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# BETA-CONJUGATES OF REAL ALGEBRAIC NUMBERS AS PUISEUX EXPANSIONS

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**Abstract**

The beta-conjugates of a base of numeration $\beta > 1$, $\beta$ being a Parry number, were introduced by Boyd, in the context of the Rényi-Parry dynamics of numeration system and the beta-transformation. These beta-conjugates are canonically associated with $\beta$. Let $\beta > 1$ be a real algebraic number. A more general definition of the beta-conjugates of $\beta$ is introduced in terms of the Parry Upper function $f_\beta(z)$ of the beta-transformation. We introduce the concept of a germ of curve at $(0, 1/\beta) \in \mathbb{C}^2$ associated with $f_\beta(z)$ and the reciprocal of the minimal polynomial of $\beta$. This germ is decomposed into irreducible elements according to the theory of Puiseux, gathered into conjugacy classes. The beta-conjugates of $\beta$, in terms of the Puiseux expansions, are given a new equivalent definition in this new context. If $\beta$ is a Parry number the (Artin-Mazur) dynamical zeta function $\zeta_\beta(z)$ of the beta-transformation, simply related to $f_\beta(z)$, is expressed as a product formula, under some assumptions, a sort of analog to the Euler product of the Riemann zeta function, and the factorization of the Parry polynomial of $\beta$ is deduced from the germ.

## 1. Introduction

For $\beta > 1$ a Parry number, Boyd [9] introduced the notion of the beta-conjugates of $\beta$ in the context of the Rényi - Parry numeration system [25] [21] [7] [15]. As he has shown it in numerous examples, the investigation of beta-conjugates is an important question. These beta-conjugates, up till now defined for Parry numbers, are canonically associated to $\beta$ and to the dynamics of the beta-transformation. Our aim is to show that their definition can be given in a larger context, namely for any algebraic number $\beta > 1$, and that the theory of Puiseux provides a geometric origin to the beta-conjugates of $\beta$; for doing it, once $\beta$ is given by its minimal polynomial, we first put into evidence that a germ of curve "at $1/\beta$" does exist and develop new tools deduced from the canonical decomposition of this germ in order to express the beta-conjugates of $\beta$ in terms of the Puiseux expansions [24] [11] of the germ.

Though the existence of this germ of curve was discovered by the author some years ago, the present note is the first account on it and its potential applications. It establishes a deep relation between the theory of singularities of curves in Algebraic Geometry and the dynamical system of numeration $([0,1], T_\beta)$ where $\beta > 1$ is an algebraic number and $T_\beta$ is the beta-transformation. The existence of this germ of curve brings new tools to the Rényi-Parry numeration system, namely the Puiseux series associated to the germ, and defines new directions of research for old questions. For instance, if $(\beta_i)$ is a sequence of Salem numbers which converges to a real number $\beta$, then it is known [5] that $\beta$ is a Pisot or a Salem number, but how is distributed the collection of the beta-conjugates and the Galois conjugates of $\beta_i$, with $i$ large enough, with respect to that of the limit $\beta$ ? This question is merely a generalization of the classical question of how is distributed the collection of the Galois conjugates of $\beta_i$ with respect to that of $\beta$ ? Why should we add the beta-conjugates ? Because a new phenomenon appears which generally does not exist with only the Galois conjugates: under some assumptions the collections of Galois- and beta- conjugates may have equidistribution limit properties on the unit circle (§3.6 in [33]) if the two collections of conjugates are simultaneously considered. Both collections of conjugates are expected to play a role in limit and dynamical properties of convergent sequences of real algebraic numbers $> 1$ in general. A basic question is then to understand the role and the relative density of the beta-conjugates in this possible equidistribution process, in particular if the limit $\beta$ is an integer $\geq 2$ or is equal to 1 (context of the Conjecture of Lehmer).

Conversely the curve canonically associated with this numeration dynamical system is of interest for itself (critical points, monodromy, ...). It will be studied elsewhere.

In this first contribution we obtain useful expressions for the beta-conjugates as Puiseux expansions of $\beta$ and of the minimal polynomial of $\beta$, towards this goal.

As usual now we use the new terminology, which is in honor of W. Parry. The old terminology used by W. Parry himself in [21] transforms as follows: we now call *Parry number* a $\beta$-number [21], and *Parry polynomial* of a Parry number $\beta$ the characteristic polynomial [21] of the $\beta$-number $\beta$. As previously a *simple* Parry number $\beta$ is a Parry number $\beta$ for which the Rényi $\beta$-expansion $d_\beta(1)$ of unity is finite (i.e. ends in infinitely many zeros). The exact definitions are given in Section 3.

If $\beta$ is a Parry number, the roots of the Parry polynomial of $\beta$, denoted by $\beta^{(i)}$, are called the conjugates of $\beta$. A conjugate of $\beta$ is either a Galois conjugate of $\beta$ or a beta-conjugate, if the collection of beta-conjugates of $\beta$ is not empty.

Let $\beta > 1$ be a real number and $d_\beta(1) = 0.t_1t_2t_3\ldots$ be the Rényi $\beta$-expansion of 1. Since this Rényi $\beta$-expansion of 1 controls the language in base $\beta$ [20], the properties of the analytic function constructed from it, called Parry Upper function at $\beta$, defined by $f_\beta(z) := -1 + \sum_{i\geq 1} t_i z^i$, is of particular importance.

Ito and Takahashi [17] have shown that the Parry Upper function at a Parry number $\beta$, of the complex variable $z$, is related to the (Artin-Mazur) dynamical zeta function

$$\zeta_\beta(z) \;:=\; \exp\left(\sum_{i\geq 1} \frac{\#\{x \in [0,1] \mid T_\beta^n(x) = x\}}{n} z^n\right) \tag{1}$$

of the beta-transformation $T_\beta$ (Artin and Mazur [2], Boyd [9], Flatto, Lagarias and Poonen [14], Verger-Gaugry [32] [33]). Namely, if $\beta$ is a nonsimple Parry number, with $d_\beta(1) = 0.t_1 t_2 \dots t_m (t_{m+1} \dots t_{m+p+1})^\omega$ (where $(\,)^\omega$ means infinitely repeated),

$$f_\beta(z) \;=\; -\frac{1}{\zeta_\beta(z)} \;=\; -\frac{P^*_{\beta,P}(z)}{1 - z^{p+1}} \tag{2}$$

where $P^*_{\beta,P}(X) = (-1)^{d_P}\left(\prod_{i=1}^{d_P} \beta^{(i)}\right) \times \prod_{i=1}^{d_P}(X - \frac{1}{\beta^{(i)}}) = X^{d_P} P_{\beta,P}(1/X)$ is the reciprocal of the Parry polynomial $P_{\beta,P}(X)$ of $\beta$, of degree $d_P = m+p+1$ ($m$ is the preperiod length and $p+1$ is the period length in $d_\beta(1)$, if $\beta$ is a nonsimple Parry number, with the convention $p+1 = 0$ for a finite Rényi $\beta$-expansion of unity (for $\beta$ a simple Parry number), with the convention $m = 0$ if $d_\beta(1)$ is a purely periodic expansion [33]); if $\beta$ is a simple Parry number, with $d_\beta(1) = 0.t_1 t_2 \dots t_m$, then

$$f_\beta(z) \;=\; -\frac{1 - z^m}{\zeta_\beta(z)} \;=\; -P^*_{\beta,P}(z). \tag{3}$$

The zeros of $f_\beta(z)$ are the poles of $\zeta_\beta(z)$. The set of zeros of $f_\beta(z)$ is the set $(1/\beta^{(i)})_i$ of the reciprocals of the conjugates $(\beta^{(i)})_i$ of $\beta$. The geometry of the conjugates $(\beta^{(i)})_i$ of $\beta$ was carefully studied by Solomyak [29] [32]: these conjugates all lie in Solomyak's fractal $\Omega$, a compact connected subset of the closed disc $\overline{D}(0, \frac{1+\sqrt{5}}{2})$ in the complex plane (Figure 1), having a cusp at $z = 1$, a spike on the negative real axis, symmetrical with respect to the real line [29] [33].

If $\beta > 1$ is an algebraic number but not a Parry number, some relations are expected between $f_\beta(z)$ and $\zeta_\beta(z)$, though not yet determined. Indeed, on one hand, $f_\beta(z)$ is an analytic function on the open unit disc which admits $|z| = 1$ as natural boundary by Szegő-Carlson-Polyá's Theorem [12] [33]; $f_\beta(z)$ admits $1/\beta$ as zero of multiplicity one, which is its only zero in the interval $(0,1)$. On the other hand $\zeta_\beta(z)$ is an analytic function defined on the open unit disc $D(0, 1/\beta)$, which admits a nonzero meromorphic continuation on $D(0,1)$, by [16] [22] [26], or by Baladi-Keller's Theorem 2 in [3]. Whether the zeros of $f_\beta(z)$ correspond to poles of $\zeta_\beta(z)$ is unknown. The behaviour of the dynamical zeta function $\zeta_\beta(z)$ on the unit circle remains unknown, i.e. we do not know whether $|z| = 1$ is a natural boundary for $\zeta_\beta(z)$ or not. But the multiplicity of the pole $1/\beta$ of $\zeta_\beta(z)$ is known to be one [16] [22] [26]. For $\beta > 1$ an algebraic number, as a consequence of Theorem 1 in [3], the coefficients in (1) obey the following asymptotics of growth (Pollicott, §5.2 in [23]) : for any $\delta > 0$ there exist an integer $M > 0$ and constants $(i)$ $\lambda_{1,\beta}, \lambda_{2,\beta}, \ldots, \lambda_{M,\beta}$, with $|\lambda_{i,\beta}| > 1 + \delta$ $(i = 1, \ldots, M)$, and $(ii)$ $C_{1,\beta}, C_{2,\beta}, \ldots, C_{M,\beta} \in \mathbb{C}$, such that

$$\#\{x \in [0,1] \mid T_\beta^n(x) = x\} \;=\; \sum_{i=1}^{M} C_{i,\beta}\lambda_{i,\beta}^n + O((1+\delta)^n). \tag{4}$$

In the case where $\beta > 1$ is a Parry number, $\zeta_\beta(z)$ is a rational fraction and, from (2) and (3), (4) transforms into the following exact formula (after Pollicott, §1 in [23]):

$$\#\{x \in [0,1] \mid T_\beta^n(x) = x\} \;=\; \sum_{i=1}^{k} (\rho_i)^n - \sum_{i=1}^{d_P} (\beta^{(i)})^n, \tag{5}$$

where $(\rho_i)_i$ is the collection of $k$-th roots of unity, $(k, d_P) = (p+1, m+p+1)$ if $\beta$ is nonsimple with $d_\beta(1) = 0.t_1t_2\ldots t_m(t_{m+1}\ldots t_{m+p+1})^\omega$ and $(k, d_P) = (m, m)$ if $\beta$ is simple with $d_\beta(1)$ of length $m$ (i.e. $d_\beta(1) = 0.t_1t_2\ldots t_m$). Moreover, $\beta$ is a Perron number since it is a Parry number (Lind, [20]): hence the asymptotic growth of (5) is dictated by the geometry and the moduli of the beta-conjugates of $\beta$, all being algebraic integers lying in Solomyak's fractal $\Omega$, of modulus less than or equal to $(1+\sqrt{5})/2$, and by the geometry and the moduli of the Galois conjugates of $\beta$, all being less than $\beta$, by definition.

Our objective consists in showing that a germ of curve exists in a neighbourhood of the point $(0, 1/\beta)$ in $\mathbb{C}^2$ (this point being the origin of this germ) each time $\beta > 1$ is a real algebraic number, that is, roughly speaking, a germ of curve located at the reciprocal $1/\beta$ of the base of numeration $\beta$. The construction of this germ of curve comes from a (unique) writting of the one-variable analytic function $f_\beta(z)$ as a (unique) two-variable analytic function parametrized by $P_\beta^*(z)$ and $z - 1/\beta$, where $P_\beta^*(X) = X^{\deg\beta}P_\beta(1/X)$ is the reciprocal of the minimal polynomial $P_\beta(X)$ of $\beta$:

$$f_\beta(z) \;=\; G\Big(P_\beta^*(z), z - 1/\beta\Big), \tag{6}$$

where $G = G_\beta(U, Z) \in \mathbb{C}[[U]][Z]$, $\deg_Z(G_\beta(U, Z)) <$ deg $\beta$, is convergent, with coefficients in $\mathbb{C}$, possibly in some cases in the algebraic number field $\mathbb{K}_\beta := \mathbb{Q}(\beta)$, or in a finite algebraic extension of $\mathbb{K}_\beta$.

The existence of this germ of curve arises from the fact that $\beta > 1$ is a real number which is an algebraic number, since it is constructed from the imposed parametrization $(P_\beta^*(z), z - 1/\beta)$, which makes use of the minimal polynomial of $\beta$. This parametrization of $G_\beta(U, Z)$ leads to the identity (6).

Applying the theory of Puiseux [11] [13] to (6) provides a canonical decomposition of this germ into irreducible curves, conjugacy classes, as stated in Theorem 8. This decomposition brings to light several new features of the Parry Upper function $f_\beta(z)$:

(i) a new definition of the beta-conjugates of $\beta$ in terms of the Puiseux expansions of the germ (Definition 9),

(ii) the explicit relations between the field of coefficients of the Puiseux series of the germ $G_\beta$, and the beta-conjugates,

(iii) a product formula, as given by (26); in particular, if $\beta$ is a Parry number, from (2) and (3), this product gives an analog of the Euler product of the Riemann zeta function for the dynamical zeta function $\zeta_\beta(z)$, where the product is taken over the different rational conjugacy classes of the germ (as given by (33)).

In addition to the usual Galois conjugation relating the roots of the minimal polynomial of $\beta$, a new conjugation relation, called “Puiseux-conjugation”, among the beta-conjugates, is defined.

The reader accustomed to numeration systems and to the theory of Puiseux for germs of curves can skip Section 3 and Section 4 to proceed directly to beta-conjugates in Section 5.

## 2. Origin of the Work

The present note finds its origin in [27], for the parametrization by $(P^*_\beta(z), z - \frac{1}{\beta})$, and in the two articles [8] [4], for the idea of developping a two-variable analytic function canonically associated with the beta-transformation and the minimal polynomial of the base of numeration $\beta$. Let us recall them.

In Theorem IV in [27], for constructing convergent families of Salem numbers $(\tau_m)_m$ for which the limit is a (nonquadratic) Pisot number $\theta$, Salem introduces polynomials of the following type

$$Q_m(z) = z^m P_\theta(z) + P^*_\theta(z) \quad \text{or} \quad Q_m(z) = (z^m P_\theta(z) - P^*_\theta(z))\,/(z-1) \qquad (7)$$

where $Q_m(\tau_m) = 0$ and $P_\theta(X)$ is the minimal polynomial of the limit $\theta$. We may consider $Q_m(z)$ in (7), in one or the other form, as parametrized by the couple $(P^*_\theta(z), z)$ (ordered pair). This parametrization, and its consequences, were developped and extended by Boyd [8] to a more general form, by adding ingeniously and in a “profitable” way a second variable $t$, as follows

$$Q(z,t) = z^n P_\theta(z) \pm\, t\, z^k\, P^*_\theta(z)$$

with $n, k$ integers. The advantage of introducing a second variable $t$, as “continuous parameter”, lies in the fact that an algebraic curve $z = Z(t)$ is associated to $Q(z,t) = 0$, with a finite number of branches and multiple points [10]. Boyd [8] shows that the existence of this curve gives a deep insight into the geometry of the roots of $Q(z,t) = 0$, for some values of $t$, in particular those roots on the unit circle. Using these polynomials Bertin and Boyd [4] explore the interlacing of the Galois conjugates of Salem numbers with the roots of associated polynomials (Theorem A and Theorem B in [4]).

### 3. Functions of the Rényi-Parry Numeration System in Base $\beta > 1$

A Salem number is an algebraic integer greater than 1 for which all the Galois conjugates lie in the closed unit disc, with at least one conjugate on the unit circle; the degree of a Salem number is even, greater than 4, and its minimal polynomial is reciprocal (a Salem number is Galois-conjugated to its inverse) [5]. A Perron number is either 1 or an algebraic integer $\beta > 1$ such that all its Galois conjugates $\beta^{(i)}$ satisfy: $|\beta^{(i)}| < \beta$ for $i = 1, 2, \ldots, \deg(\beta) - 1$, if the degree of $\beta$ is denoted by $\deg(\beta)$ (with $\beta^{(0)} = \beta$). A Pisot number $\beta$ is a Perron number $\neq 1$ which has the property: $|\beta^{(i)}| < 1$ for $i = 1, 2, \ldots, \deg(\beta) - 1$ (with $\beta^{(0)} = \beta$).

Let $\beta > 1$ be a real number and define the beta-transformation $T_\beta : [0,1] \to [0,1], x \to \{\beta x\}$ ($\lceil x \rceil$, resp. $\lfloor x \rfloor$, denotes the closest integer to the real number $x$, $\geq x$, resp. $\leq x$, and $\{x\}$ its fractional part). Denote $T_\beta^0 =$Id, $T_\beta^j = T_\beta(T_\beta^{j-1})$, and $t_j = t_j(\beta) := \lfloor \beta T_\beta^{j-1}(1) \rfloor$, $j \geq 1$ (the dependency of each $t_j$ to $\beta$ will not be indicated in the sequel). The digits $t_j$ belong to the finite alphabet $\mathcal{A}_\beta = \{0, 1, \ldots, \lceil \beta - 1 \rceil\}$. The Rényi $\beta$-expansion of unity is denoted by

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots \qquad \text{and corresponds to} \qquad 1 = \sum_{j \geq 1} t_j \beta^{-j} \tag{8}$$

obtained by the Greedy algorithm applied to 1 by the successive negative powers of $\beta$. The set of successive iterates of 1 under $T_\beta$, hence the sequence $(t_i)_{i \geq 1}$, has the important property that it controls the admissibility of finite and infinite words written in base $\beta$ over the alphabet $\mathcal{A}_\beta$, that is the language in base $\beta$, by the so-called Conditions of Parry [15] [20] [33].

A Parry number $\beta$ is a real number $> 1$ for which the sequence of digits $(t_i)_{i \geq 1}$ in the Rényi $\beta$-expansion of unity $d_\beta(1) = 0.t_1 t_2 t_3 \ldots$ either ends in infinitely many zeros, in which case $d_\beta(1)$ is said to be finite and $\beta$ is said a simple Parry number, or is eventually periodic. In the second case, if the preperiod length is zero, $d_\beta(1)$ is said to be purely periodic. The set of Parry numbers is denoted by $\mathbb{P}_P$.

Let $\overline{\mathbb{Q}}$ be the set of algebraic numbers. Denote by $\mathbb{T}$, resp. $\mathbb{S}$, resp. $\mathbb{P}$, the set of Salem numbers, resp. Pisot numbers, resp. Perron numbers. After Bertrand-Mathis [6], Schmidt [28], Lind [20], the following inclusions hold

$$\mathbb{S} \subset \mathbb{P}_P \subset \mathbb{P} \subset \overline{\mathbb{Q}}.$$

The question of the dichotomy $\mathbb{P} = \mathbb{P}_P \cup (\mathbb{P} \setminus \mathbb{P}_P)$ is an important open question, which amounts to finding a method for discrimating when a Perron number $> 1$ is a Parry number or not. In particular, for Salem numbers, though conjectured to be nonempty with a positive density [10], the set $\mathbb{T} \setminus \mathbb{P}_P$ is not charaterized yet. For now, it is a fact that all the small Salem numbers, for instance those given by Lehmer in [19], and many others known, are Parry numbers [8] [10]. The set of simple Parry numbers contains $\mathbb{N} \setminus \{0, 1\}$ and is dense in $(1, +\infty)$ [21].

Let $\beta$ be a Parry number, with $d_\beta(1) = 0.t_1 t_2 \dots t_m (t_{m+1} \dots t_{m+p+1})^\omega$. If $m \neq 0$, the integer $m$ is the preperiod length of $d_\beta(1)$; if $p+1 \geq 1$, the period length of $d_\beta(1)$ is $p+1$. The iterates of 1 under $T_\beta$ are polynomials: $T_\beta^n(1) = \beta^n - t_1\beta^{n-1} - t_2\beta^{n-2} \dots - t_n$ (by induction). This observation allows Boyd in [9] to define uniquely the Parry polynomial of $\beta$. Indeed, writting $r_n(X) = X^n - t_1 X^{n-1} - t_2 X^{n-2} \dots - t_n$, we have $r_n(\beta) = T_\beta^n(1)$ and $\beta$ satisfies the polynomial equation $P_{\beta,P}(\beta) = 0$, where

$$P_{\beta,P}(X) := \begin{cases} r_{m+p+1}(X) - r_m(X) & \text{if } m > 0 \ \ (p+1 \geq 1), \\ r_{p+1}(X) - 1 & \text{if } m = 0 \ \ (p+1 \geq 1, \text{“purely periodic”}), \\ r_m(X) & \text{if } m \geq 1 \ \ (p+1 = 0, \text{“simple”}). \end{cases} \tag{9}$$

The Parry polynomial $P_{\beta,P}(X)$ of the Parry number $\beta$, monic, of degree $d_P = m+p+1$, multiple of the minimal polynomial $P_\beta(X)$ of $\beta$, can also be defined from the rational fraction $\zeta_\beta(X)$: its reciprocal $P^*_{\beta,P}(z)$, of the complex variable $z$, is the denominator of the meromorphic function $\zeta_\beta(z)$, given in both cases by (2) and (3) (“simple” case). Boyd [9] defines the beta-conjugates of $\beta$ as being the roots of $P_{\beta,P}(X)$, canonically attached to $\beta$, which are not the Galois conjugates of $\beta$. Beta-conjugates are algebraic integers.

For any real number $\beta > 1$, from the sequence $(t_i = t_i(\beta))_{i\geq 1}$ we form the Parry Upper function $f_\beta(z) := -1 + \sum_{i\geq 1} t_i z^i$ at $\beta$, of the complex variable $z$. The terminology “Parry Upper” comes from the fact that $(t_i)_{i\geq 1}$ gives the upper bound for admissible words in base $\beta$, where being lexicographically smaller than this upper bound, with all its shifts, means satisfying the Conditions of Parry for admissibility [15] [20] [33].

When $\beta$ is a Parry number, the inverses $\xi^{-1}$ of the zeros $\xi$ of the analytic function $f_\beta(z)$ are exactly the roots of the Parry polynomial $P_{\beta,P}(X)$ of $\beta$ (from (2), (3); [33]). In particular we have $f_\beta(1/\beta) = 0$ by (8). The multiplicity of the root $1/\beta$ in $f_\beta(z)$ is one by the fact that $f'_\beta(1/\beta) = \sum_{i\geq 1} i t_i \beta^{i-1} > 0$. Hence in the factorization of $P_{\beta,P}(X)$ the multiplicity of the minimal polynomial $P_\beta(X)$ of $\beta$ is one. But the determination of the multiplicity of a beta-conjugate of $\beta$ and of the factorization of the Parry polynomial of $\beta$ is an open problem [9] [33]. We give a partial solution to this problem by showing how this factorization can be deduced from the germ of curve “at $1/\beta$” and the theory of Puiseux.

Though the degree $d_P$ of the Parry polynomial $P_{\beta,P}(X)$ of a Parry number $\beta$ be somehow an obscure function of $\beta$, the Parry polynomial $P_{\beta,P}(X)$, say $= \sum_{i=0}^{d_P} a_i X^i$, has the big advantage, as compared to the minimal polynomial $P_\beta(X)$ of $\beta$, to exhibit a naive height $\mathrm{H}(P_{\beta,P}) = \max_{i=0,1,\dots,d_P} |a_i|$ in $\{\lfloor\beta\rfloor, \lceil\beta\rceil\}$ [33]. This control of the height by the base of numeration $\beta$ has an important consequence: given a convergent family of Parry numbers $(\beta_j)_j$, an Equidistribution Limit Theorem for the conjugates $(\beta_j^{(i)})_{i,j}$ holds with a limit measure which is the Haar measure on the unit circle [33], under some assumptions. Solomyak's fractal $\Omega$ is densely occupied by all the conjugates of all the Parry numbers [29], with a major concentration of conjugates occuring in a neighbourhood of the unit circle.

Beta-conjugates are then equivalently defined either as roots of $P_{\beta,P}(X)$, as inverses of zeros of $f_\beta(z)$, or as inverses of poles of the dynamical zeta function $\zeta_\beta(z)$. The three equivalent definitions arise from the relations (2) and (3) ("simple" case), deduced from [17] [14].

The Galois- and beta- conjugates $\beta^{(i)}$ of a Parry number $\beta$ all lie in Solomyak's fractal [29], represented in Figure 1. The left extremity of the spike on the real negative axis is $-(1+\sqrt{5})/2$ and the general bound $|\beta^{(i)}| \leq (1+\sqrt{5})/2$ holds for all $i$ and all Parry numbers $\beta$; this upper bound was also found by Flatto, Lagarias and Poonen [14].

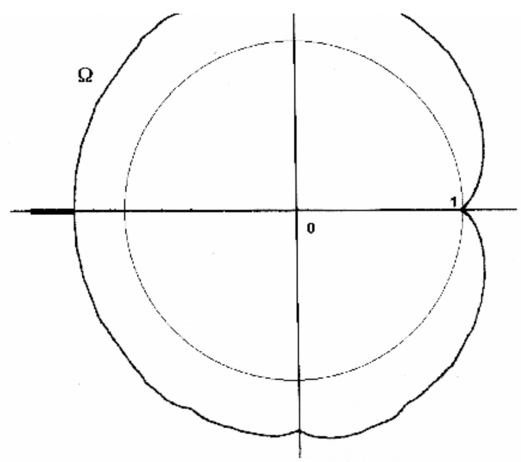


Figure 1: Solomyak's fractal $\Omega$.

Let $\beta$ be a Parry number. The three following assertions are obviously equivalent: (i) $\beta$ has no beta-conjugate, (ii) the Parry polynomial of $\beta$ is irreducible, (iii) the Parry polynomial of $\beta$ is equal to the minimal polynomial of $\beta$. For some families of Parry numbers [18] [33] it is possible to deduce the irreducibility of their Parry polynomials.

By Szegő-Carlson-Polyá Theorem [12], the Parry Upper function $f_\beta(z)$ is a rational fraction if and only if $\beta$ is a Parry number [33]. If $\beta > 1$ is an algebraic number, but not a Parry number, $f_\beta(z)$ is an analytic function on the open unit disc with the unit circle as natural boundary.

For $\beta > 1$ any algebraic number, except a Parry number, we define a beta-conjugate of $\beta$ as the inverse of a zero of the function $f_\beta(z)$, if it exists. A priori, it may happen that $f_\beta(z)$ admits the only zero $1/\beta$ in its domain of definition $D(0,1)$, with $|z| = 1$ as natural boundary. The problem of the existence of zeros of $f_\beta(z)$ in $D(0,1)$ is linked to the gappiness (the terminology *gappiness* was introduced in [31] as a notion which is much weaker than that of *lacunarity*; indeed *lacunarity* is classically associated to Hadamard gaps) of the sequence $(t_i)$ and its Diophantine approximation properties [31] [1]; this gappiness cannot be too large at infinity and the Ostrowski "quotients of the gaps" are dominated by $\log \mathrm{M}(\beta)/\log\beta$, where $\mathrm{M}(\beta)$ is the Mahler measure of $\beta$.

By a Theorem of Fuchs [32], if $f_\beta(z)$ is such that $(t_i)$ admits Hadamard gaps, then the number of zeros of $f_\beta(z)$ is infinite in $D(0,1)$. This occurence, of having Hadamard gaps, is conjectured to be true for infinitely many transcendental numbers $\beta > 1$ but to be impossible as soon as $\beta > 1$ is an algebraic number. If $\beta > 1$ is an algebraic number, the number of zeros of $f_\beta(z)$ in $D(0,1)$, i.e. the number of beta-conjugates of $\beta$ of modulus $> 1$, is conjectured to be finite. This finiteness property of the number of beta-conjugates would be in agreement with the existence of an integer $M \geq 1$ in (4), in the context of the dynamical zeta function.

## 4. Fractionary Power Series and Puiseux Expansions for Germs of Curves

In the sequel, we will follow Casas-Alvero [11], Duval [13], Walker [34], Walsh [35] and restrict ourselves to what is needed for the application of the theory of Puiseux to beta-conjugates of algebraic numbers $> 1$, to fix notations. The terminology "fractionary" is taken from [11]. Let $k$ be a (commutative) field of characteristic zero and let $G(X,Y) \in k[[X,Y]]$. We consider the formal equation

$$G(X,Y) = 0$$

and are interested in solving it for $Y$, that is we want to find some sort of series in $X$, say $Y(X)$, with coefficients in $k$, such that

$$G(X, Y(X)) = 0, \tag{10}$$

$G(X,Y(X))$ being the series in $X$ obtained by substituting $Y(X)$ for $Y$ in $G$. The series $Y(X)$ is called a $Y$-root of $G$. When $k = \mathbb{C}$, this general problem was considered by Newton. In the following we will consider $k = \mathbb{C}$ and will consider rationality questions over smaller fields $k$ in Section 6.

For solving (10), we need to deal with series in fractionary powers of $X$. First, let us define the field of fractionary power series over $\mathbb{C}$. Denote $\mathbb{C}((X))$ the field of the formal Laurent series

$$\sum_{i=d}^{\infty} a_i X^i, \qquad d \in \mathbb{Z}, a_i \in \mathbb{C}.$$

An element of $\mathbb{C}((X^{1/n}))$ has the form

$$s = \sum_{i \geq r} a_i X^{i/n}.$$

The field of fractionary power series is denoted by $\mathbb{C}\ll X \gg$ and by definition is the direct limit of the system

$$\left\{ \mathbb{C}((X^{1/n})), \iota_{n,n'} \right\},$$

where, for $n$ dividing $n'$ (with $n' = dn$),

$$\iota_{n,n'} : \mathbb{C}((X^{1/n})) \to \mathbb{C}((X^{1/n'})), \quad \sum a_i X^{i/n} \ \to \ \sum a_i X^{di/dn}.$$

A Puiseux series is by definition a fractionary power series

$$s = \sum_{i \geq r} a_i X^{i/n}$$

for which the order in $X$

$$o_X(s) := \frac{\min\{i \mid a_i \neq 0\}}{n}$$

is (strictly) positive. A natural representant of its class in the direct limit is such that $n$ and $\gcd\{i \mid a_i \neq 0\}$ have no common factor; then $n$ is called the ramification index (or polydromy order) of $s$, denoted by $\nu(s)$.

If $s \in \mathbb{C}((X^{1/n}))$ is a Puiseux series, with $n = \nu(s)$ its ramification index, the series $\sigma_\epsilon(s), \epsilon^n = 1$, will be called the conjugates of $s$, where

$$\sigma_\epsilon(s) = \sum_{i \geq r} \epsilon^i a_i X^{i/n}.$$

The set of all (distinct) conjugates of $s$ is called the conjugacy class of $s$. The number of different conjugates of $s$ is $\nu(s)$.

Let us recall the Newton polygon of a two-variable formal series. Let

$$G = G(X, Y) = \sum_{i>0, j>0} A_{i,j} X^i Y^j \qquad \in \mathbb{C}[[X, Y]]$$

and obtain the discrete set of points with nonnegative integral coefficients

$$\Delta(G) := \{(i, j) \mid A_{i,j} \neq 0\},$$

called the Newton diagram of $G$. Let $(\mathbb{R}^+)^2 := \{(x, y) \mid x \geq 0, y \geq 0\}$ be the first quadrant in the plane $\mathbb{R}^2$ and consider

$$\Delta'(G) := \Delta(G) + (\mathbb{R}^+)^2.$$

Then the convex hull of $\Delta'(G)$ admits a border which is composed of two half-lines (a vertical one, an horizontal one, parallel to the coordinate axes) and a polygonal line, called the Newton polygon of $G$, joining them, denoted by $\mathcal{N}(G)$. The height $h(\mathcal{N}(G))$ of $G$ is by definition the maximal ordinate of the vertices of the Newton polygon $\mathcal{N}(G)$.

If $y(X) = \sum_{q\geq 1} a_q \left(X^{1/\nu(y)}\right)^q$ is a Puiseux series, write $G_y = G_y(X, Y) = \prod_{i=1}^{\nu(y)}(Y - y_i(X))$, the $y_i$, $i = 1, \ldots, \nu(y)$, being the conjugates of $y$. The series $G_y$ is irreducible in $\mathbb{C}[[X, Y]]$. The theory of Puiseux allows a formal decomposition as follows.

**Theorem 1.** *For any $G = G(X, Y) \in \mathbb{C}[[X, Y]]$,*

(i) *there are Puiseux series $y_1, y_2, \ldots, y_m$, $m \geq 0$, in $\mathbb{C}\ll X\gg$ so that $G$ decomposes in the form*

$$G = u\, X^r\, G_{y_1}\, G_{y_2} \ldots G_{y_s}$$

*where $r \in \mathbb{Z}$, and $u$ is an invertible series in $\mathbb{C}[[X, Y]]$,*

(ii) *the height of the Newton polygon of $g$ is the sum of the ramification indices*

$$h(\mathcal{N}(G)) = \nu(y_1) + \nu(y_2) + \cdots + \nu(y_s)$$

*and the $Y$-roots of $G$ are the conjugates of the $y_j(X)$, $j = 1, \ldots, s$.*

The Newton-Puiseux algorithm applied to the Newton polygon $\mathcal{N}(G)$ of $G$ allows to compute all the $Y$-roots of $G(X, Y)$ and the ramification indices [11] [13] [34].

**Definition 2.** Let $k$ be a (commutative) field of characteristic zero and $g(X, Y) \neq 0$ an element of $k[[X, Y]]$ such that $g(0, 0) = 0$. A parametrization of $g$ is an ordered pair $(\mu_1(T), \mu_2(T))$ of elements of $k[[T]]$ which satisfies

(i) $\mu_1$ and $\mu_2$ are not simultaneously identically zero,

(ii) $\mu_1(0) = \mu_2(0) = 0$,

(iii) $g(\mu_1(T), \mu_2(T)) = 0 \in k[[T]]$.

Denote $\mathbb{C}\{x_1, x_2, \ldots, x_q\}$ the ring of convergent power series, and turn to convergence questions. Let $s = \sum_{i\geq 0} a_i X^{i/n}$ be a fractionary power series, with $a_i \in \mathbb{C}$. We say that $s$ is a convergent fractionary power series if and only if the ordinary power series

$$s(t^n) = \sum_{i\geq 0} a_i t^i$$

has nonzero convergence radius. This condition does not depend upon the integer $n$ and the set of convergent fractionary power series $\mathbb{C}\{X\}$ is a subring of $\mathbb{C}\ll X \gg$.

If $s$ is convergent, with $\nu(s) = n$, one may compose the polydromic (multivalued) function $z \to z^{1/n}$ and the analytic function defined by $s(t^n)$ in a neighbourhood of $t = 0$: we obtain a polydromic function $\overline{s}$, defined in a neighbourhood of $z = 0$, which we call the (polydromic) function associated with $s$. If $s$ is convergent, all its conjugates are also convergent and any of them defines the same polydromic function $\overline{s}$ as $s$. If $s$ is convergent, the associated function $\overline{s}$ takes $\nu(s)$ different values on each $z_0 \neq 0$ in a suitable neighbourhood of 0.

In the context of convergent series the theory of Puiseux makes Theorem 1 more accurate as follows.

**Theorem 3.** *If $G(x, y) \in \mathbb{C}\{x, y\}$ is a convergent series, then all its $y$-roots are convergent, and there are an invertible series $v \in \mathbb{C}\{x, y\}$ and a nonnegative integer $r$, both uniquely determined by $G$, and convergent Puiseux series $y_1, y_2, \ldots, y_s$, uniquely determined by $G$ up to conjugation, so that*

$$G = vx^r G_{y_1} G_{y_2} \ldots G_{y_s}. \tag{11}$$

If $G$ is a polynomial in $Y$, i.e., if $G \in \mathbb{C}[[X]][Y]$, and if the coefficients $a_q$ of the Puiseux expansions involved in its decompositon are algebraic numbers, denote by $L = \mathbb{Q}(a_1, a_2, \ldots)$ the number field generated by the coefficients. Assume $[L : \mathbb{Q}] < +\infty$ and let $r := [L : \mathbb{Q}]$. Let $\sigma_1, \sigma_2, \ldots, \sigma_r$, the $r$ embeddings of $L$ into $\overline{\mathbb{Q}}$. Denote

$$C = C(y(X)) := \left\{ \sum_{q \geq 1} \sigma_i(a_q) \left( \zeta^j_{\nu(y)} X^{1/\nu(y)} \right) \mid i = 1, \ldots, r \, , \, j = 0, 1, \ldots, \nu(y) \right\}$$

the $L$-rational conjugacy class of $y(X)$. By Proposition 2.1 in Walsh [35], assuming that all the Puiseux expansions of $X$ in $G$ are distinct,

$$\prod_{i=1}^{\nu(y)} (Y - y_i(X))$$

is irreducible in $\overline{\mathbb{Q}}((X))[Y]$, of degree $\nu(y)$ in $Y$, and

$$\prod_{y_i \in C} (Y - y_i(X)) \tag{12}$$

is irreducible in $\mathbb{Q}((X))[Y]$ of degree $\nu(y)r/r_0$ in $Y$ where

$$r_0 := \{\sigma : L \to \overline{\mathbb{Q}} \mid \exists t \in \mathbb{Z} \text{ such that } \sigma(a_q) = a_q \zeta^{tq}_{\nu(y)} \text{ for all } q \geq 1\}.$$

If, in addition, $G$ is assumed convergent, gathering the Puiseux expansions by $L$-rational conjugacy classes, whose number is (say) $e$, the collection of such classes being $(C_j)_{j=1,\dots,e}$, allows to write $G$ in the form of the product of a unit $v \in \mathbb{C}[[x,y]]$ by a nonnegative power $x^r$ of the first variable $x$ and a product of $e$ irreducible polynomials in $\mathbb{Q}[[x]][y]$ as follows:

$$G = vx^r \prod_{j=1}^{e} \prod_{y_i \in C_j} (y - y_i(x)). \tag{13}$$

## 5. Beta-Conjugates as Puiseux Expansions

Let $\beta > 1$ be an algebraic number, not necessarily a Parry number. In the sequel we will not consider the case where $\beta > 1$ is a rational integer: indeed, in this case, $\beta$ has no Galois conjugate unequal to $\beta$, and $f_\beta(z) = -1 + \beta z$ is a polynomial having only the root $1/\beta$; therefore $\beta$ has no beta-conjugate.

The key observation, that the three functions $z - 1/\beta$, $P_\beta^*(z)$, $f_\beta(z)$ cancel at $1/\beta$, each of them with multiplicity one, leads to consider the point $(0, 1/\beta)$ of $\mathbb{C}^2$ as natural origin of the germ of curve. Therefore we consider the new variable $Z := z - 1/\beta$ and make the change of variable $z \to Z$ into $f_\beta(z)$ and $P_\beta^*(z)$, as follows:

$$\widetilde{f_\beta}(Z) := f_\beta(z), \qquad \widetilde{P_\beta^*}(Z) := P_\beta^*(z).$$

**Lemma 4.** *Let $\beta > 1$ be a real number. Then*

$$\widetilde{f_\beta}(Z) \;=\; \sum_{j \geq 1} \lambda_j Z^j \tag{14}$$

*with* $\lambda_j = \lambda_j(\beta) := \sum_{q \geq 0} t_{j+q} \begin{pmatrix} j+q \\ j \end{pmatrix} \left(\frac{1}{\beta}\right)^q$.

*Proof.* Expanding $f_\beta(z) = -1 + \sum_{i \geq 1} t_i (z - \frac{1}{\beta} + \frac{1}{\beta})^i$ as a function of $Z = z - 1/\beta$ readily gives (14). □

Let $\beta > 1$ be any real number. The series $\lambda_j = \lambda_j(\beta), j \geq 1$, have nonegative terms and, by Stirling's formula applied to the binomial coefficients, are convergent.

**Proposition 5.** *Let $\beta > 1$ be a real number. For all $j \geq 1$, the map $(1, +\infty) \to \mathbb{R}^+, \beta \to \lambda_j(\beta)$ is right-continuous. The set of discontinuity points is contained in the set of simple Parry numbers.*

*Proof.* Assume $\beta > 1$ a real number which is not an integer. Let us fix $j \geq 1$. There exists $u > 0$ such that the open interval $(\beta - u, \beta + u)$ contains no integer. Then any $\beta' \in (\beta - u, \beta + u)$ is such that its Rényi $\beta'$-expansion $d_{\beta'}(1)$ of 1 has digits

$t_q(\beta')$ within the same alphabet which is $\mathcal{A}_\beta = \{0, 1, \ldots, \lfloor\beta\rfloor\}$. Let $\epsilon > 0$. Then there exists $q_0 \geq j$ such that

$$\sum_{q>q_0} \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta - u}\right)^{q-j} < \frac{\epsilon}{4\lfloor\beta\rfloor}.$$

Then, for all $\beta' \in (\beta - u, \beta + u)$, since $1/\beta' \leq 1/(\beta - u)$, the following uniform inequality holds:

$$\sum_{q>q_0} t_q(\beta') \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta'}\right)^{q-j} < \frac{\epsilon}{4}. \tag{15}$$

Now there are are two cases: either $\beta$ is a simple Parry number, or not.

(i) Assume $\beta > 1$ is not a simple Parry number. Then the sequence $(t_i(\beta))_i$ is infinite (does not end in infinitely many zeros). There exists $\eta > 0, \eta < u$, small enough such that $t_1(\beta') = t_1(\beta), t_2(\beta') = t_2(\beta), \ldots, t_{q_1}(\beta') = t_{q_1}(\beta)$ for all $\beta' \in (\beta - \eta, \beta + \eta)$ with $q_1 = q_1(\beta') > q_0, t_{q_1+1}(\beta') \neq t_{q_1+1}(\beta)$, for which, since $\beta' \to \beta'^{q-j}, q = j, j+1, \ldots, q_0$, are all continuous,

$$\left| \sum_{q=j}^{q_0} t_q(\beta) \left( \begin{array}{c} q \\ j \end{array} \right) \left( \left(\frac{1}{\beta'}\right)^{q-j} - \left(\frac{1}{\beta}\right)^{q-j} \right) \right| < \epsilon/2. \tag{16}$$

In this nonsimple Parry case, recall [21] that the function $\beta' \to q_1(\beta')$ is monotone increasing and locally constant when the variable $\beta'$ tends to $\beta$ (i.e. $d_{\beta'}(1)$ and $d_\beta(1)$ start by the same string of digits $t_1 t_2 \ldots t_{q_1}$ when $\beta'$ is close to $\beta$).

(ii) Assume that $\beta > 1$ is a simple Parry number. Let $d_\beta(1) = 0.t_1 t_2 \ldots t_N$ be its Rényi $\beta$-expansion of unity ($N \geq 1$). If $N > q_0$, there exists $\eta > 0$, $\eta < u$, such that $|\beta' - \beta| < \eta \Longrightarrow t_q(\beta') = t_q(\beta)$ for all $q = 1, \ldots, N-1$, and (16) also holds. If $j \leq N \leq q_0$, we express $\beta$ in base $\beta$ and $\beta'$ in base $\beta'$ in the sense of Rényi: then we deduce that there exists $\eta > 0$, $\eta < u$, such that $\beta \leq \beta' < \beta + \eta$ implies

$$\left| \sum_{q=N+1}^{q_0} t_q(\beta') \left(\frac{1}{\beta'}\right)^{q-j} \right| < \frac{\epsilon}{4} \frac{1}{\max_{q=N+1,\ldots,q_0}\{\left( \begin{array}{c} q \\ j \end{array} \right)\}}$$

and

$$\left| \sum_{q=j}^{N} t_q(\beta) \left( \begin{array}{c} q \\ j \end{array} \right) \left( \left(\frac{1}{\beta'}\right)^{q-j} - \left(\frac{1}{\beta}\right)^{q-j} \right) \right| < \epsilon/4; \tag{17}$$

in this case,

$$\left| \sum_{q=N+1}^{q_0} t_q(\beta') \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta'}\right)^{q-j} \right| < \epsilon/4. \tag{18}$$

If $q_0 \leq N$, we deduce, for all $\beta' \in (\beta, \beta+\eta)$,

$$|\lambda_j(\beta) - \lambda_j(\beta')| \leq \left| \sum_{q=j}^{q_0} t_q(\beta) \left( \begin{array}{c} q \\ j \end{array} \right) \left( \left(\frac{1}{\beta'}\right)^{q-j} - \left(\frac{1}{\beta}\right)^{q-j} \right) \right| +$$

$$\left| \sum_{q>q_0} t_q(\beta') \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta'}\right)^{q-j} - \sum_{q>q_0} t_q(\beta) \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta}\right)^{q-j} \right| < \epsilon/2 + 2\epsilon/4 = \epsilon, \quad (19)$$

and, in the case $j \leq N \leq q_0$, we decompose the sum $\sum_{q=j}^{q_0}$ as $\sum_{q=j}^{N} + \sum_{q=N+1}^{q_0}$ in the upper bound (19), using (17) and (18), to obtain $|\lambda_j(\beta) - \lambda_j(\beta')| < \epsilon$ as well. If $j > N$, then $\lambda_j(\beta) = 0$; there exists $\eta > 0$, $\eta < u$, such that $\beta \leq \beta' < \beta + \eta$ implies

$$\left| \sum_{q=j}^{q_0} t_q(\beta') \left(\frac{1}{\beta'}\right)^{q-j} \right| < \frac{3\epsilon}{4} \frac{1}{\max_{q=j,\ldots,q_0}\{\left( \begin{array}{c} q \\ j \end{array} \right)\}}. \quad (20)$$

Hence, using (15) and (20), for $\beta \leq \beta' < \beta + u$,

$$|\lambda_j(\beta')| \leq \left| \sum_{q=j}^{q_0} t_q(\beta') \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta'}\right)^{q-j} \right| + \left| \sum_{q>q_0} t_q(\beta') \left( \begin{array}{c} q \\ j \end{array} \right) \left(\frac{1}{\beta'}\right)^{q-j} \right| < \frac{3\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

and the right-continuity $\lim_{\beta' \to \beta^+} \lambda_j(\beta') = 0$ for $j > N$.

Let us now assume that $\beta > 1$ is an integer. Then $d_\beta(1) = 0.\beta$, $t_1(\beta) = \beta$, $\lambda_1(\beta) = \beta$ and $t_j(\beta) = 0, \lambda_j(\beta) = 0$ for $j \geq 2$. The same arguments as in (ii), with $N = 1$, lead to the result. □

**Lemma 6.** *If $\beta > 1$ is an algebraic number of minimal polynomial $P_\beta(X) = a_0 + a_1 X + a_2 X^2 + \ldots + a_d X^d$, $a_i \in \mathbb{Z}, a_0 a_d \neq 0$, then*

$$\widetilde{P^*_\beta}(Z) = \gamma_1 Z + \gamma_2 Z^2 + \ldots + \gamma_d Z^d, \quad (21)$$

*with $\gamma_q = \sum_{j=q}^{d} a_{d-j} \left( \begin{array}{c} j \\ q \end{array} \right) \left(\frac{1}{\beta}\right)^{j-q} \in \mathbb{K}_\beta, \gamma_d = a_0 \neq 0, \gamma_1 = P^{*'}_\beta(1/\beta) \neq 0.$*

*Proof.* The relation $\widetilde{P^*_\beta}(Z) = P^*_\beta(z - \frac{1}{\beta} + \frac{1}{\beta})$ leads to

$$\widetilde{P^*_\beta}(Z) = \sum_{j=0}^{d} \sum_{q=0}^{j} a_{d-j} \left( \begin{array}{c} j \\ q \end{array} \right) \left(\frac{1}{\beta}\right)^{j-q} Z^q = \sum_{q=0}^{d} \sum_{j=q}^{d} a_{d-j} \left( \begin{array}{c} j \\ q \end{array} \right) \left(\frac{1}{\beta}\right)^{j-q} Z^q.$$

The constant term is zero since $P_\beta(\beta) = \sum_{j=0}^{d} a_j \beta^j = 0$. □

**Theorem 7.** *Let $\beta > 1$ be an algebraic number and $P_\beta(X)$ its minimal polynomial. Then there exists a unique polynomial $G = G_\beta(U, Z) \in \mathbb{C}[[U]][Z]$ in $Z$, $\deg_Z G < \deg \beta$, such that $(\widetilde{P^*_\beta}(Z), Z)$ is a parametrization of $G - \widetilde{f_\beta} \in \mathbb{C}[[U, Z]]$, i.e. such that*

$$G_\beta(\widetilde{P^*_\beta}(Z), Z) - \widetilde{f_\beta}(Z) \;=\; 0. \tag{22}$$

*Proof.* Uniqueness. Assume that $G^{(1)}$ and $G^{(2)}$ are such that $G^{(1)} - \widetilde{f_\beta}$ and $G^{(2)} - \widetilde{f_\beta}$ are both parametrized by $(\widetilde{P^*_\beta}(Z), Z)$. Then $(G^{(1)} - G^{(2)})(\widetilde{P^*_\beta}(Z), Z) = 0$ with $G^{(1)} - G^{(2)} \in \mathbb{C}[[U]][Z]$, $\deg_Z (G^{(1)} - G^{(2)}) < d$. Assume $G^{(1)} \neq G^{(2)}$ and $G^{(1)} - G^{(2)}$ irreducible in $Z$ (no loss of generality). Then this equation defines a plane curve

$$\mathcal{C}_\beta := \{(u, z) \in \mathbb{C}^2 \mid (G^{(1)} - G^{(2)})(u, z) = 0\}$$

along with a ramified covering $\pi : \mathcal{C}_\beta \to \mathbb{C}$ of the complex plane. Above all but finitely many points $u$ of the $U$-plane, the fiber $\pi^{-1}(u)$ has cardinality $\leq d - 1$. The implicit function theorem states that there exist $\delta$ analytic functions $z_1(u), \ldots, z_\delta(u)$, $\delta \leq d - 1$, such that $\pi^{-1}(u) = \{z_i(u) \mid i = 1, \ldots, \delta\}$ and $(G^{(1)} - G^{(2)})(u, z_i(u)) = 0$ for $i = 1, \ldots, \delta$. Each of them parametrizes one sheet of the covering in a neighbourhood of $u$. The contradiction comes from the fact that the polynomial $P^*_\beta(z)$ is irreducible, of degree $d$, that the parametrization $(\widetilde{P^*_\beta}(Z), Z)$ is imposed. Therefore the number of sheets $\delta$ should be equal to $d$. Contradiction.

Existence: by construction. Let $U := \widetilde{P^*_\beta}(Z)$. From (21),

$$U = \gamma_1 Z + \gamma_2 Z^2 + \ldots + \gamma_d Z^d \Rightarrow Z^d = \frac{1}{\gamma_d} U - \left(\frac{\gamma_1}{\gamma_d} Z + \frac{\gamma_2}{\gamma_d} Z^2 + \ldots + \frac{\gamma_{d-1}}{\gamma_d} Z^{d-1}\right).$$

It follows that $Z^d \in \mathbb{K}_\beta[U][Z]$, with $\deg_Z(Z^d) < d$. The idea consists in replacing all powers $Z^j, j \geq d$, in $\widetilde{f_\beta}(Z)$ by polynomials in $Z$, of degree $< d$, with coefficients in $\mathbb{K}_\beta[U]$. Let us prove recursively that $Z^h \in \mathbb{K}_\beta[U][Z]$, with $\deg_Z(Z^h) < d$, for all $h \geq d$: assume $Z^h := \sum_{i=0}^{d-1} v_{i,h} Z^i$ with $v_{i,h} \in \mathbb{K}_\beta[U]$ and show $Z^{h+1} \in \mathbb{K}_\beta[U][Z]$, with $\deg_Z(Z^{h+1}) < d$. Indeed,

$$Z^{h+1} := \sum_{i=0}^{d-1} v_{i,h+1} Z^i = (Z^h) Z = \sum_{i=0}^{d-2} v_{i,h} Z^{i+1} + v_{d-1,h} Z^d$$

$$= \sum_{i=0}^{d-2} v_{i,h} Z^{i+1} + v_{d-1,h} \left[\frac{1}{\gamma_d} U - \left(\frac{\gamma_1}{\gamma_d} Z + \frac{\gamma_2}{\gamma_d} Z^2 + \ldots + \frac{\gamma_{d-1}}{\gamma_d} Z^{d-1}\right)\right].$$

Hence

$$v_{0,h+1} = \frac{1}{\gamma_d} v_{d-1,h} U \quad \text{and} \quad v_{i,h+1} = v_{i-1,h} - \frac{\gamma_i}{\gamma_d} v_{d-1,h},\ 1 \leq i \leq d-1, \tag{23}$$

and the result. We deduce

$$\widetilde{f_\beta}(Z) = \sum_{h\geq 1}\lambda_h Z^h = \sum_{h=1}^{d-1}\lambda_h Z^h + \sum_{h\geq d}\lambda_h Z^h = \sum_{i=1}^{d-1}\lambda_i Z^i + \sum_{h\geq d}\lambda_h\left(\sum_{i=0}^{d-1} v_{i,h}Z^i\right)$$

$$= \sum_{i=0}^{d-1}\left(\lambda_i + \sum_{h\geq d}\lambda_h v_{i,h}\right) Z^i \ \in \mathbb{C}[[U]][Z]. \tag{24}$$

□

Equation (22) is exactly (6) with the usual variable $z$.

We call $G_\beta$ the germ associated with the analytic function $f_\beta(z)$, or with the base of numeration $\beta$.

Following Theorem 3 and the relations (23) and (24), the decomposition of the germ $G_\beta$ shows that the coefficients of its Puiseux series do possess a "right - continuity" property, with $\beta$, via the functions $\lambda_j$ (Proposition 5), and an "asymptotic" property, linked to the invariants of the companion matrix form of (23). This will be developped further elsewhere. The interest of such a remark may consist in studying globally the properties of the family of germs $(G_\beta)$ when $\beta > 1$ varies in the set of algebraic numbers.

**Theorem 8.** *Let $\beta > 1$ be an algebraic number, $P_\beta(X)$ its minimal polynomial and $G_\beta$ the germ associated with the Parry Upper function $f_\beta(z)$. Then*

$$G_\beta(U,Z) \ = \ v\,U\,G_{y_1}G_{y_2}\ldots G_{y_s} \tag{25}$$

*where $v = v(U,Z) \in \mathbb{C}\{U,Z\}$ is an invertible series, and the convergent Puiseux series*

$$y_1(U) = \sum_{i\geq 1} a_{i,1}U^{i/\nu(y_1)}, \ldots, y_s(U) = \sum_{i\geq 1} a_{i,s}U^{i/\nu(y_s)}$$

*are uniquely determined by $G_\beta$, up to conjugation, with*

$$G_\beta(P_\beta^*(z), z-\frac{1}{\beta}) \ = \ f_\beta(z) \ =$$

$$v(P_\beta^*(z), z-\frac{1}{\beta})\,P_\beta^*(z)\ \prod_{i=1}^{\nu(y_1)}\big(z-\frac{1}{\beta}-y_{i,1}(P_\beta^*(z))\big)\ldots\prod_{i=1}^{\nu(y_s)}\big(z-\frac{1}{\beta}-y_{i,s}(P_\beta^*(z))\big), \tag{26}$$

*and*

$$h(\mathcal{N}(G_\beta)) \ = \ \sum_{i=1}^{s}\nu(y_i) \ < \ \deg\beta. \tag{27}$$

*Proof.* Theorem 3 is applied to the germ $G_\beta(U,Z)$. Since $f_\beta(z)$ is convergent in a neighbourhood of $1/\beta$, $G_\beta$ and all the Puiseux expansions involved in its decomposition are convergent in this neighbourhood. The power of $U$ in (32) is necessarily equal to 1 since $f'_\beta(1/\beta) > 0$, i.e. 0 is a simple zero of $G_\beta(\widetilde{P^*_\beta}(Z), Z)$.

Since $\deg_Z G_\beta(U,Z) < \deg\, \beta$, by the definition of the height of the Newton polygon of the germ $G_\beta$, we readily deduce (27) from Theorem 1 (ii). □

For $\beta > 1$ any algebraic number, a beta-conjugate $\xi$ of $\beta$ is by definition a complex number such that (i) $\xi^{-1}$ is a zero of $f_\beta(z)$ which lies in its domain of definition, (ii) $\xi$ is not a Galois conjugate of $\beta$.

For Parry numbers $\beta$, (2) and (3) show that this definition is exactly the usual one which uses the Parry polynomial of $\beta$ [9].

Equation (26) gives the exhaustive list of zeros of $f_\beta(z)$, and therefore suggests the following alternate definition of the beta-conjugates of $\beta$ (where the natural boundary $|z| = 1$ of $f_\beta(z)$ is taken into account, if $\beta$ is not a Parry number).

**Definition 9.** Let $\beta > 1$ be an algebraic number.

(i) A complex number $\xi$ which satisfies

$$0 \ = \ \xi^{-1} - \beta^{-1} - \sum_{i\geq 1} a_i \left(P^*_\beta(\xi^{-1})\right)^{i/n}, \tag{28}$$

where $y(U) = \sum_{i\geq 1} a_i U^{i/n}$, $n = \nu(y)$ is any $Z$-root, is called a *cancellation point* of the germ $G_\beta(U,Z)$. We say that the cancellation point $\xi$ lies on the $Z$-root $y(U)$. The set of cancellation points is denoted by $\mathcal{S}_\beta$. Equation (28) has to be understood as the composition of the (convergent) two analytic functions $z \to z - \beta^{-1} - \sum_{i\geq 1} a_i z^i$ and $z \to P^*_\beta(z)$ with the multivalued (polydromic) analytic function $z \to \Big(P^*_\beta(z)\Big)^{1/n}$. Since $\xi$ is not a Galois conjugate of $\beta$ the function $z \to P^*_\beta(z)$ does not cancel on a small neighbourhood of $\xi^{-1}$; this give a sense to (28).

Since the Puiseux expansions in (28) are convergent, truncating them to a few terms transforms (28) into a finite collection of equations whose solutions provide the geometry of the beta-conjugates of $\beta$ with a certain approximation, controlled by the error terms. This approach will be continued elsewhere.

(ii) If $\beta$ is a Parry number, a beta-conjugate of $\beta$ is a cancellation point of the germ. The set $\mathcal{S}_\beta$ is the set of beta-conjugates of $\beta$, and $\mathcal{S}_\beta \subset \Omega$ Solomyak's fractal.

(iii) If $\beta$ is not a Parry number, a beta-conjugate of $\beta$ is a cancellation point $\xi \in \mathcal{S}_\beta$ of the germ such that $|\xi| > 1$.

(iv) A cancellation point $\xi \in \mathcal{S}_\beta$, lying on the $Z$-root $y(U)$, is said Puiseux-conjugated to another cancellation point $\xi' \in \mathcal{S}_\beta$ if $\xi'$, lying on a $Z$-root $y'(U)$, is such that $y(U)$ and $y'(U)$ belong to the same conjugacy class of the germ $G_\beta(U,Z)$.

If $\beta > 1$ is an algebraic number which is not a Parry number the natural boundary $|z| = 1$ of $f_\beta(z)$ is the natural boundary of at least one of the factors in (26), but not necessarily of all of them a priori. In other terms it may occur that Puiseux-conjugation may be addressed to cancellation points of the germ $G_\beta$ which lie beyond the natural boundary of $f_\beta(z)$, some branches possibly spiraling outside the domain of definition of $f_\beta(z)$.

## 6. Rationality, Descent Over $\mathbb{Q}$, and Factorization of the Parry Polynomial of a Parry Number

Let $\beta$ be a Parry number, with $m$ as preperiod lenght and $p+1$ as period length in $d_\beta(1)$. Then the Parry polynomial of $\beta$ is, for non-simple Parry numbers,

$$P_{\beta,P}(X) = X^{m+p+1} - t_1 X^{m+p} - t_2 X^{m+p-1} - \ldots - t_{m+p} X - t_{m+p+1}$$
$$- X^m + t_1 X^{m-1} + t_2 X^{m-2} + \ldots + t_{m-1} X + t_m \tag{29}$$

and

$$P_{\beta,P}(X) = X^{p+1} - t_1 X^p - t_2 X^{p-1} - \ldots - t_p X - (1 + t_{p+1}) \tag{30}$$

in the case of pure periodicity. For simple Parry numbers, the Parry polynomial is

$$P_{\beta,P}(X) = X^m - t_1 X^{m-1} - t_2 X^{m-2} - \ldots - t_{m-1} X - t_m \tag{31}$$

with $m \geq 1$ [15] [20] [33]. The height (= maximum of the moduli of the coefficients) of the Parry polynomial lies in $\{\lfloor\beta\rfloor, \lceil\beta\rceil\}$; if $\beta$ is a simple Parry number, then it is equal to $\lfloor\beta\rfloor$ [33]. In the decomposition of $P_{\beta,P}(X)$ as the product of irreducible polynomials with coefficients in $\mathbb{Q}$, as $P_{\beta,P} = P_\beta\,\pi_1\,\pi_2 \ldots \pi_\sigma$, we may identify the irreducible factors $\pi_j$ as arising from the conjugacy classes of the germ $G_\beta$. This requires some assumptions.

**Theorem 10.** *Let $\beta > 1$ be a Parry number, $P_\beta(X)$ its minimal polynomial, $P_{\beta,P}(X)$ its Parry polynomial decomposed as $P_{\beta,P} = P_\beta\,\pi_1\,\pi_2 \ldots \pi_\sigma$ into irreducible factors. Let $G_\beta$ be the germ associated with $\beta$ and $L$ be the field of coefficients of the Puiseux series of $G_\beta$. Assume that all Puiseux expansions of $X$ in $G_\beta$ are distinct. Assume $[L : \mathbb{Q}] < +\infty$ and, for each $L$-rational conjugacy class $C$, the product*

$$\prod_{y_i \in C} (Y - y_i(X)) \qquad \textit{lies in } \mathbb{Q}[X][Y].$$

*If $e$ is the number of $L$-rational conjugacy classes $(C_j)_{j=1,\ldots,e}$, then*

(i) *$e = \sigma < \deg\beta$, and*

(ii) *up to the order,*

$$\pi_j^*(X) \;=\; \prod_{y_i \in C_j} \Big(X - \frac{1}{\beta} - y_i(P_\beta^*(X))\Big)\,, \quad j = 1, \ldots, e. \tag{32}$$

*Proof.* This is a consequence of Proposition 2.1 in Walsh [35]. Under the present assumptions $\pi_i \neq \pi_j$ if $i \neq j$ and the decomposition of $G_\beta$, as given by (13), allows to write $f_\beta(z)$ as a product of distinct irreducible factors in $\mathbb{Q}[X][Y]$. From (2) and (3) the identification of the factors readily gives $e = \sigma$, the irreducible factors $\pi_j^*$ and the unit $v = -(1 - z^k)^{-1}$, with $k = m$ if $\beta$ is simple, with $d_\beta(1)$ of length $m$, and $k = p + 1$ if $\beta$ is not simple, with $d_\beta(1)$ of period length $p + 1$.

From (27), the number $\sigma$ of irreducible factors which arises from $L$-rational conjugacy classes of Puiseux expansions is smaller than deg $\beta$. □

## 7. A Product Formula for $\zeta_\beta(z), \beta$ a Parry Number

Using (2) and (3) and assuming the hypotheses of Theorem 10 we obtain the following reformulation of the dynamical zeta function $\zeta_\beta(z)$ as a finite product over the $e$ $L$-rational conjugacy classes, $e <$ deg $\beta$,

$$\zeta_\beta(z) \;=\; v\,\frac{1}{P_\beta^*(z)} \prod_{j=1}^{e} \left( \frac{1}{\prod_{y_i \in C_j} \left( z - \frac{1}{\beta} - y_i(P_\beta^*(z)) \right)} \right). \tag{33}$$

The unit $v$ is equal to $(1 - z^k)$ with $k = m$ if $\beta$ is simple, with $d_\beta(1)$ of length $m$, and $k = p + 1$ if $\beta$ is not simple, with $d_\beta(1)$ of period length $p + 1$. The poles of $\zeta_\beta(z)$ are either the reciprocals $\xi^{-1}$ of the cancellation points $\xi$ of the germ $G_\beta$ of $\beta$, or the reciprocals of the Galois conjugates of $\beta$.

The assumptions in Theorem 10 could probably be weakened, for obtaining the same decomposition (33).

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