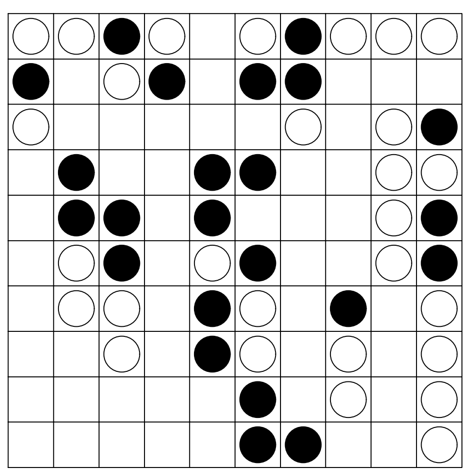


Figure 2: Problem: Black to move and win.

4 Clobber is NP-hard

In this section, we will consider graphs with only one white token. Consider a graph G and the corresponding game G_v in which all but vertex v is occupied by black tokens and v is occupied by a white token.

A *blocked v-path* in G is any maximal path $P = \{v_0 = v, v_1, v_2, \ldots, v_r\}$ starting at v. That is, every vertex adjacent to v_r is in P so that the path cannot be extended from v_r. A *blocked split* v-path, or split path for short, is a blocked path $P = \{v_0 = v, v_1, v_2, \ldots, v_r\}$ for which there exists vertices x, y such that $Q = \{v_0 = v, v_1, v_2, \ldots, v_{r-1}, x, y\}$ is also a blocked path. The *length*, $l(P)$, of a blocked path $P = \{v_0 = v, v_1, v_2, \ldots, v_r\}$ depends on whether the path is split:

$$l(P) = \begin{cases} r+1 & \text{if } P \text{ is not a split path} \\ r & \text{if } P \text{ is a split path} \end{cases}$$

It may be that the two paths in a split path only have v in common but then $l(P) = 1$: for example, ●○●●. Let

$$l(G_v) = \min\{l(P) \mid P \text{ is a blocked path or a split path of } G_v\}.$$

For example,

$$\begin{aligned} l\left(\begin{smallmatrix}\bullet & \bullet & \bullet \\ & \circ & \end{smallmatrix}\right) &= 3 \\ l\left(\begin{smallmatrix}\bullet & \bullet & \bullet & \bullet \\ & \circ & & \end{smallmatrix}\right) &= 2 \\ l\left(\begin{smallmatrix}\bullet & \bullet & \bullet & \bullet & \bullet \\ & \circ & & & \end{smallmatrix}\right) &= 3 \end{aligned}$$

Note, that if G is a row of $n \geq 2$ vertices with v at one end then Lemma 3.1 gives that $\mathrm{aw}(G) = \mathrm{aw}((n-2).{\uparrow} + n.{*}) = n-2$. Also, note that aw(●○●●) $= \mathrm{aw}(\downarrow *) = -1$, whereas aw(●●○●●●) $= 1$. At the end of a split path, i.e. at ●○●●, Right has the option of moving to 0 or to ○● $= *$, both of which have atomic weight 0. This choice between two apparently small games is actually the difference of playing to a first-player-win and a second-player-win game. This gives Right a local advantage that is reflected in the atomic weights.

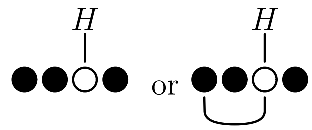
Lemma 4.1 *For a graph G with $v \in V(G)$ having at least one neighbor,*

$$\mathrm{aw}(G_v) = l(G_v) - 2$$

(In the exceptional case when v has no neighbor, $G_v = 0$.)

Proof: We prove this by induction on the number of vertices, and assume throughout that G is connected and v has a neighbor. Note that Left's options are all identically 0.

First, suppose $l(G_v) = 1$. Since v has a neighbor, the blocked path of length 1 must be split, so G_v is of the form



where H is arbitrary and possibly empty. Right has moves to $*$ and to 0. A move to a vertex in H by induction has atomic weight ≥ -1. In the preliminary atomic weight calculation of G_v,

$$G_0 = \{0 - 2 \mid 0 + 2, \ldots\}$$

where the omitted right options could be from $\{-1+2, 0+2, 1+2, 2+2, \ldots\}$ and the atomic weight of G_v is either -1, 0, or (possibly) 1, depending on how G_v compares with . But

$$G_v = \{0 \mid 0, *, \ldots\} \leq \{0 \mid 0, *\} = \downarrow *,$$

which has atomic weight -1. So $\mathrm{aw}(G_v) = -1$.

If $l(G_v) = 2$, then in the preliminary atomic weight calculation of G_v,

$$G_0 = \{0 - 2 \mid -1 + 2, \ldots\}$$

and $\mathrm{aw}(G_v)$ is either -1 or 0. We'll confirm $G_v +$ 0, and hence $\mathrm{aw}(G_v)$ must be 0: Left can win moving first on $G_v +$ by moving to 0, and then moving G_v to 0 at Left's next opportunity. (Right can't move to G_v to 0 because $l(G_v) = 2$.)

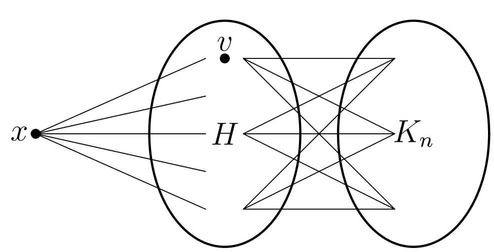
Henceforth, assume $l(G_v) > 2$. We know that $G_v = \{0 \mid (G - v)_w, w \text{ adjacent to } v\}$ There is at least one vertex b that is the next vertex on a shortest blocked path from v and $l((G - v)_b) = l(G_v) - 1$. So, the preliminary atomic weight of G_v is,

$$G_0 = \{0 - 2 \mid l((G - v)_b - 2 + 2\} = \{-2 \mid l(G_v) - 1)\},$$

and $\mathrm{aw}(G_v)$ is -1, 0, or $l(G_v) - 2$, depending on how G_v compares with . The same argument as the $l(G_v) = 2$ case shows Left can win moving first on $G_v +$. But Left can also win moving second by moving to 0 as soon as possible. Even given two consecutive moves on G_v, Right cannot move G_v to 0 since $l(G_v) > 2$. ■

Theorem 4.2 *Determining the atomic weight of a clobber graph, G_v, is NP-hard.*

Proof: We reduce from Hamiltonian path which is NP-complete [8, p. 199 (GT39)]. Let (H, v) be an instance of Hamiltonian path, where H is an undirected graph on n vertices, and $v \in V(H)$. For the clobber position, construct the following graph on $2n + 1$ vertices which has H as a subgraph:



The new graph, G, consists of a new vertex x, H, and a clique on n vertices, K_n. Every vertex of H is connected by an edge to x and to every vertex of K_n. So the new graph has $n + |E(H)| + n^2 + \binom{n}{2}$ edges.

The reader can confirm that G_v has no blocked paths of length n and that any blocked path of length $n+1$ in G_v must visit all vertices of H, and then end at x. Further, G_v has no split paths of length n. Hence, the atomic weight of G_v is $n-1$ if and only if H has a Hamiltonian path originating at v. ■

Corollary 4.3 *Determining who wins from a clobber position is NP-hard.*

Proof: Define G_v as in proof of the last Theorem and let

$$C = \circ\overbrace{\bullet\bullet\bullet\cdots\bullet}^{n}.$$

If H has a Hamilton cycle, then $G_v \leq C$, for the right options from positions in C are a subset of the right options from positions in G_v. If H has no Hamilton path, then $G_v \; C$, since $\mathrm{aw}(G_v) \geq n$ and $\mathrm{aw}(C) = n-1$. Hence, White wins moving second on $G_v - C$ if and only if H has a Hamilton path. ■

We leave open whether clobber is NP-hard when played on a grid graph.

5 For the Clobber Museum

The following 3×7 rectangle is fascinating:

$$G = \begin{array}{|c|c|c|c|c|c|c|}\hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \circ & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \end{array}$$

In particular, a remarkably large number of paths which the white stone can take through the rectangle are canonical. Figure 3 includes G as one component, along with a second component, call it H, which is G's negative. Furthermore, H has no dominated or reversible options; it is in canonical form.

Since the sum of the two components has value zero, the second player wins. If Black moves first (typically on H), White can find his winning response on G by observing the

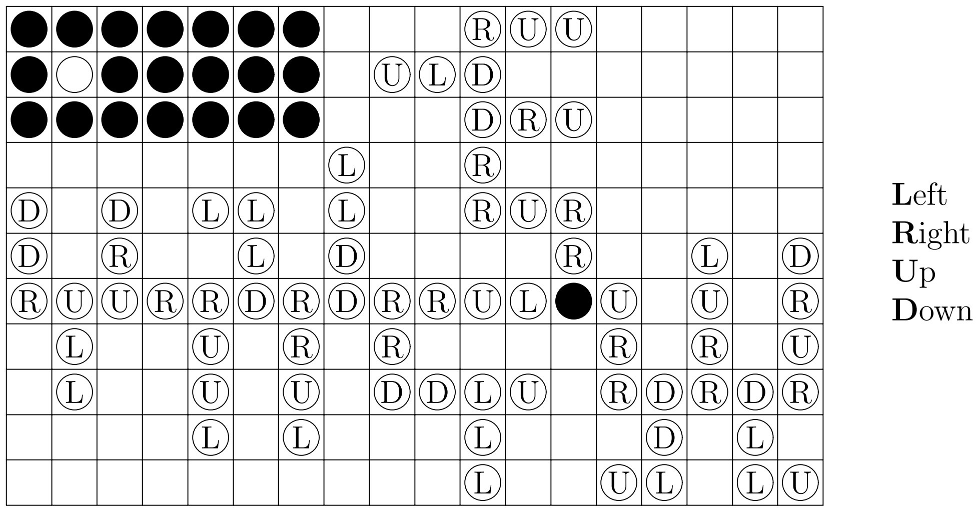


Figure 3: A position of value zero. The markings provide guidance on how to win moving second.

direction (Up, Down, Left, or Right) indicated on the stone which Black captured. Except when the 3×7 component is symmetric, all winning responses are unique.

By Lemma 4.1, the atomic weight of the 3×7 component is 6; to achieve this, White's moves are U-R-R-D-D-L-U.

The position has another curious property. Augment H by lengthening or shortening any of its tendrils by one to get a new position H'. In that case, $G + H'$ is incomparable with 0. Black wins moving first by making a bee-line towards the augmented tendril until either (1) White fails to follow the markings on the stones, or (2) Black gets near the end of a tendril. In either exceptional case, with practice the reader can confirm that Black can win.

Similarly, White can win moving first by moving on the 3×7 component, following the letters on the marked stones along the augmented tendril to coax Black along.

Lastly, lengthening any tendril by two white stones yields a position in which White wins moving first or second.

References

[1] Elwyn R. Berlekamp, John H. Conway, and Richard K. Guy. *Winning Ways.* A K Peters, Ltd., Natick, Massachusetts, 2nd edition, 2001. First edition published in 1982 by Academic Press.

[2] John H. Conway. *On Numbers and Games.* A K Peters, Ltd., Natick, Massachusetts,

2nd edition, 2001. First edition published in 1976 by Academic Press.

[3] E. D. Demaine, M. L. Demaine, and R. Fleischer. Solitaire clobber. *Theoretical Computer Science*, 313(3):325–338, February 2004.

[4] Adam Duffy and Garrett Kolpin. By hook or by crook. `http://www.gustavus.edu/~wolfe/games/clobber`.

[5] Michael D. Ernst. Playing konane mathematically: A combinatorial game-theoretic analysis. *UMAP Journal*, 16(2):95–121, Spring 1995.

[6] R. Fleischer and R. J. Nowakowski, editors. *Algorithmic Combinatorial Game Theory, Dagstuhl, Germany, February 2002*, volume 313(3):417–425 of *Theoretical Computer Science*, February 2004.

[7] Aviezri Fraenkel. Complexity, appeal and challenges of combinatorial games. *Theoretical Computer Science*, 313(3):393–415, February 2004.

[8] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman and Company, 1979.

[9] J. P. Grossman. Periodicity in one-dimensional peg duotaire. *Theoretical Computer Science*, 313(3):417–425, February 2004.

[10] J. P. Grossman. Report on the first international clobber tournament. *Theoretical Computer Science*, 313(3):533–537, February 2004.

[11] Christopher Moore and David Eppstein. 1-dimensional peg solitaire and duotaire. In Richard Nowakowski, editor, *More Games of No Chance*, pages 341–350. Cambridge University Press, MSRI Publications 42, 2002.

[12] Richard Nowakowski, editor. *Games of No Chance: Combinatorial Games at MSRI, 1994*. Cambridge University Press, MSRI Publications 29, 1996.

[13] Ivars Peterson. Getting clobbered. *Science News*, 161(17), April 27 2002.

[14] B. Ravikumar. Peg-solitaire, string rewriting systems and finite automata. *Proceedings of the 8th International Symposium on Algorithms and Computation*, pages 233–242, 1997.