VERTEX DELETION GAMES WITH PARITY RULES

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Abstract

We consider a group of related combinatorial games all of which are played on an undirected graph. A move is to remove a vertex but which vertices are available depend on parity conditions. In the main game Left removes a vertex of even degree and Right removes a vertex of odd degree. This game is special case of Odd-Even games introduced, for the first time, in this paper. The obvious two variants are considered briefly at the end. One is completely trivial but very little is known about the other.

Key words: game, vertex deletion, partizan octal, graph.

1. Introduction

There are many games defined on graphs which involve adding or deleting edges. Some come under the heading of avoidance- or achievement-games. In such games, edges are added until the graph has a certain property. In an avoidance game, the person making the last move loses, in an achievement game that player would win. See [6], [8] and [11] for examples. In [11], it is noted that many of these games have a periodicities of order 4, although the author does show the existence of arbitrary length periods for some of these games. Another edge game is Hackenbush[2]. The graph is rooted and each edge is colored red, blue or green. Left on his move, deletes a blue or green edge and Right removes a red or green edge. Any component disconnected from the ground also is deleted. Even Dot-and-Boxes [1] generalized as Strings-and-Coins involves deleting edges.

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Vertex deletion games have appeared in disguise. Most typically as subtraction or octal games [2,5,10]. For example, Kayles [2] is played with a row of skittles where a player removes one or two adjacent pins. The translation is to play on a path where each player can remove one or two adjacent vertices. In partizan octal games, translated to playing on a path, the legal moves for each player would be different. For example, Left could remove a vertex to leave either zero heaps or two non-empty heaps and Right could remove one vertex to leave one non-empty heap. (In octal notation [2], Left plays **0.5** and Right **0.2**.) Playing on a path, these rules say that Left can remove a vertex of even degree and Right a vertex of odd degree. Partizan subtraction games are known to be periodic in their outcome classes, [5], as a function of the size of the heap (i.e. length of the path) but nothing is known about partizan octal games. (Also see [7], Problem 2.)

Here, we consider the generalization of the example in the last paragraph. Given a simple, finite, undirected graph, Left deletes vertices of even degree, including those of degree zero, and Right those of odd degree.

The game is biased in favor of Left. There are no graphs with value less than 0 (Corollary 7), there are graphs with value 0, e.g. K_2, and Right can win going first, e.g. on $K_{1,2}$. The analysis of complete, multipartite graphs and generalized paths (paths with a $K_{1,n}$ attached at one end), is presented in Section 3. Although we do not have a full analysis of complete, multipartite graphs, they do seem to have a regularity of order 2 (Theorems 11, 12). Theorem 10 helps ease the analysis, by giving a process that allows deletion of certain subgraphs without changing the game value. The generalized paths, surprisingly, have a regularity of order 3 in the length of the path and one of order 2 in the size of the star (Theorem 13). This analysis does include the path or the partizan octal game Left plays **0.5** and Right **0.2**. Not only is this periodic in terms of its outcome classes but the values are arithmetic periodic.

Before the analysis of this specific game, in Section 2, we introduce the notion of Odd-Even games. Roughly speaking, Even games only have Odd options and Odd games only have Even options. A caveat is that 0 never appears in the Odd games. This must be a consequence of the rules of the game. This is true for this vertex deletion game. The trivial game, 'she-loves-me-she-loves-me-not' [2] is an Odd-Even game. Less trivially, Corridors in Go [3] are also in this class although Go itself is not. One consequence, given in Theorem 5, of being an Odd-Even game is that the only infinitesimal values that occur are 0 and $*$ and that no fractional values occur either.

Section 4 considers briefly, the games where Left and Right both remove even vertices, which turns out to be trivial; and where they both take odd vertices.

The graphs are finite, simple and undirected. Changing to directed graphs will generalize the game and take away the Left bias. Many of the results presented here are easily translated into the directed cases. This, we leave for the reader to persue. Some of these can be found in [9]. The work there also lays the foundations for this paper.

By convention, Left will be refered as 'he' and Right as 'she'.

2. Odd-Even Games

The following is new class of games that has relevance here and in other games. There is a subtlety in the definition of the Odd games. The value 0 never occurs as an Odd game even if it can be constructed with strictly Even options.

Definition: Let $E_0 = \{0\}$ and $O_{-1} = \emptyset$. Recursively, let $E_{2k} = \{G|G^L, G^R \subseteq O_{2k-1}\}$ and $O_{2k+1} = \{G|G^L, G^R \subseteq E_{2k}\} - \{0\}$. Let

$$\mathcal{E} = \cup_{k\geq 0} E_{2k} \cup O_{2k+1}$$

A game G is an *Odd-Even* game if G and all followers of G are in $\mathcal{E}$.

If G is an Odd-Even game, we abuse this definition slightly by calling G *Even*, if $G \in E_{2k}$ for some k, *Odd* if $G \in O_{2k+1}$.

The restriction given for the sets O_k are a natural consequence of many games. For example, in the Vertex Deletion game where players may delete vertices of opposite parity we will find that games with value 0 can never occur when there are an odd number of vertices left in the graph.

Lemma 1. $E_{2k} \subset E_{2k+2}$ and $O_{2k-1} \subset O_{2k+1}$ for $k \geq 0$.

Proof: We will proceed by induction. First we see that $O_1 = \{-1, *, 1\}$ and so $O_{-1} \subset O_1$. E_2 contains $\{-1|1\} = 0$ and so $E_0 \subset E_2$.

Now, assume the statement is true up to k. Since $O_{2k-1} \subset O_{2k+1}$ then we know that $E_{2k} \subset E_{2k+2}$ since we generate the games in E_{2k+2} by taking all possible games formed by taking options from O_{2k+1}. A similar argument also holds when we consider O_{2k+1}. ∎

Theorem 2. The sets E_{2k} and O_{2j-1} are disjoint for all k, j.

Proof: Assume there is a game G that is both Even and Odd. By Lemma 1, we may let k be the smallest value such that G is in both E_{2k} and O_{2k-1}. By the definition of Odd-Even games, we must have $k > 0$. Then we can write $G = \{H_1|\, H_2\} = \{K_1|\, K_2\}$ where $H_1, H_2 \in O_{2k-1}$ and $K_1, K_2 \in E_{2k-2}$.

Now, we find the simplest form of G by removing its dominated and reversed options using both representations. If a Left or Right option of G is dominated, it is simply removed. If a Left option is reversed, it is replaced by the Left options of Right's best response. Therefore, if the Left option is an Even game, it is again replaced by options which are Even games. The same holds when we deal with Odd games. Similarly, if a reversed Right option is an

Even game, it is replaced by the Right options of Left's best response. As before, if the Right option is an Even game, it is replaced by options which are Even games (and again, this holds for Odd games as well).

After performing these operations to both of G's representations we know that both new representations must have the exact same options since the simplest form of a game is unique ([3], Theorem 69). Let this simplest form be $\{P|Q\}$. By our reduction process, we know that any element of P or Q must be in E_{2k-2} and O_{2k-3} but this contradicts that k is the smallest value such that there is a game in both E_{2k} and O_{2k-1}. Therefore, we have that P and Q are both the empty set, i.e. $G = \{|\} = 0$ which contradicts the fact that 0 never occurs in O_k for any k. Therefore, such a game G cannot exist and the sets E_k and O_k are disjoint for all k. ■

It should be noted that examining the simplest form of a game involves deleting and bypassing options available to both players. As demonstrated in the previous proof, this does not affect the property of a game being Even or Odd. The following results make use of this fact.

Lemma 3. If a game G is infinitesimal and positive then: (i) Left can eventually move to a 0 regardless of whether he plays first or second; and (ii) Left is unable to move to any positive number if Right plays optimally.

Proof: Left will win this game so he, at some point, must move to a position with value ≥ 0. Assume for a contradiction that Left can at some point move to a game equal to a number $z > 0$. Then consider the game $G - \frac{z}{2}$. Left will still win this game since he can move to $z - \frac{z}{2} > 0$ which implies $G \rhd \frac{z}{2}$ and therefore contradicts that G is infinitesimal. ■

Lemma 4. Let G be an infinitesimal game which is fuzzy with 0 but not equal to $*$. Then G is not an Even or Odd game.

Proof: Assume for a contradiction that G is an Even or an Odd game. We may assume that G is the game with the earliest birthday such that this is true. We will examine Left's options in this game. Similar arguments can be made for Right's options.

Case 1: Left has an option which is a positive number. That is, $G^L = z > 0$. As in Lemma 3, we can see that $G - \frac{z}{2} \rhd 0$ which implies that G cannot be infinitesimal.

Case 2: G has a positive infinitesimal Left option. That is, $0 < G^L < z$. Then by Lemma 3, we know a 0 position can be reached by playing first or playing second. Recursively, this tells us either G has options where one is Even and another is Odd or G has an option from which a position of value 0 can be reached by playing first or second. At some point we reach a game which is not Even or Odd which tells us that G cannot be Even or Odd.

From cases 1 and 2, we conclude that Left has no options which are positive. We know that since $G||0$, Left can win going first. Since he has no options greater than 0, he must have an option equal to 0.

Case 3: Left has negative options. That is, $G^L < 0$. This cannot occur because these options are dominated (Lemma 3 says Left, eventually, must be able to play to 0) and hence removed.

Case 4: Left has an option which is fuzzy with 0. That is, $G^L||0$. If this option is $*$ then G is not an Even or Odd game since he has options to 0 and $*$ which are Even and Odd respectively. If the option is not $*$, then we must have a game with an earlier birthday which satisfies the above properties. This cannot occur due to our original assumption about G.

Finally, if G has no other options besides 0, then we find that $G = \{0 \mid 0\} = *$ which is again a contradiction. Therefore, such a game G does not exist. ∎

Theorem 5. The sets E_{2k} and O_{2k-1} do not have any games with values that contain fractions or any infinitesimals besides 0 and $*$.

Proof: Lemma 4 rules out any infinitesimals besides 0 and $*$.

Fractions, in their simplest form, only have other fractions or integers as their options (by our definition of numbers). Also, by the definition of simplicity, we know that if a game has a fractional value, its options can differ by at most 1. Recursively we can show that if a fraction has integer options it is not Even or Odd since the options must be consecutive and hence one option is Even and the other is Odd. If a fraction has options which are not both integers when in simplest form, then it must have a fractional option. This option is not Even or Odd, so the game itself is not Even or Odd. ∎

3. Even/Odd

In this version of the Vertex Deletion game, Left may only remove vertices of even degree and Right may only remove vertices of odd degree. This is a partizan game. Since vertices of degree 0 can be removed by Left this game looks advantageous to him. In fact, the next result shows that there are no Right-win graphs, but as K_2 shows, it is possible for Right to win if she plays second.

Lemma 6. Let G be a graph. If $|G(V)|$ is odd then $G \in L \cup N$. If $|G(V)|$ is even, then $G \in L \cup P$.

Proof. First, suppose that $|G(V)|$ is odd. Since $|G(V)|$ is odd then there is a vertex of even degree. Left, therefore always has a legal move. On each of his subsequent turns, there will again be an odd number of vertices. Thus, after each of Right's turns, Left must have a legal move. He will eventually make the last move and win the game.

Suppose $|G(V)|$ is even. If Right plays first, she must move to a position which has an odd number of vertices. By the result in the previous paragraph, this is in $L \cup N$ so Right will lose. Therefore, G cannot be in R or N. ■

Corollary 7. For any graph G, $G \not< 0$ and if $|V(G)|$ is odd then $G \neq 0$.

The next result shows that this game is in the class of Odd-Even games.

Theorem 8. If a graph G has an odd number of vertices, $G \in O_{2k-1}$ for some k. Likewise, if G has an even number of vertices, $G \in E_{2k}$ for some k.

Proof. The graph on 0 vertices has value 0 and is hence an Even game. Also, the graph with 1 vertex has value 1 and is therefore an Odd game. Assume that all graphs on $2k$ vertices are Even games. Every option from a graph G on $2k+1$ vertices is Even since a legal move for either player is to delete exactly one vertex. Also, by Corollary 7, we know $G \neq 0$. Therefore G is an Odd game. Now, we know that all graphs on $2k+1$ vertices are Odd games so a graph on $2k+2$ vertices can only have Odd options. Therefore, it must be an Even game. By induction, we now know that all graphs with an odd number of vertices are Odd games and all graphs with an even number of vertices are Even games. ■

Theorem 8 and Theorem 5, together give the next result.

Corollary 9. No graph has values that contain fractions or any infinitesimals besides 0 and $*$.

We now turn to actually evaluating games. First we have a reduction theorem. The conditions appear technical but the remarks after the proof give an application.

Theorem 10. Let G be a graph. Let H, K be isomorphic, induced subgraphs of G with $f : H \to K$ an isomorphism which satisfies

(i) $H \cap K = \emptyset$;

(ii) $f(a)$ is adjacent to a;

(iii) $N(h) = N(f(h))$ in $G - (H \cup K)$ for all $h \in H$;

then the game played on G has the same value as the game played on $G' = G - (H \cup K)$.

Proof: To show that $G = G'$ we can simply play the game $G - G'$ and show that the second player has a winning strategy. For any $h \in H$, h and $f(h)$ have the same neighbourhoods in $G - (H \cup K)$ and the same cardinality of neighborhoods in $H \cup K$. Therefore, the parity of the degree of all other vertices is the same in G as it is in G'.

It is important to remember that the Left player will be able to remove even degree vertices in G but odd degree vertices in $-G'$ since the roles of the players will be reversed. If the first player deletes a vertex which is not part of H or K in one game, the second player can certainly delete the corresponding vertex in the other game since the degree of the vertices have the same parity. On the other hand if the first player deletes vertex $h \in V(H)$, for instance, then the second player (in the same graph) can delete $f(h)$ since its parity will have changed due to deleting the edge between h and $f(h)$. Therefore, regardless of the first player's moves, the second player always has a legal move to make in response. Since the game must eventually end we know it must be the second player who makes the last move. Thus, $G = G'$. ■

This theorem can be used to simplify certain graphs. In the complete graph, K_n, vertices can be paired off and deleted by Theorem 10. Therefore, $K_n = 0$ if n is even and 1 if n is odd. Also, when dealing with complete multi-partite graphs, if there are two partitions that have the same number of vertices, then these two partitions are isomorphic and each vertex of them has the same neighbourhood outside of the two partitions. Therefore, by the previous theorem we can remove both partitions from the graph and not change the game value. We include the complete graph value in the next result which gives the values of complete, bipartite graphs.

Theorem 11. The complete bipartite graphs have the following values:

$$K_{2n+1,2m+1} = 0; \qquad K_{2n,2m} = \begin{cases} 0 & \text{if } n > 0, m > 0 \\ 2n & \text{if } m = 0 \\ 2m & \text{if } n = 0 \end{cases}$$

$$K_{2n,2m+1} = \begin{cases} \{2n|0\} & \text{if } n > 0, m = 0 \\ 2m+1 & \text{if } n = 0 \\ * & \textit{otherwise} \end{cases}$$

Proof. In the graph $K_{2n+1,2m+1}$, Left has no legal moves since all vertices will have odd degree which implies $K_{2n+1,2m+1} \notin L$. Also, this graph has an even number of vertices so by Lemma 6, it must be in $L \cup P$. Therefore, the game must be in P and has value 0.

Now, $K_{2n,2m} = \{K_{2n-1,2m}, K_{2n,2m-1} \mid \}$ when $n > 0$ and $m > 0$. From either of Left's options, Right can move to $K_{2n-1,2m-1}$ which has value 0. Therefore, the first player can never win and this graph has value 0. If $n = 0$ or $m = 0$, then $K_{2n,2m}$ is just a set of isolated vertices which is the disjunctive sum of $2m$, respectively $2n$, games of value 1.

Finally, for the game $K_{2n,2m+1}$. If $n = 0$, this is a game with $2m + 1$ isolated vertices and has value $2m+1$. If $n > 0$ we see that $K_{2n,2m+1} = \{K_{2n,2m} | K_{2n-1,2m+1}\}$. Right's option always has value 0 and Left's option has value 0 when $m > 0$ and has value $2n$ when $m = 0$. Therefore, $K_{2n,2m+1} = \{0|0\} = *$ for $m > 0$ and $K_{2n,2m+1} = \{2n|0\}$ for $m = 0$. ■

Theorem 12. Let G be a complete multi-partite graph with $2k$ partitions with each partition containing at least 2 vertices. Then, the value of the game played on this graph is 0 if there are an even number of vertices and $*$ otherwise.

Proof: To begin, we examine the cases where every partition has an odd number of vertices. We know this game is in $L \cup P$ but Left has no legal moves since every vertex has odd degree. Therefore, this game has value 0.

Now consider the complete multi-partite graph where each partition has an even number of vertices. Right has no legal move, and would therefore lose going first. If Left goes first, he changes the parity of exactly one partition. On Right's turn, she must remove a vertex from a partition that contains an even number of vertices since these are the only ones that have odd degree. On Left's subsequent turn, he must also remove a vertex from a partition containing an even number of vertices since they are now the only ones with even degree. After exactly $2k$ moves, every partition will have had exactly 1 vertex removed. This leaves a graph where every partition has an odd number of vertices. We have already shown this has value 0. Since there were an even number of moves to get to this point, it would now be Left's turn and he would lose. Therefore, if every partition has an even number of vertices we again have a game with value 0.

Now consider any complete multi-partite graph with an even number of vertices. We know that there must be an even number of partitions that have an odd number of vertices. If Left begins, he must remove a vertex from a partition with an even number of vertices. Then Right must remove a vertex from a partition with an even number of vertices (since they now have odd degree). This continues until all partitions have odd degree. We know that Right has made the last move since we began with an even number of partitions with an even number of vertices. Since this position has value 0, Left has lost going first. If Right plays first we find that they will alternately remove a vertex from partitions with an odd number of vertices. After an even number of moves (so Left plays last) we have a position with value 0. Therefore, Right also loses playing first. Thus, all such graphs have value 0.

Finally, if we have a complete multi-partite graph with an odd number of vertices, there must be an odd number of partitions with an odd number of vertices. Therefore, both players have legal moves. Furthermore, both players' options are to graphs that have an even number of vertices and therefore have value 0. Thus, this game must have value $\{0|0\} = *$. ■

In the cases of an even number of partitions with one of size 1 or an odd number partitions, the values do appear alternate. However, the 'border' of exceptions grows. For example, the next tables list the values for $K_{m,n}$ and $K_{1,m,n}$ with $m \leq n$.

$m \backslash n$	1	2	3	4	5	6	
1	0	$\{2\|0\}$	0	$\{4\|0\}$	0	$\{6\|0\}$	0
2		0	$*$	0	$*$	0	$*$
3			0	$*$	0	$*$	0
4				0	$*$	0	$*$
5					0	$*$	0
6						0	$*$
7							0

Table 1: Values of $K_{m,n}$

$m \backslash n$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2		1	$\{3\|*\}$	1	$\{5\|*\}$	1	$\{7\|*\}$
3			1	$\{1\|*\}$	1	$\{1\|*\}$	1
4				1	$\{1\|*\}$	1	$\{1\|*\}$
5					1	$\{1\|*\}$	1
6						1	$\{1\|*\}$

Table 2: Values of $K_{1,m,n}$

3. Generalized Paths

Definition: A sequence $\{t_n\}_{n\geq 0}$ is **arithmetic periodic** with *period* p and *saltus* s if there is some $N \geq 0$ such that $t_{n+p} = t_n + s$ for all $n \geq N$.

Let G be a graph, $H \subseteq V(G)$ and P_n be a path with n vertices. Let $(G, H; P_n)$ be the graph formed by the disjiont union of G and P_n together with all the edges from one end vertex of P_n to all the vertices in H.

Conjecture: Given a graph G and a subset of vertices H, the values of (G, H, P_n) are arithmetic periodic.

As evidence for this conjecture, we do prove it for a restricted class of graphs that includes the paths.

Definition: Let $P(n,k)$ be the graph $\{w, v_1, v_2, \ldots, v_k, x_1, x_2, \ldots, x_n\}$ where $\{w, v_1, v_2, \ldots, v_k\}$ induces a star (i.e. $K_{1,k}$) with w as the center of the star; $wx_1 \in E(G)$ and $\{x_1, x_2, \ldots, x_n\}$ form a path.

Note that $P(n,0) = P(n-1,1)$ since we assume that $w \in P(n,k)$ for all n and k. When Left makes a move in $P(n,k)$ in the middle of the generalized path, it is natural to regard the new game as a disjunctive sum of $P(n',k)$, $n' < n$ and a path rather than the disjunctive sum of $P(n',k)$ and $P(n-n'-1,1)$. We, therefore, denote the path component as $P(m)$ and note that $P(m) = P(m-2,1)$.

The values, as we shall show in the next theorem, form patterns. In particular, as can be seen in Table 3, which was generated by CGSuite[9], there is a regularity of order 2 in k and arithmetic periodicity of length 3, saltus 1, in n.

$k\backslash n$	0	1	2	3	4	5	6	7
0	1	0	$\{2\|0\}$	2	1	$\{3\|1\}$	3	2
1	0	$\{2\|0\}$	2	1	$\{3\|1\}$	3	2	$\{4\|2\}$
2	$\{2\|0\}$	0	$\{3\|1\|\|0\}$	$1*$	$\{2\|1*\}$	2	$2*$	$\{3\|2*\}$
3	0	$\{4\|0\}$	$\{3\|1\}$	$\{5\|3\|\|1*\}$	$\{5\|\|2\|1*\}$	$\{4\|2\}$	$\{6\|4\|\|2*\}$	$\{6\|\|3\|2*\}$
4	$\{4\|0\}$	0	$\{5\|1\|\|0\}$	$1*$	$\{2\|1*\}$	2	$2*$	$\{3\|2*\}$
5	0	$\{6\|0\}$	$\{5\|1\}$	$\{7\|5\|\|1*\}$	$\{7\|\|2\|1*\}$	$\{6\|2\}$	$\{8\|6\|\|2*\}$	$\{8\|\|3\|2*\}$
6	$\{6\|0\}$	0	$\{7\|1\|\|0\}$	$1*$	$\{2\|1*\}$	2	$2*$	$\{3\|2*\}$
7	0	$\{8\|0\}$	$\{7\|1\}$	$\{9\|7\|\|1*\}$	$\{9\|\|2\|1*\}$	$\{8\|2\}$	$\{10\|8\|\|2*\}$	$\{10\|\|3\|2*\}$
8	$\{8\|0\}$	0	$\{9\|1\|\|0\}$	$1*$	$\{2\|1*\}$	2	$2*$	$\{3\|2*\}$

Table 3: Values of $P(n,k)$.

Theorem 13: The values for $P(n,k)$ are:

$P(0,2j) = P(1,2j-1) = \{2k \mid 0\}$ and $P(0,2j+1) = P(1,2j) = 0$;

$P(0,2) = \{2|0\}$, $P(1,2) = 0$, $P(2,2) = (3 \mid 1 \parallel 0)$; and, for $k > 2$, $P(2,2j+1) = \{2j+1 \mid 1\}$ and $P(2,2j+2) = \{2j+1 \mid 0\}$;

$P(3m) = \{m+1 \mid m-1\}$, $P(3m+1) = m+1$, and $P(3m+2) = m$;

and the values of $P(n,k)$ for $n \geq 3$, $k \geq 2$ are

n	$P(n,1)$	$P(n,2)$	$P(n,3)$
$3m$	m	$m*$	$\{m+4 \mid m+2 \parallel m*\}$
$3m+1$	$\{m+2 \mid m\}$	$\{m+1 \mid m*\}$	$\{m+4 \parallel m+1 \mid 1*\}$
$3m+2$	$m+2$	$m+1$	$\{m+3\|m+1\}$

n	$P(n,2j)$	$P(n,2j+1)$
$3m$	$m*$	$\{m+2j+2 \mid m+2j \parallel m*\}$
$3m+1$	$\{m+1 \mid m*\}$	$\{m+2j+2 \parallel m+1 \mid m*\}$
$3m+2$	$m+1$	$\{m+2j+1 \mid m+1\}$

Proof: The values of $P(n,k)$ for $k=0,1,2$ are arithmetic-periodic, period 3 and saltus 1 but they do not have the same description as those for $k>2$. As exceptions and as base cases we first obtain the values for $n=0,1,2$ and also for $k=0,1,2,3$.

For $n=0$ and $n=1$, the graph is $K_{1,k}$ and $K_{1,k+1}$ respectively. This gives that $P(0,2j) = P(1,2j-1) = \{2k \mid 0\}$ and $P(0,2j+1) = P(1,2j) = 0$.

For $n=2:$ $P(2,0) = 2 \mid 0$; $P(2,1) = \{1 \parallel 2 \mid 0\} = 2$; $P(2,2) = \{3 \mid 1 \parallel 0\}$. For $j>0$,

$$P(2,2j+1) = \{2j+1, (2j+1 \mid 0)+1 \parallel P(2,2j), P(1,2j+1)\} = \{2j+1 \parallel\mid (2j+1 \mid 1 \parallel 0)\}$$

after eliminating dominated options and thus $P(2,2j+1) = (2j+1 \mid 1)$ after replacing the reversible options. Also

$$P(2,2j+2) = \{\{2j+1 \mid 0\}+1 \parallel 2j+1, P(2,2j+1), P(1,2j+2)\} = \{2j+1 \mid 0\}.$$

We now consider $k=1,2$, and, as a basis for induction for the rest of the table, $k=3$.

$\mathbf{P(n)}$: Recall that $P(n) = P(n-1,0) = P(n-2,1)$. The period starts at $n=1$. It is straightforward to verify that $P(1)=1$, $P(2)=0$,, $P(3)=\{2|0\}$, $P(4)=2$, and $P(5)=1$. By the assumption of arithmetic-periodicity, the Left options of $P(n)$ all reduce to one of $P(1)+P(n-2)$, $P(2)+P(n-3)$, and $P(3)+P(n-4)$, $n-4>0$. The Right option is $P(n-1)$. For $m>1$,

$$P(3m) = \{m+1, m \mid m-2 \parallel m-1\} = \{m+1 \mid m-1\}.$$

Similarly, we find that

$$P(3m+1) = \{m+1 \mid m-1, m \parallel m+1 \mid m-1\} = \{m \mid \ \} = m+1,$$

since Left's first option and Right's option both reverse. Finally, $P(3m+2) = \{m+2 \mid m, m-1 \parallel m+1\} = m$.

The rest of the arguments in this proof are similar in nature. To abbreviate them, we present 'one-line' arguments. The first equality gives the distinct options, the second equality gives the undominated options and lastly, if necessary, reversible options are reversed to give the final value.

$\mathbf{P(n,2)}$: The period starts at $n=3$. We already have that $P(0,2)=\{2|0\}$, $P(1,2)=0$, and $P(2,2)=\{3 \mid 1 \parallel 0\}$. It is straightforward to verify that $P(3,2)=1*$, $P(4,2)=(2 \mid 1*)$, $P(5,2)=(4|2)$, and $P(6,2)=2*$.

By the assumption of arithmetic-periodicity, the Left options of $P(n, 2)$ are the exceptional values $P(0, 2) + P(n-1)$, $P(1, 2) + P(n-2)$, and $P(2, 2) + P(n-3)$ and the rest of the options, all of which, by the induction hypothesis, reduce to one of $P(3, 2) + P(n-4)$, $P(4, 2)+P(n-5)$, and $P(5, 2)+P(n-6)$. Right has three options: $P(n, 1) = P(n+2)$—take a leaf from the star; $P(n-1, 2)$—take a leaf from the path; or $2 + P(n)$—take the central vertex of the star. For, $m > 1$:

$$\begin{aligned} P(3m, 2) &= \{(m+1|m-1), m, ((3|1||0) + (m|m-2)), (m+1|m*) \\ &\qquad \| \ m, (m+3|m+1)\} \\ &= \{m, (m+1|m-1) \mid m\} \\ &= m* . \end{aligned}$$

$$\begin{aligned} P(3m+1, 2) &= \{m+1, m-1, (m+3|m+1||m), (m+1*|m-1*), |m-1*, m+1 \\ &\qquad \| \ (m+2 \mid m-1), m*\} \\ &= \{m+1 \mid m*\}. \end{aligned}$$

$$\begin{aligned} P(3m+2, 2) &= \{(m+2|m), (m+1|m-1), (m+2|m||m-1), m*, (m+1*|m), m \\ &\qquad \| \ m+2, (m+1|m*)\} \\ &= \{(m+2 \mid m), m \ \| \ m+2, (m+1 \mid m*)\} = \{m \mid \ \} \\ &= m+1. \end{aligned}$$

$\mathbf{P(n, 3)}$: The period does not start at $n = 0$ but at $n = 2$. It is easy to verify that $P(4, 3) = \{5 \ \| \ 2 \mid 1*\}$ and $P(5, 3) = \{4 \mid 2\}$. For the induction, we can now assume that $n > 5$. The Left options of $P(n, 3)$ are the exceptional values $3 + P(n)$, $P(n-1)$, $(4 \mid 0) + P(n-2)$ and those in the period which all reduce to one of $P(2, 3) + P(n-3)$, $P(3, 3) + P(n-4)$, and $P(4, 3) + P(n-5)$.

$$\begin{aligned} P(3m, 3) &= \{(m+4|m+2), m-1, (m+4|m), m+1*, (m+3|m+1 \ \| \ m-1*), \\ &\qquad (m+4 \ \| \ m+2|1*+m) \ \||\ m*, (m+2|m)\} \\ &= \{m+4 \mid m+2 \ \| \ m*\} \\ P(3m+1, 3) &= \{m+4, (m+1|m-1), (m+3|m-1), (m+3|m+1), \\ &\qquad (5||3|1*+(m|m-2)), (m+3||m|m-1*) \\ &\qquad \||| \ (m+4||m+2|m*), (m+1|m*)\} \\ &= \{m+4||m+1|m*\} \\ P(3m+2, 3) &= \{3+m, m+1, (m+3|m+1), (m+2|m), (m+5|m+3 \ \| \ m+1*), \\ &\qquad (5 \ \| \ 2|1*+m|m-2) \ \||\ m+1, (m+4 \ \| \ m+1 \mid 1*)\} \\ &= \{m+3 \mid m+1\} \end{aligned}$$

$\mathbf{P(n, k)}$, $k > 3$: We now cover this case by induction on $n + k$. Again, the periods do not start at $k = 0$ but at $k = 2$. We also have to divide the argument into k even and k odd.

The Left options of $P(n,2j)$ are the exceptional values $\{2j\mid\}0+P(n-1)$, $\{2j\mid 1\parallel 0\}+P(n-2)$ and those in the period which all reduce to one of $P(3,2j)+P(n-3)$, $P(4,2j)+P(n-4)$, and $P(5,2j)+P(n-5)$, where we require $n-5>0$. In the cases $n=3m+1,3m+2$ the last two Left options are dominated so the analysis does cover the cases $n=4,5$. The Right options are $2j+P(n)$, $P(n-2j-1)$, and $P(3n-1,2j)$.

$$\begin{aligned}
P(3m,2j) &= \{(2j+m-1|m-1), m, ((m|m-2)+(2j+1|1||0)),\\
&\quad m-1*, (m+1|m*),\\
&\quad ||| \ (m+2j+1|m+2j-1), (m+2j|m+2j-2||m+2j-4*), m\}\\
&= \{(2j+m-1|m-1), ((m|m-2)+(2j+1|1||0)), (m+1|m*) \ ||| \ m\}\\
&= \{m\mid m\}\\
&= m*
\end{aligned}$$

$$\begin{aligned}
P(3m+1,2j) &= \{(m+2j-1|m+1), m-1, (m+2j+1|m+1||m),\\
&\quad (m+1*|m-1*), (m|m-1*), m+1,\\
&\quad ||| \ (m+2j+1|m+2j-1), (m+2j||m+1|m*), m*\}\\
&= \{(m+2j-1|m+1), m+1 \ ||| \ m*\}\\
&= m+1|m*
\end{aligned}$$

$$\begin{aligned}
P(3m+2,2j) &= \{(m+2j+1|m+1), (m+1|m-1), (m+2j|m||m-1), m+1*,\\
&\quad (m+1*|m), m \ ||| \ m+2j, (m+2j-1\mid m), (m+1|m*)\}\\
&= \{(m+2j+1|m+1), m+1* \ \| \ (m+2j-1\mid m), (m+1|m*)\}\\
&= \{m+1* \ \| \ m+1\mid m*\} = m+1
\end{aligned}$$

The Left options of $P(n,2j+1)$ are the exceptional values $2j+1+P(n)$, $P(n-1)$, $(2j+2\mid 0)+P(n-2)$, and those in the period which all reduce to one of $P(2,2j+1)+P(n-3)$, $P(3,2j+1)+P(n-4)$, and $P(4,2j+1)+P(n-5)$, where we require $n-5>0$. However, in the cases $n=3m+1,3m+2$ the last two Left options are dominated, so the analysis does cover the cases $n=4,5$. The Right options are $P(n,2j)$ and $P(n-1,2j+1)$.

$$\begin{aligned}
P(3m,2j+1) &= \{(2j+m+2|m+2j), m-1, (m+2j+2|m), (m+2j+1|m+1),\\
&\quad (m+2j+2|m+2j||m), (m+2j+2||m+1|m*)\\
&\quad ||| \ m*, (m+2j|m)\}\\
&= \{2j+m+2|m+2j \ \| \ m*\}
\end{aligned}$$

$$\begin{aligned}
P(3m+1,2j+1) &= \{m+2j+2, (m+1|m-1), (m+2j+1|m+1),\\
&\quad ((2j+3|2j+1||1)+(m|m-2)), (n+2j+1||m|m-1*)\\
&\quad ||| \ (m+1|m*), (m+2j+2|m+2j||m*)\}\\
&= \{m+2j+2 \ \| \ (m+1|m*)\}
\end{aligned}$$

$$\begin{aligned}
P(3m+2,2j+1) &= \{m+2j+1, m+1, (m+2j+1|m+1), (m+2j|m),\\
&\quad (m+2j+3|m+2j+1||m+1), ((2j+3||2|1*)+(m|m-2))\\
&\quad ||| \ m+1, (m+2j+2 \ \| \ m+1\mid m*)\}\\
&= \{m+2j+1, (m+2j+1|m+1) \ \| \ m+1\}\\
&= \{m+2j+1\mid m+1\}.
\end{aligned}$$

This concludes the proof. ■

4. Variants

4.1. Even/Even

In this version of the game, both Left and Right play on an undirected graph and can only remove vertices which have even degree. Since both players have the same options from a given position, this is an impartial game. Therefore, the values we can obtain are limited to 0 and the nimbers $*n$ for any positive integer n.

Lemma 14. If $|V(G)|$ is odd then $G \in N$.

Proof: If $|V(G)|$ is odd then there must be a vertex of even degree. Therefore, there exists some legal move in G. On each of the first player's subsequent turns there will again be an odd number of vertices. Thus, on each of their turns, a legal move exists. Since the first player will always have a legal move, the game can only end if the second player is unable to move. That is, the first player will always win regardless of the moves he makes throughout the game and thus $G \in N$. ■

Lemma 15. If $|V(G)|$ is even then $G \in P$.

Proof. If the first player has a legal move it must be to a graph with an odd number of vertices. So, by Lemma 14, the next player (which is the second player in the original game) will win. Since the first player can never win, $G \in P$. ■

Theorem 16. When both players can remove only even degree vertices, the game is trivial and

$$G = \begin{cases} 0 & \text{if } |V(G)| \text{ is even;} \\ * & \text{if } |V(G)| \text{ is odd.} \end{cases}$$

Proof. By Lemma 15, we know that when $|V(G)|$ is even, the game has value 0. When $|V(G)|$ is odd, the only options available to each player are to games of value 0. Therefore, $G = \{0|0\} = *$. ■

4.2. Odd/Odd

In this version of the game, both players may only remove vertices which have odd degree. Much like the Even/Even version, both players have the same options from a particular position which makes this an impartial game as well.

Lemma 17 The path on n vertices, P_n has value $\begin{cases} * & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$

Proof. Since only the end vertices can be removed this is a version of she-loves-me-she-loves-me-not and is trivial. ∎

K_n has value $*$ if n is even and 0 if n is odd and $K_{m,n} = *$ if n and m are both odd and 0 otherwise. Many other graphs have value $*$ or 0. At the time of writing, there are only 2 known connected graphs with value $*2$.

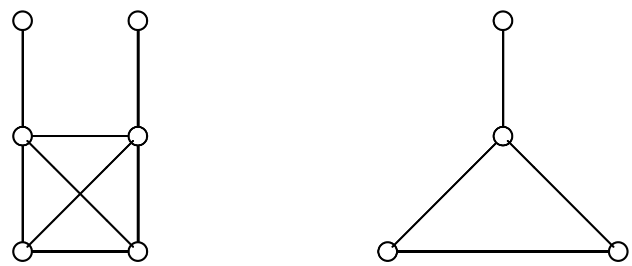

Figure 1: Graphs with value $*2$.

We can, however, use the disjunctive sum of two connected graphs with values $*$ and $*2$ to create a graph with value $*3$. This is too little evidence to suggest that the nim-values are restricted. We leave this as a question..

Question: For any $n \geq 0$ is there a graph G with $G = *n$?

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