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DYNAMICAL ZETA FUNCTIONS

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Abstract

These notes are a rather subjective account of the theory of dynamical zeta functions. They correspond to three lectures presented by the author at the "Numeration" meeting in Leiden in 2010.

1 A Selection of Zeta Functions

In its various manifestations, a zeta function $\zeta(s)$ is usually a function of a complex variable $s \in \mathbb{C}$. We will concentrate on three main types of zeta function, arising in three different fields, and try to emphasize the similarities and interactions between them.

There are three basic questions which apply equally well to all such zeta functions:

Question 1: Where is the zeta function defined and where does it have an analytic or meromorphic extension?

Question 2: Where are the zeros of $\zeta(s)$? What are the values of $\zeta(s)$ at particular values of s?

Question 3: What does this tell us about counting quantities?

We first consider the original and best zeta function.

1.1 The Riemann Zeta Function

We begin with the most familiar example of a zeta function.

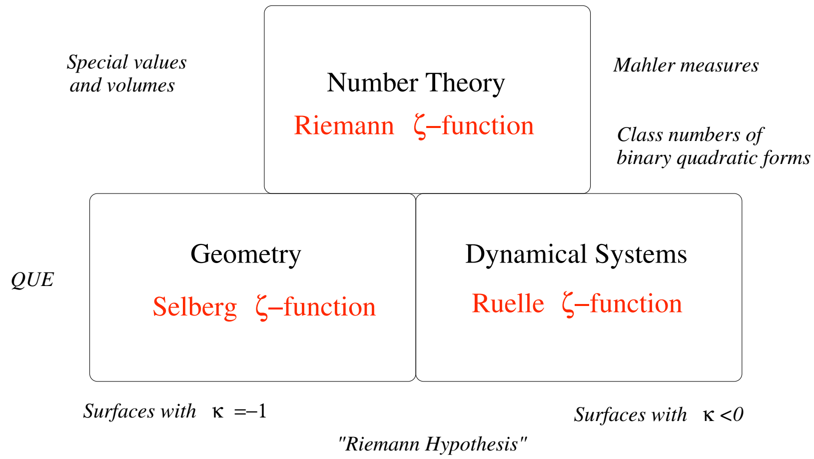


Figure 1: Three different areas and three different zeta functions

Figure 2: Riemann (1826-1866): His only paper on number theory was a report in 1859 to the Berlin Academy of Sciences on "On the number of primes less than a given magnitude" discussing $\zeta(s)$

The Riemann zeta function is the complex function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

which converges for $Re(s) > 1$. It is convenient to write this as an Euler product

$$\zeta(s) = \prod_p \left(1 - p^{-s}\right)^{-1}$$

where the product is over all primes $p = 2, 3, 5, 7, 11, \cdots$.

Theorem 1 (Basic properties). *The following basic properties are well know*

1. *$\zeta(s)$ has a simple pole at $s = 1$;*

2. $\zeta(s)$ otherwise has an analytic extension to the entire complex plane $\mathbb{C}$;

3. $\zeta(s)$ satisfies a functional equation (relating $\zeta(s)$ and $\zeta(1-s)$);

4. $\zeta(s)$ has zeros $\zeta(-2k) = 0$, for $k \geq 1$.

A brief and readable account of the Riemann zeta function is contained in the book [15].

1.2 Other Zeta Functions

Let us recall a few other well known zeta functions.

Example 2 (Weil zeta function from number theory). Let X be a (n-dimensional) projective algebraic variety over the field F_q with q elements. Let F_{q^m} be the degree m extension of F_q and let N_m denote the number of points of X defined over F_{q^m}. We define a zeta function by

$$\zeta_X(s) = \exp\left(\sum_{m=1}^{\infty} \frac{N_m}{m}(q^{-s})^m\right).$$

For the particular case of the projective line we have that $N_m = q^m + 1$ and $\zeta(s) = 1/(1-q^{-s})(1-q^{1-s})$. The reader can fine more details in [7].

Theorem 3 (Weil Conjectures: Grothendieck-Dwork-Deligne). *The function $\zeta_X(s)$ is rational in q^s. We can write $\zeta_X(s) = \prod_{i=0}^{2n} P_i(q^{-s})^{(-1)^i}$ where $P_i(\cdot)$ are polynomials and the zeros $\zeta_X(s)$ of occur where $Re(s) = \frac{k}{2}$.*

An important ingredient of the above theory is a version of the Lefschetz fixed point theorem. The original version of the Lefschetz fixed point theorem can also be used to study another zeta function.

Example 4 (The Lefschetz zeta function from algebraic topology). Let $T : \mathbb{R}^d/\mathbb{Z}^d \to \mathbb{R}^d/\mathbb{Z}^d$ be a linear hyperbolic toral automorphism

$$T(\underline{x} + \mathbb{Z}^d) = (A\underline{x} + \mathbb{Z}^d)$$

where $A \in SL(d, \mathbb{Z})$. Let $N_k = \mathrm{tr}(A^k)$ be the number of fixed points for T^k, for $k \geq 1$. Then we define the Lefschetz zeta function by

$$\zeta(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} N_k\right)$$

which converges for $|z|$ sufficiently small.

Lemma 5. *The function $\zeta(z)$ has a meromorphic extension to $\mathbb{C}$ as a rational function $P(z)/Q(z)$ with $P, Q \in \mathbb{R}[z]$*

The proof is an easy exercise if one knows the Lefschetz fixed point theorem (otherwise it is still relatively easy) cf., [27] . The Lefschetz fixed point theorem actually counts fixed (or periodic) points with weights ± 1, but in the present example the counting is the usual one.

Corollary 6. *If $\zeta(z)$ has simple poles and zeros α_i, $i = 1, 2, \cdots, n$, say, then there exist c_i, $i = 1, 2, \cdots, n$, such that for any $\theta > 0$,*

$$N_k = \sum_{i=1}^{n} c_i \alpha_i^{-k} + O(\theta^k),$$

The corollary is easily proved.

Exercise 7. Derive the Corollary from the Lemma.

Question 8. What happens if the poles or zeros are not necessarily simple?

We recall an exam question from Warwick University.

Exercise 9 (Example from MA424 exam (Warwick, May 2010)). Compute the zeta function for the hyperbolic toral automorphism $T : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ associated to

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Solution: In this case $N_m = \mathrm{tr}(A^m) - 2$, for $m \geq 1$. Thus

$$\zeta(z) = \exp\left(\sum_{m=1}^{\infty} \frac{z^m}{m}(\mathrm{tr}(A^m) - 2)\right) = \frac{1 - 2z}{\det(I - zA)}.$$

This is a rational function $\zeta(z) = \frac{1-2z}{2z^2-3z}$.

Remark 10. It would appropriate here to mention also the Milnor-Thurston zeta function (and kneading matrices) [13] and the Hofbauer-Keller zeta function for interval maps [6].

Finally, we recall a third type of zeta function.

Example 11 (Artin Mazur zeta function for Subshifts of finite type). Let $\sigma : \Sigma_A \to \Sigma_A$ be a mixing subshift of finite on the space of sequences

$$\Sigma_A = \{x = (x_n) \in \{1, \cdots, k\}^{\mathbb{Z}} : A(x_n, x_{n+1}) = 1 \text{ for } n \in \mathbb{Z}\}$$

where A is a $k \times k$ aperiodic matrix with entries either 0 or 1. Let $N_k = \mathrm{tr}(A^k)$ be the number of periodic points of period k. Then we define a discrete zeta function by

$$\zeta(z) = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k} N_k\right)$$

which converges for $|z|$ sufficiently small. In this case, the following is a simple, but fundamental, result [2].

Lemma 12. *The function $\zeta(z)$ has a meromorphic extension to $\mathbb{C}$ as the reciprocal of a polynomial, i.e., $\zeta(z) = 1/\det(I - zA)$.*

This follows easily by expressing the periodic points in terms of the traces $\mathrm{tr}(A^n)$ of the powers A^n.

Exercise 13. Prove the Lemma.

Using this lemma we can easily compute explicitly the zeta functions.

Exercise 14. Compute the zeta function for the hyperbolic toral automorphism of $\mathbb{R}^2/\mathbb{Z}^2$ associated to

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Corollary 15. *If A has simple eigenvalues α_i, $i = 1, 2, \cdots, k$, say, then there exist c_i, $i = 1, 2, \cdots, k$, such that for any $\theta > 0$:*

$$N_k = \sum_{i=1}^{k} c_i \alpha_i^{-k} + O(\theta^k)$$

This follows by considering the power series expansions.

Exercise 16. Derive the Corollary from the Lemma.

We recall another exam question from Warwick University.

Exercise 17 (Example from MA424 exam (Warwick, May 2009)). Compute the zeta function for the subshift of finite type associated to

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Solution: In this case $N_m = \mathrm{tr}(A^m)$, for $m \geq 1$. Thus

$$\zeta(z) = \frac{1}{\det(I - zA)}.$$

This is a rational function $\zeta(z) = \frac{-1}{z^3 - 3z^2 + z + 2}$.

Finally, a simple variant of this zeta function is to not merely count the closed orbits, but to also weight the orbits by some function.

Example 18 (Ruelle zeta function). Let $\sigma : \Sigma_A \to \Sigma_A$ be a mixing subshift of finite and let $f : \Sigma_A \to \mathbb{R}$ be Hölder, with respect to the metric

$$d(x, y) = \sum_{n=-\infty}^{\infty} \frac{e(x_n, y_n)}{2^n}$$

Given $s \in \mathbb{C}$ we let

$$Z_n(-sf) = \sum_{\sigma^n x = x} \exp\left(-s \sum_{i=0}^{n-1} f(\sigma^i x)\right)$$

be the weighted sum over periodic orbits of period n. Then we define a zeta function of two variables $s, z \in \mathbb{C}$ by

$$\zeta(z, s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(-sf)\right)$$

which converges for $|z|$ sufficiently small. In particular, if $\sigma \in \mathbb{R}$ and we define

$$P(-\sigma f) := \limsup_{n \to +\infty} \frac{1}{n} \log Z_n(-\sigma f)$$

then $\zeta(z, s)$ is analytic for $|z| < e^{P(-Re(s)f)}$ [20].

Lemma 19. *There exists* $0 < \theta < 1$ *such that* $\zeta(z, s)$ *has a meromorphic extension to* $|z| < e^{P(-Re(s)f)}/\theta$.

A particularly illuminating example is where the function is locally constant.

Exercise 20. Let $f : \Sigma_A \to \mathbb{R}$ depend only on the zeroth and first coordinates x_0 and x_1 (i.e., $f(x) = f(x_0, x_1)$). How does one write $\zeta(z, s)$ in terms of the matrix $A_s = (A(i, j)e^{-sf(i,j)})$?

In the particular case that f is identically zero, we recover the Artin-Mazer zeta function.

The Ruelle zeta function is a useful bridge between certain zeta functions for flows and those for discrete maps. We will return to this when we discuss suspension flows. However, in order to give a more coherent presentation we will typically present the Ruelle zeta function in the context of geodesic flows for surfaces of variable negative curvature, giving a natural generalization of the Selberg zeta function for surfaces of constant negative curvature.

2 Selberg Zeta Function

A particularly important zeta function in both analysis and geometry is the Selberg zeta function. A comprehensive account appears in [5]. A lighter account appears in [28].

2.1 Hyperbolic Geometry and Closed Geodesics

Perhaps one of the easier routes into understanding dynamical zeta functions for flows is via geometry. Assume that V is a compact surface with constant curvature $\kappa = -1$.

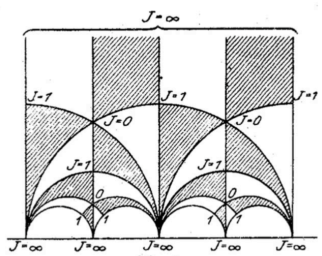


Fig. 7.

Figure 3: The upper half plane $\mathbb{H}^2$ with the Poincaré metric. In this hand-drawn figure from Felix Klein's 1878 paper the triangles have the same size and only seem smaller as we approach the boundary in the Euclidean metric. (The values at the vertices are those of a related modular form.)

The covering space is the upper half plane $\mathbb{H}^2 = \{z = x + iy : y > 0\}$ with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

(with constant curvature $\kappa = -1$). In particular, compared with the usual Euclidean distance, the distance in the Poincaré metric tends to infinity as y tends to zero, to the extent that geodesics never actually reach the boundary.

Lemma 21. *There are a countable infinity of closed geodesics.*

Proof. There is a one-one correspondence between closed geodesics and conjugacy classes in the fundamental group $\pi_1(V)$. The group $\pi_1(V)$ is finitely generated and countable, as are its conjugacy classes. $\square$

We denote by γ one of the countably many closed geodesics on V. We can write $l(\gamma)$ for the length of γ. We can also interpret the closed geodesics dynamically.

Example 22 (Geodesic flow). Let V be a compact surface with curvature $\kappa = -1$, for $x \in V$. Let

$$M = SV = \{(x, v) \in TM : \|v\|_x = 1\}$$

denote the unit tangent bundle. then we let $\phi_t : M \to M$ be the geodesic flow, i.e., $\phi_t(v) = \dot{\gamma}(t)$ where $\gamma : \mathbb{R} \to V$ is the unit speed geodesic with $\dot{\gamma}(0) = (x, v)$.

This also corresponds to parallel transporting tangent vectors along geodesics.

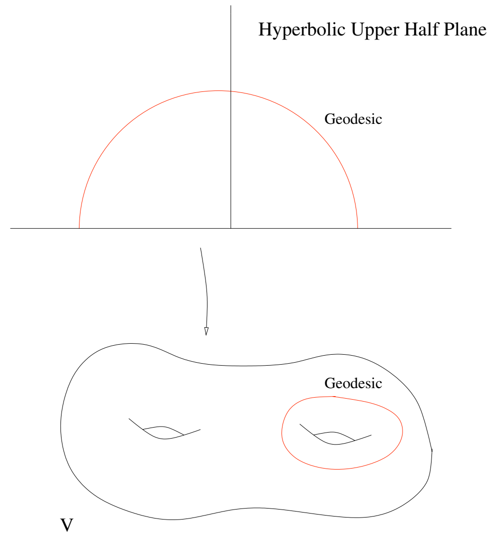


Figure 4: The upper half plane $\mathbb{H}^2$ is the universal cover for V. Geodesics on the upper half plane $\mathbb{H}^2$ project down to geodesics on V. Sometimes they may be closed geodesics.

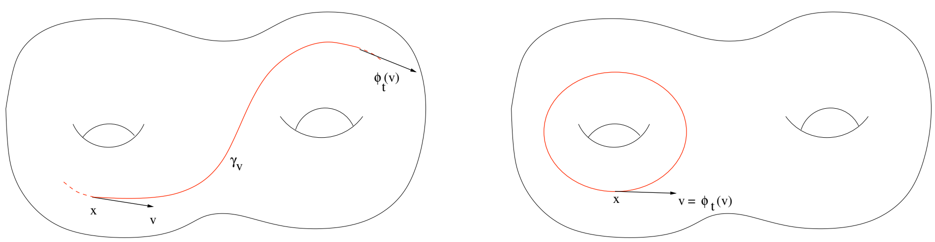


Figure 5: Periodic orbits for the geodesic flow correspond to closed geodesics on surface V.

2.2 Definition of the Selberg Zeta Function

We now come to the definition of the zeta function.

Definition 23. The Selberg zeta function is a function of $s \in \mathbb{C}$ formally defined by

$$Z(s) = \prod_{n=0}^{\infty} \prod_{\gamma} \left(1 - e^{-(s+n)l(\gamma)}\right).$$

The (first) main result on the Selberg zeta function is the following:

Theorem 24. *$Z(s)$ converges for $Re(s) > 1$ and extends analytically to $\mathbb{C}$.*

The original proof used the Selberg trace formulae [5], but there are alternative proofs using continued fraction transformations, as we will see later.

Figure 6: Selberg (1917-2007): In 1956 he proved a trace formula giving a correspondence between the lengths of closed geodesics on a compact Riemann surface and the eigenvalues of the Laplacian.

The "extra" product over n is essentially an artifact of the method of proof. However, it can easily be put back into a form closer to that of the Riemann zeta function.

Remark 25. If we denote $\zeta_V(s) = Z(s+1)/Z(s)$ then we can write

$$\zeta_V(s) = \prod_{\gamma} \left(1 - e^{-sl(\gamma)}\right)^{-1}$$

which is perhaps closer in appearance to the Riemann zeta function

$$\zeta(s) = \prod_{\gamma} \left(1 - p^{-s}\right)^{-1}.$$

We can summarize this in the following table:

Riemann (primes p)	Selberg (closed orbit γ)
p $\zeta(s)$	$e^{l(\gamma)}$ $\zeta_V(s) = \frac{Z(s+1)}{Z(s)}$
pole at $s = 1$ extension to $\mathbb{C}$	pole at $s = 1$ extension to $\mathbb{C}$

3 Riemann Hypothesis

We return to the Riemann zeta function and one of best known unsolved problems in mathematics. The following conjecture was formulated by Riemann in 1859 (repeated as Hilbert's 8th problem).

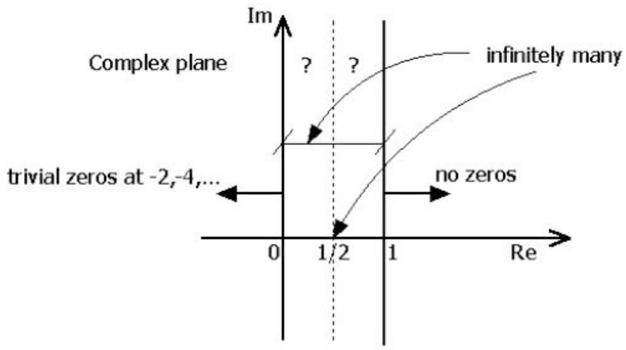


Figure 7: The Riemann zeta function extends to the entire complex plane, but the location of the zeros remains to be understood.

Question 26 (Riemann Hypothesis). The non-trivial zeros lie on $Re(s) = \frac{1}{2}$?

There is some partial evidence for this conjecture. An early result was the following.

Theorem 27 (Hardy). *There are infinitely many zeros lie on the line* $Re(s) = \frac{1}{2}$.

Figure 8: G.H. Hardy (1877-1947) proved that there are infinitely many zeros of $\zeta(s)$ on $Re(s) = \frac{1}{2}$ in 1914.

In 1941, Selberg improved this to show that at least a (small) positive proportion of zeros lie on the line.

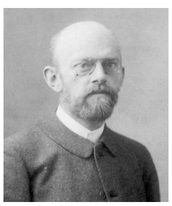
Remark 28. The Riemann Conjecture topped Hardy's famous wish list from the 1920s:

1. Prove the Riemann Hypothesis.

2. Make 211 not out in the fourth innings of the last test match at the Oval.
3. Find an argument for the nonexistence of God which shall convince the general public.
4. Be the first man at the top of Mount Everest.
5. Be proclaimed the first president of the U.S.S.R., Great Britain and Germany.
6. Murder Mussolini.

3.1 The Hilbert-Polya Approach to the Riemann Hypothesis

Hilbert and Polya are associated with the idea of trying to understand the location of the zeros of the Riemann zeta functions in terms of eigenvalues of some (as of yet) undicovered self-adjoint operator whose necessarily real eigenvalues are related to the zeros.


Hilbert (1862-1943)


Polya (1887-1985)

This idea has yet to reach fruition for the Riemann zeta function (despite interesting work of Berry-Keating, Connes, etc.) but the approach works particularly well for the Selberg Zeta function. (Interestingly, Selberg won the Field's medal, in part, for showing that the Riemann zeta function was not needed in counting primes.)

Remark 29. There are also interesting connections with random matrices and moments of zeta functions (Keating-Snaith).

However, this philosophy has been particularly successful in the context of the Selberg zeta function, as we shall see in the next subsection.

3.2 Zeros of the Selberg Zeta Function

The use of the Laplacian in extending the zeta function leads to a characterization of its zeros. For simplicity, assume again that $V = \mathbb{H}^2/\Gamma$ is compact.

Let $\Delta : L^2(V) \to L^2(V)$ be the Laplacian (or Laplace-Beltrami operator) given by

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

(i.e., the operator is actually defined on the $C^\infty(M)$ functions and extends to the square integrable functions in which they are dense)

Lemma 30. *The Laplacian is a self-adjoint operator. In particular, its spectrum lies on the real line.*

We are interested in the solutions

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

for the eigenvalue equation $\Delta\phi_n + \lambda_n\phi_n$. The location of the zeros s_n for $Z(s)$ (i.e., $Z(s_n) = 0$) are described in terms of the eigenvalues λ_n for the Laplacian by the following result (cf. [5]).

Theorem 31. *The zeros of the Selberg zeta function $Z(s)$ can be described by:*

1. *$s = 1$ is a zero.*
2. *$s_n = \frac{1}{2} \pm i\sqrt{\frac{1}{4} - \lambda_n}$, for $n \geq 1$, are "spectral zeros".*
3. *$s = -m$, for $m = 0, 1, 2, \cdots$, are "trivial zeros".*

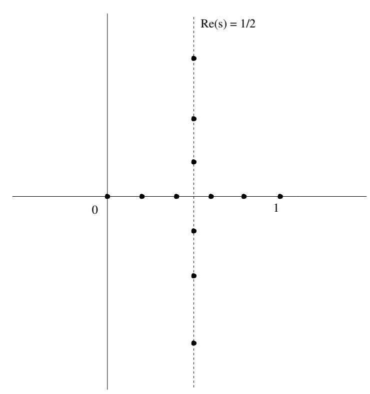


Figure 9: Spectral zeros of the Selberg Zeta function The spectral zeros lie on the "cross" $[0, 1] \cup [\frac{1}{2} - i\infty, \frac{1}{2} + i\infty]$.

Remark 32. A somewhat easier setting for studying Laplacians is on the standard flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. When $d = 1$ then $\Delta = \frac{d^2}{dx^2}$ and self-adjointness is essentially integration by parts, i.e., $\int(\Delta\psi)\phi dx = \int\psi(\Delta\phi)dx$, the eigenvalues are $\{n^2\}$ and the eigenfunctions are $\psi_{nm}(x, y) = e^{2\pi inx}$. When $d = 2$ then $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}$ then the eigenvalues are $\{n^2 + m^2\}$ and the eigenfunctions are $\psi_{nm}(x, y) = e^{2\pi i(nx+my)}$.

Remark 33. The asymptotic estimate on growth eigenvalues of the Laplace operator comes from Weyl's theorem.

The proof of these results is usually based on using the Selberg trace formula to extend the logarithmic derivative $Z'(s)/Z(s)$ [5]. In an appendix we outline the strategy to proving the theorem.

If V is non-compact, but of finite area (such as the modular surface), then there are extra zeros which correspond to those for the Riemann zeta function. In view of the Riemann hypothesis, this illustrates the difficulty is understanding the zeros in that case.

4 Dynamical Approach

We would like to use a more dynamical approach to study the Selberg zeta function. This allows us to recover some of the classical results (such as the existence of an analytic extension to $\mathbb{C}$) and this different perspective also allows us to get additional bounds on the zeta function. With this perspective, the ideas are a little clearer if we study the special case of the modular surface.

We begin with a little motivation for this viewpoint, which includes a connection between closed geodesic for the modular flow and periodic points for the continued fraction transformation.

4.1 Binary Quadratic Forms

Let us begin with a simple connection between closed geodesics and number theory. Consider quadratic forms

$$Q(x, y) = Ax^2 + Bxy + Cy^2$$

where $A, B, C \in \mathbb{Z}$.

Definition 34. Two quatratic forms $Q_1(x, y)$ and $Q_2(x, y)$ are said to be equivalent if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$ with $\alpha\delta - \beta\gamma = 1$ and $Q_1(x, y) = Q_2(\alpha x + \beta y, \gamma x + \delta y)$.

A fundamental quantity in number theory is that of the Class Number. We recall the definition.

Definition 35. Given $d > 0$, the class number $h(d)$ is the number of inequivalent quadratic forms with given discriminant $d := B^2 - 4AC$.

There is a correspondence between equivalence classes of quadratic forms and closed geodesics. In particular, the distinct roots x, x' of $Ax^2 + Bx + C = 0$ determine a unique a unique geodesic $\widehat{\gamma}$ on $\mathbb{H}^2$ with $\widehat{\gamma}(+\infty) = x$ and $\widehat{\gamma}(-\infty) = x'$ which quotients down to a closed geodesic γ on V.

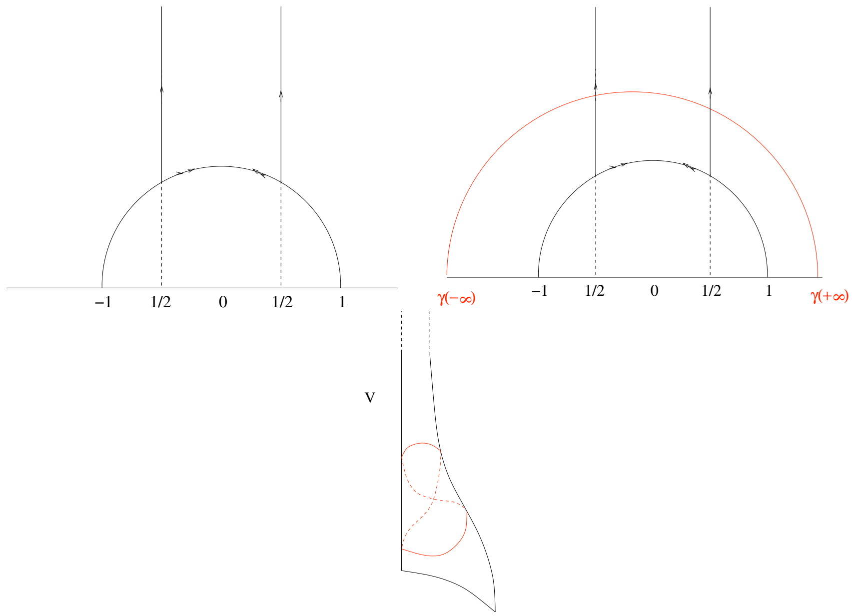


Figure 10: Geodesics on the Poincaré upper half plane are determined by their two endpoints. Moreover, they project down to geodesics on the modular surface. Of particular interest is when this image geodesic is closed.

On the other hand, the group $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{\pm I\}$ acts isometrically on $\mathbb{H}^2$ as linear fractional transformations, i.e.,

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ acts by } g : z \mapsto \frac{az+b}{cz+d}.$$

The quotient space $V = \mathbb{H}^2/PSL(2,\mathbb{Z})$ is the modular surface.

The following result dates back to Sarnak [21].

Lemma 36. *Equivalence classes $[Q]$ of quadratic forms $Q(x,y) = Ax^2+Bxy+Cy^2$ (of discriminant d) correspond to closed geodesics γ of length*

$$l(\gamma) = 2\log\left(\frac{t+u\sqrt{d}}{2}\right).$$

where $t^2 - du^2 = 4$ and (t,u) is a fundamental solution. The class number $h(d)$ corresponds the multiplicity of distinct closed geodesics with the same length.

Of course, the lengths are constrained by the fact that $e^{l(\gamma)} + e^{-l(\gamma)} \in \mathbb{Z}^+$ (since it is the trace of an element of $SL(2,\mathbb{Z})$).

Exercise 37. Show that closed geodesics can have a high multiplicity.

Remark 38. Gauss observed that the dependence $h(d)$ on d is very irregular. Siegel showed that:

$$\sum_{d \in D, d \leq x} h(d) \log \epsilon_d \sim \frac{\pi^2 x^{3/2}}{18\zeta(3)} \text{ as } x \to +\infty.$$

However, using the above correspondence Sarnak [21] gave an asymptotic formula for sums of class numbers:

$$\frac{1}{\mathrm{Card}(\{d \in D : \epsilon_d \leq x\})} \sum_{\{d \in D \,:\, \epsilon_d \leq x\}} h(d) \sim \frac{8}{35} \frac{x}{2 \log x} \text{ as } x \to +\infty.$$

It suffices to restrict our attention to considering the lifts $\widehat{\gamma}$ of closed geodesics γ for which the endpoints satisfy $\widehat{\gamma}(+\infty) > 1$ and $0 < \widehat{\gamma}(-\infty) < 1$. In particular, we can write

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}}$$

This is suggestive of the connection with the Gauss map. (The values a_n have a geometric interpretation whereby they reflect the number of times the geodesic wraps around the cusp on each excursion into the cusp.)

4.2 Discrete Transformations

We can briefly describe the properties of the general class of maps we want to study. Let $X \subset \mathbb{R}$ be a countable union of closed intervals $X = \coprod_i X_i$ and let $T : X \to X$ be a map such that:

1. The restrictions $T|X_i$ are all real analytic.
2. The map T is expanding, i.e., there exists $\beta > 1$ such that $\inf_{x \in X} |T'(x)| \geq \beta$. (More generally, we can assume the map T is (eventually) expanding, i.e., there exists $\beta > 1$ and $N \geq 1$ such that $\inf_{x \in X} |(T^N)'(x)| \geq \beta$.)
3. The map is Markov (i.e., each image $T(X_i)$ is a finite union of sets from $\{X_j\}$).

The main example we are interested in is particularly well known:

Example 39 (Gauss map, or Continued Fraction Transformation). Let $T : [0, 1] \to [0, 1]$ be the well known Gauss map, or continued fraction map, defined by the fractional part of the reciprocal:

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right] \in [0, 1].$$

(We ignore the complication with the point 0, in the great tradition of ergodic theory). In this example $X = \coprod_n X_n$ where $X_n = [\frac{1}{n+1}, \frac{1}{n}]$. In particular,

$$T'(x) = -\frac{1}{x^2}$$

and we can check that T is eventually expanding since $|(T^2)'(x)| \geq 4$. Furthermore, for each X_i we have $TX_i = [0,1] = X$ confirming the Markov property.

Lemma 40 (Artin, Hedlund, Series [26]). *There is a bijection between the periodic orbits of T (of even period) and the closed geodesics for $\mathbb{H}^2/\Gamma$ where $\Gamma = PSL(2,\mathbb{Z})$.*

A natural generalization of the Gauss map is the following.

Example 41 (Hurewicz-Nakada continued fractions). We can first consider the Hurewicz continued fraction maps $f : \left[-\frac{1}{2}, \frac{1}{2}\right] \to \left[-\frac{1}{2}, \frac{1}{2}\right]$ before by

$$f(x) - \frac{1}{x} - \langle -\frac{1}{x} \rangle$$

where $\langle \xi \rangle$ is the nearest integer to ξ (cf. [3]). More generally, for $q = 4, 5, 6 \cdots$, let $\lambda_q = 2\cos(\pi/q)$. We define

$$f : \left[-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}\right] \to \left[-\frac{\lambda_q}{2}, \frac{\lambda_q}{2}\right]$$

by

$$f(x) = -\frac{1}{x} - \frac{1}{x}\langle -\frac{1}{x} \rangle_q \lambda_q$$

where $\langle \xi \rangle_q := [\xi/\lambda_q + 1/2]$.

Lemma 42 (Mayer and Mühlenbruch). *There is a bijection between the periodic orbits of f and the closed geodesics for $\mathbb{H}^2/\Gamma$ where $\Gamma = \langle S, T_q \rangle$ is the Hecke group generated by*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } T_q = \begin{pmatrix} 1 & \lambda_q \\ 0 & 1 \end{pmatrix}.$$

Let us next consider a different type of generalization of the modular surfaces and the continued fraction transformation.

Example 43 (Principle congruence subgroups). For each $N \geq 1$ we can consider the subgroups $\Gamma(N) < PSL(2,\mathbb{Z})$ such that

$$\Gamma(N) = \{A \in PSL(2,\mathbb{Z}) : A = I \text{ (mod N)}\}.$$

The geodesic flow on $\mathbb{H}^2/\Gamma(N)$ is a finite extension of the geodesic flow on the modular surface with the finite group

$$G := \Gamma/\Gamma(N) = SL(2, \mathbb{Z}/N\mathbb{Z}).$$

The new transformation is a skew product over the continued fraction map: $\widehat{T} : [0,1] \times G \to [0,1] \times G$, i.e., $\widehat{T}(x,g) = (Tx, g(x)g)$ where $g : [0,1] \to G$.

Lemma 44 (Chung-Mayer, Kraaikamp-Lopes). *There is a bijection between the periodic orbits of $\widehat{T}$ and the closed geodesics for $\mathbb{H}^2/\Gamma(N)$.*

One can also consider higher dimensional analogues of continued fraction transformations [22]. Let $X \subset \mathbb{R}^d$ be a subset of Euclidean space and let $X = \coprod_i X_i$ be a union of sets (where $\mathrm{int}X_i \cap \mathrm{int}X_j = \emptyset$ and $X_i = \mathrm{cl}(\mathrm{int}X_i)$). Let $T : X \to X$ be a map such that:

1. The restrictions $T_i = T|X_i$ are real analytic.

2. The map T is expanding, i.e., there exists $\beta > 1$ such that $\|DT(v)\|/\|v\| \geq \beta$.

3. The map is Markov (i.e., the image $T_i(X_i)$ is a finite unions of sets from $\{X_i\}$).

The following example is useful in studying both Interval Exchange Transformations and Teichmüller flows for translation surfaces.

Example 45 (Rauzy-Veech-Zorich). Let us consider a continued fraction transformation on a finite union of copies of the $(n-1)$-dimensional simplices Δ. Let $\mathcal{R}$ be (a subset of) the permutations on k-symbols and partition $\Delta \times \mathcal{R} = \cup_{\pi \in \mathcal{R}} \Delta_\pi^+ \cup \Delta_\pi^-$ where

$$\Delta_\pi^+ = \{(\lambda, \pi) \in \Delta \times \{\pi\} : \lambda_n > \lambda_{\pi^{-1}n}\} \text{ and}$$
$$\Delta_\pi^- = \{(\lambda, \pi) \in \Delta \times \{\pi\} : \lambda_n < \lambda_{\pi^{-1}n}\}$$

(up to a set of zero measure). Define a map $\mathcal{T}_0 : \Delta \times \mathcal{R} \to \Delta \times \mathcal{R}$ by

$$\mathcal{T}_0(\lambda, \pi) = \begin{cases} \left(\frac{(\lambda_1, \cdots, \lambda_{n-1}, \lambda_n - \lambda_k)}{1 - \lambda_k}, a\pi \right) & \text{if } \lambda \in \Delta_\pi^- \\ \left(\frac{(\lambda_1, \cdots, \lambda_{k-1}, \lambda_k - \lambda_n, \lambda_n, \lambda_{k+1}, \cdots, \lambda_{n-1})}{1 - \lambda_n}, b\pi \right) & \text{if } \lambda \in \Delta_\pi^+ \end{cases}$$

where $a\pi$ and $b\pi$ are suitably defined new permutations Actually, this does not quite satisfy the expansion assumption and so one accelerate the map following Zorich, Bufetov and Marmi-Yoccoz, etc.

The Rauzy-Veech-Zorich transformation is also natural generalization of the Gauss map by virtue of its analogous geometric interpretation. We first recall that the modular surface $V = \mathbb{H}^2/SL(2,\mathbb{Z})$ gives a classical description of the space of flat metrics on a torus. In particular, we can consider the vectors 1 and $\xi = x + iy \in \mathbb{H}^2$ and associate the geometric torus $T = \mathbb{R}^2/(\mathbb{Z} + \xi\mathbb{Z})$. However, replacing ξ by $\gamma(\xi)$, with $\gamma \in PSL(2,\mathbb{Z})$ then we get the same surface, so the moduli space of metrics corresponds to $\mathbb{H}^2/SL(2,\mathbb{Z})$.

More generally, we can replace the torus by a surface of higher genus with a flat metric (i.e., translation surfaces). Then the continued fraction transformation

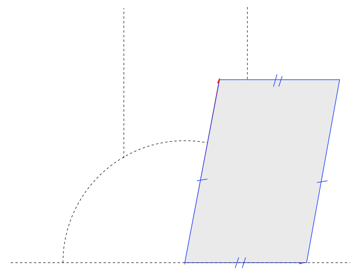


Figure 11: The metrics on the tori correspond to points on the Modular surface

is replaced by the Rauzy-Veech-Zorich transformation, which encodes orbits for an associated flow (the Teichmüller flow). Actually, this does not quite satisfy all of the properties we might want, and so one modifies the map (essentially by inducing) following ideas of Bufetov or Marmi-Moussa-Yoccoz, etc.

The associated zeta function is clearly related (in a way which is still not well understood) to the closed geodesics for the Teichmüller flow on the space of translation surfaces (by analogy with the geodesic flow on the Modular surface).

Exercise 46. Is there a connection with the asymptotic results of Athreya-Bufetov-Eskin-Mirzakhani for Teichmüller space?

When one discusses higher dimensional continued fraction transformations, it is natural to mention the most famous example:

Example 47 (Jacobi-Perron). We can define this two dimensional analogue $T : [0,1]^2 \to [0,1]^2$ of the Gauss map by

$$T(x,y) = \left(\frac{y}{x} - \left[\frac{y}{x}\right], \frac{1}{x} - \left[\frac{1}{x}\right]\right).$$

Unfortunately, it is not clear at the present time how the zeta function gives interesting insights into this map (e.g., the invariant density, which is not explicitly known, or the Lyapunov exponents).

4.3 Suspension Flows

A basic construction in ergodic theory is to associate to any discrete transformation a flow on the area under the graph of a suitable positive function. This flow is called the suspension flow (or special flow). More precisely, given a function $r : X \to \mathbb{R}^+$ we define a new space by

$$Y = \{(x,u) \in X \times \mathbb{R} : 0 \le u \le r(x)\}$$

with the identification of $(x, r(x))$ with $(T(x), 0)$. The (semi)-flow is defined on the space Y locally by $\psi_t(x, u) = (x, u + t)$, subject to the identification, i.e,

$$\psi_t(x, u) = (T^n x, t + u - \sum_{i=0}^{n-1} r(T^i x)) \text{ where } 0 \leq t + u - \sum_{i=0}^{n-1} r(T^i x)) \leq r(x).$$

In the present context, a natural choice of function is given by $r(x) = \log |T'(x)|$ (or, more generally, in higher dimensions $r(x) = \log |\mathrm{Jac}(T)(x)|$).

Lemma 48. *If $r(x) = \log |T'(x)|$ then the flow ψ has entropy $h(\psi) = 1$ and the measure of maximum entropy is absolutely continuous.*

This is a simple exercise using the variational principle for pressure [14].

Exercise 49. Prove the lemma.

We can now apply this to the continued fraction transformation.

Example 50. The primary example is the continued fraction transformation $T(x) = \{1/x\}$ and then $T'(x) = 1/x^2$ and thus $r(x) = -2 \log x$.

4.4 Transfer Operators

In order to extend the domain of the zeta function by dynamical methods it is convenient to introduce a family of bounded linear operators.

Definition 51. Given $T : X \to X$ and $s \in \mathbb{C}$ we can associate the transfer operator $\mathcal{L}_s : C^0(X) \to C^0(X)$ by

$$\mathcal{L}h(x) = \sum_{T(X_i) \supset X_j} |\mathrm{Jac}(DT_i)(x)|^s f(T_i x) \text{ where } x \in X_j$$

provided the series converges.

We want to consider a smaller space of functions, in order that the spectrum should be sufficiently small to proceed further. To begin, assume that we choose a complex neighbourhood $U \supset X$. We can then consider the space of analytic functions $h : U \to \mathbb{C}$ which are analytic and bounded. This forms a Banach space, which we denote by $\mathcal{A}(U)$ with the supremum norm $\| \cdot \|_\infty$.

Definition 52. We recall that a nuclear operator $\mathcal{L}$ (or order zero) is one for which there are families of functions $w_n \in \mathcal{A}(U)$ and functionals $l_n \in \mathcal{A}(U)^*$ (with $\|w_n\| = 1 = \|l_n\|$) such that

$$\mathcal{L}(\cdot) = \sum_{n=0}^{\infty} \alpha_n w_n l_n(\cdot).$$

where $|\alpha_n| = O(\theta^n)$, for some $0 < \theta < 1$.

In the case that there are only finitely many branches we have the following result of Ruelle [23].

Lemma 53 (Ruelle). *Assume that whenever $T(X_i) \supset X_j$ we have that the closure of the image $T_i(U)$ is contained in U. It then follows that each operator $\mathcal{L}_s : \mathcal{A}(U) \to \mathcal{A}(U)$ as defined above is nuclear.*

The easiest way to understand this lemma is through considering simple examples. The basic idea can be seen in the following.

Exercise 54. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $|\lambda| < 1$. By considering the power series expansion about $z = 0$ that the operator $\mathcal{M} : \mathcal{A}(U) \to \mathcal{A}(U)$ defined by $\mathcal{M}f(z) = f(\lambda z)$ is nuclear.

In particular, nuclear operators have countably many eigenvalues and are trace class. We can now use the following result:

Lemma 55 (Grothendieck). *We can write*

$$det(I - z\mathcal{L}) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} tr(\mathcal{L}_s^n)\right)$$

Moreover, we have the following characterizations of the traces due to Ruelle [23].

Lemma 56 (Ruelle). *We have the following:*

1. *The trace of $\mathcal{L}_s : \mathcal{A}(U) \to \mathcal{A}(U)$ is*

$$tr(\mathcal{L}_s) = \sum_{Tx=x} \frac{|Jac(DT)(x)|^{-s}}{det(I - D_x T^{-1})}.$$

2. *For each $n \geq 1$ the trace of $\mathcal{L}_s^n : \mathcal{A}(U) \to \mathcal{A}(U)$ is*

$$tr(\mathcal{L}_s^n) = \sum_{Tx=x} \frac{|Jac(DT)(x)|^{-s}}{det(I - D_x(T^n)^{-1})}.$$

Of particular interest is the one dimensional case.

Corollary 57. *In the particular case that X is one dimensional we have that:*

$$tr(\mathcal{L}_s) = \sum_{Tx=x} \frac{|T'(x)|^{-s}}{det(I - (T'(x))^{-1}}$$

Of particular interest is the special case of the continued fraction transformation, which was studied by Mayer [11].

Example 58 (Mayer). Consider the Gauss map $T : [0,1] \to [0,1]$ is defined by

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right]$$

Let $U = \{z \in \ : |z-1| < \frac{3}{2}\}$ then we can write

$$\mathcal{L}_s w(z) = \sum_{n=1}^{\infty} \frac{1}{(z+n)^{2s}} w\left(\frac{1}{z+n}\right)$$

for $w \in \mathcal{A}(U)$. In particular, we can write $\mathcal{L}_s = \sum_{n=1}^{\infty} L_{n,s}$ where

$$\mathcal{L}_{s,n} w(z) = \frac{1}{(z+n)^{2s}} w\left(\frac{1}{z+n}\right)$$

are again trace class, and thus $\mathrm{tr}(\mathcal{L}_s) = \sum_{n=1}^{\infty} \mathrm{tr}(L_{n,s})$. We can solve the eigenfunction equation $\mathcal{L}_{s,n} w(z) = \lambda w(z)$ to deduce that the eigenvalues are $\{(z_n + n)^{-2s+2m} : m \geq 0\}$ where z_n satisfies $1/(z_n+n) = z_n$, i.e., $z_n = [\overline{n}]$ (as a periodic continued fraction). This is achieved by evaluating both sides at the fixed point, and differentiating if w is zero at this point. Thus we can write

$$\mathrm{tr}(\mathcal{L}_{n,s}) = \frac{[\overline{n}]^{2s}}{1-[\overline{n}]^2} \text{ and } \mathrm{tr}(\mathcal{L}_s) = \sum_{n=1}^{\infty} \frac{[\overline{n}]^{2s}}{1-[\overline{n}]^2}.$$

More generally, we can write

$$\mathrm{tr}(\mathcal{L}_s^k) = \sum_{a_1,\cdots,a_k=1}^{\infty} \frac{[\overline{a_1,\cdots,a_k}]^{2s}}{1-[\overline{a_1,\cdots,a_k}]^2}.$$

Exercise 59. Confirm these formulae for the eigenvalues and $\mathrm{tr}(\mathcal{L}_s^k)$.

The following question is of a more general nature.

Question 60. Let Γ be any geometrically finite Fuchsian group. Can we expect to use coding of the dynamics on the boundary as in [26] to give a functional equation characterizing the zeros of the Selberg zeta function?

4.5 Zeta Functions for the Modular Surface

We now use the coding of closed geodesics by continued fractions to relate the transfer operator to the Selberg zeta function. Consider map on the disjoint union of two copies of the interval $T : [0,1] \times \{-1,+1\} \to [0,1] \times \{-1,+1\}$ given by the extension $T(x, \pm 1) = (1/x - [1/x], \mp 1))$ of the Gauss map.

We begin with the following simple rearrangement of the terms.

Lemma 61. *We can rewrite the Selberg zeta function for the Modular surface $V = \mathbb{H}^2/PSL(2,\mathbb{Z})$ as*

$$Z(s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Z_n(s)\right) \text{ where } Z_n(s) = \sum_{x \in Fix(T^n)} \frac{|(T^n)'(x)|^{-s}}{1-(T^{2n})'(x)^{-1}}.$$

In particular, we can write $Z_n(s) = tr(\mathcal{L}^n)$

The proof is a simple exercise in power series.

Exercise 62. Prove the lemma.

This zeta function a special case of the following.

Definition 63. The Ruelle dynamical zeta function of two complex variables $s, z \in \mathbb{C}$ is defined by

$$Z(z,s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(s)\right)$$

which converges for $|z| < 1$ and $Re(s) > 1$.

In particular, $Z(1,s) = Z(s)$. For our purposes, we can think of the z terms as being a useful "book-keeping" device for keeping track of the periods with respect to T. One advantage of the Ruelle dynamical zeta function formulation is that we can then formally expand

$$Z(z,s) = \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n} Z_n(s)\right) = 1 + \sum_{n=1}^{\infty} z^n a_n(s)$$

where $a_{2n-1}(s) = 0$ and $a_{2n}(s)$ only depends on periodic points T of period $\leq 2n$.

Theorem 64 (Ruelle). *There exists $c > 0$ such that for each s there exists $C > 0$ such that $|a_n(s)| \leq Ce^{-cn^2}$. In particular, the series above converges for all $s, z \in \mathbb{C}$.*

Setting $z = 1$ recovers the Selberg zeta function for the modular surface V.

Corollary 65. *We have an expression for the Selberg zeta function*

$$Z(s) = 1 + \sum_{n=1}^{\infty} a_n(s)$$

which converges for all $s \in \mathbb{C}$.

The dual characterization of the zeros of the Selberg zeta function in terms of the continued fraction transformation and also the spectrum of the Laplacian gives an interesting correspondence between the two. This bas been explored by Lewis and Zagier, and Mayer. More precisely, the correspondence between the spectral interpretation of the zeros and the use of the transfer operator leads to the study of analytic function on $|z-1| < \frac{3}{2}$ satisfying

$$h(z) - h(z+1) = (z)^{-2s} h(1 + 1/z).$$

This is called a period function equation. We conclude with the following result.

Lemma 66. *There is a bijection between zeros $s = 1/2 + it$ for the Selberg zeta function and the solutions for the period function equation in $\mathbb{C} - (-\infty, 0]$ and the eigenfunctions of the Laplacian with $u(x) = 0(1/x)$.*

These functions are closely related to the eigenfunctions of the Laplacian called non-holomorphic modular functions, or Maass wave forms.

4.6 General Principle

The dynamical method of extending the zeta function that we have described works for a variety of other surfaces of constant negative curvature, where the continued fraction transformation is replaced by other transformations. Typically, given a suitable surface we associate to a piecewise analytic expanding interval map $T : I \to I$ (replacing the continued fraction transformation) to the group (replacing $PSL(2, \mathbb{Z})$).

We can then rewrite the Selberg zeta function as

$$Z(s) = 1 + \sum_{n=1}^{\infty} a_n(s)$$

where

1. $a_n(s)$ only depends on periodic points T of period $\leq n$;
2. There exists $c > 0$ such that for each s there exists $C > 0$ such that $|a_n(s)| \leq Ce^{-cn^2}$. In particular, the series converges *for all* $s \in \mathbb{C}$.

The main conclusion is that we then have an expression for the Selberg zeta function

$$Z(s) = 1 + \sum_{n=1}^{\infty} a_n(s)$$

which converges *for all* $s \in \mathbb{C}$.

4.7 Dynamical Approach to More General Transformations and Zeta Functions

The versatility of this method is that it may be applied to more general situations than geodesic flows on surfaces. Given any suitable transformation T to which the underlying analysis applies, we can also associate a (suspension) flow and a zeta function.

For example, let $X = \coprod_{i=1}^{\infty}$ be a disjoint union of m-dimensional simplices X_i and let $T : X \to X$ be a map which:

1. Expand distances locally; and
2. $T|X_i$ is analytic; and
3. is Markov (i.e., each image $T(X_i)$ is a union of some of the other simplices).

We associate to an analytic expanding map $T : I \to I$ (replacing the continued fraction transformation) the suspension flow under the graph of the function $r : X \to \mathbb{R}^+$ defined by $r(x) = \log|\det(DT)(x)|$. We can then rewrite the Selberg zeta function as

$$Z(s) = 1 + \sum_{n=1}^{\infty} a_n(s)$$

where

1. $a_n(s)$ only depends on periodic points T of period $\leq n$;
2. There exists $c > 0$ such that for each s there exists $C > 0$ such that $|a_n(s)| \leq Ce^{-cn^{\left(1+\frac{1}{m}\right)}}$. In particular, the series converges *for all* $s \in \mathbb{C}$.

Corollary 67. *We have an expression for the Selberg zeta function*

$$Z(s) = 1 + \sum_{n=1}^{\infty} a_n(s)$$

which converges for all $s \in \mathbb{C}$.

We might then generalize the definition of the zeta function for the continued fraction transformation to:

$$Z(s) = \exp\left(-\sum_{n=1}^{\infty} \frac{1}{n} Z_n(s)\right) \text{ where } Z_n(s) = \sum_{x \in \mathrm{Fix}(T^n)} \frac{|\det(DT^n)(x)|^{-s}}{\det(I - (DT^n)'(x)^{-1})}$$

and we can conclude that $Z(s)$ has an extension to $\mathbb{C}$.

5 Special Values

We return to the basic question of what information is provided by a knowledge of the extension of the zeta function.

5.1 Special Values: Determinant of the Laplacian

We would like to find a way to define a determinant "$\det(\Delta) = \prod_n \lambda_n$". The basic complication is that the sequence $\{\lambda_n\}$ is unbounded and thus the product is undefined. In order to address this we first recall the following result on a slightly different complex function.

Lemma 68. *The series*

$$\eta(s) = \sum_{n=1}^{\infty} \lambda_n^{-s}$$

has an analytic extension to the entire complex plane.

This extension now allows us to make the following definition.

Definition 69. We define the Determinant of the Laplacian by:

$$\det(\Delta) = \exp\left(-\eta'(0)\right).$$

A nice introduction to these ideas is contained in [21].

Remark 70. Heuristically, one justifies the name by writing

$$\text{“}\eta'(0) = -\sum_{n=1}^{\infty}(\log \lambda_n)\lambda_n^{-s}|_{s=0} = -\sum_{n=1}^{\infty} \log \lambda_n.\text{”}$$

The determinant has a direct connection with the Selberg zeta function:

Theorem 71. *There exists $C > 0$ (depending only on the genus of the surface) such that $\det(\Delta) = CZ'(1)$.*

Sarnak, Wolpert and others have studied the smooth dependence of $\det(\Delta)$ on the choice of metric on the surface V. However, there appear to still remain questions about, say, the critical points.

Conjecture 72 (Sarnak). There is a unique minimum for the determinant of the laplacian among metrics of constant area.

5.2 Other Special Values: Resolvents and QUE

There is also an interesting connection between complex functions related to zeta functions and the problem of Quantum Unique Ergodicity. We begin by describing a modification to the Selberg zeta function which carries information about a fixed reference function A.

Definition 73. Let $A \in C^{\omega}(V)$ with $\int A d(\mathrm{vol}) = 0$. For each closed geodesic γ let $A(l) = \int_{\gamma} A$ denote the integral of A around the geodesic γ. We then formally define, say,

$$d(s,z) = \prod_{n=0}^{\infty} \prod_{\gamma} \left(1 - e^{zA(\gamma)-(s+n)l(\gamma)}\right) \text{ for } s, z \in \mathbb{C}$$

It is easy to see that for any given $z \in \mathbb{C}$ the function converges for $Re(s)$ sufficiently large. When $z = 0$ this reduces to the usual Selberg zeta function $Z(s) = d(0, s)$, whose spectral zeros we denote by $\{s_n\}$.

Theorem 74 (Anarathaman-Zelditch). *The function $d(s, z)$ is analytic for $s, z \in \mathbb{C}$. The logarithic derivative*

$$\eta(s) := \frac{1}{d(s,0)} \frac{\partial}{\partial z} d(z,s)|_{z=0}$$

has poles at $s = s_n$ [1].

Moreover, not only are the zeros of the function $\eta(s)$ described in terms of the spectrum of the Laplacian but the the residues $\mathrm{res}(\eta, s_n)$ are related to the corresponding eigenfunctions. This gives a reformulation of a well known conjecture.

Question 75 (Quantum Unique Ergodicity: Rudnick-Sarnak). Does $\mathrm{res}(\eta, s_n) \to 0$ as $n \to +\infty$?

There have been various important contributions to this conjecture by: Shnirelman, Colin de Verdiere, Zelditch, N. Anantharaman, and E. Lindenstrauss.

An interesting aspect of this particular formulation is that it avoids any mention of the Laplacian. A more traditional formulation is the following: Let $\int A(x)\mu_n(x) := \int A(x)|\phi_n(x)|^2 d(Vol)(x)$. Then $\lim_{n\to+\infty} \int A(x)\mu_n(x) = 0$.

Remark 76. This brings us to some vague speculation on horocycles, zeta functions and Flaminio-Forni invariants. Let $\Gamma < PSL(2, \mathbb{R})$ be a discrete subgroup such that $M = PSL(2, \mathbb{R})/\Gamma$ is compact. We define the horocycle flow $\psi_t : M \to M$ by

$$\psi_t(g\Gamma) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} g\Gamma \text{ for } t \in \mathbb{R}.$$

This flow preserves the Haar measure λ on M.

Theorem 77 (Flaminio-Forni [4]). *There are countably many obstructions $\nu_n : C^\infty(M) \to \mathbb{R}$, $n \geq 1$, such that whenever a function $A \in C^\infty(M)$ with $\int A d\lambda = 0$ satisfies $\nu_n(A) = 0$, for all $n \geq 1$, then*

1. *for all $\beta > 0$, $\frac{1}{T}\int_0^T A(\psi_t x)dt = O(T^{-\beta})$; and*
2. *there exists $B \in C^\infty(M)$ such that $A = \frac{d}{dt}B(\psi_t x)|_{t=0}$ (i.e., A is a coboundary).*

By unique ergodicity of the horocycle flow, this last statement is equivalent to the integral along each orbit being bounded.

Question 78. Can one relate these distributions correspond to the residues of the Anarathaman-Zelditch function η?

In order to generalize this to surfaces V of variable negative curvature we need to generalize the horocycle flow and find suitable distributions $\nu_n : C^\infty(M) \to \mathbb{R}$ coming from the residues of a generalization of the associated function $\eta(s)$.

5.3 Volumes of Tetrahedra

For contrast, we now mention a situation where special values of more number theoretic zeta functions have a more geometric interpretation. Consider the three dimensional hyperbolic space

$$\mathbb{H}^3 = \{x + iy + jt : z = x + iy \in \mathbb{C}, t \in \mathbb{R}^+\}$$

and the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2 + dt^2}{t^2}$$

Given $z \in \mathbb{C}$ consider the hyperbolic tetrahedron $\Delta(z)$ with vertices $\{0, 1, z, \infty\}$ for some $z \in \mathbb{C}$. Let $D(z)$ denote the volume of $\Delta(z)$ with respect to the Poincaré metric.

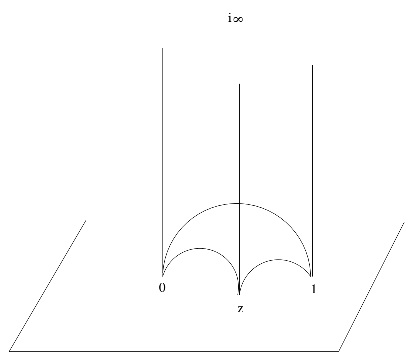


Figure 12: The ideal tetrahedron with vertices $0, 1, z, \infty$

We recall the following simple characterization of the volume of the tetrahedron.

Lemma 79 (Lobachevsky, Milnor). *If the three faces meeting at a common vertex have angles α, β, δ between them, then $D(z) = L(\alpha) + L(\beta) + L(\delta)$ where $L(\alpha) = -\int_0^\alpha \log|2\sin(t)|dt$, etc.*

These play an interesting role in, for example, Gromov's proof of the Mostow rigidity theorem.

It is easy to see that $\zeta(2) = \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$. More generally, the Dedekind zeta function $\zeta_K(s)$ is the analogue of $\zeta(s)$ for algebraic number fields $K \supset \mathbb{Q}$. We briefly recall the definition.

Definition 80. We define

$$\zeta_K(s) = \prod_{P \subset \mathcal{O}_K} \left(1 - N_{K/\mathbb{Q}}(P)^{-s}\right)^{-1}$$

where the product is over all prime ideals P of the ring of integers $\mathcal{O}_K$ of K and $N_{K/\mathbb{Q}}(P)$ is its norm.

In particular, $\zeta_K(s)$ converges for $Re(s) > 1$ and has a meromorphic extension to $\mathbb{C}$.

Theorem 81 (Zagier). *The Dedekind zeta function $\zeta_K(s)$ has values $\zeta_K(2)$ which are related to volumes of hyperbolic tetrahedra. [29]*

This is best illustrated by a specific example.

Example 82. When $K = \mathbb{Q}(\sqrt{-7})$ then the Dedekind zeta function takes the form

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(s) = \frac{1}{2} \sum_{(x,y) \neq (0,0)} \frac{1}{(x^2 + 2xy + 2y^2)^s}$$

and

$$\zeta_{\mathbb{Q}(\sqrt{-7})}(2) = \frac{4\pi^2}{21\sqrt{7}} \left(2D\left(\frac{1+\sqrt{-7}}{2}\right) + D\left(\frac{-1+\sqrt{-7}}{2}\right)\right)$$

which numerically is approximately $1 \cdot 1519254705 \cdots$.

5.4 Dynamical L-Functions

Dynamical L-functions are defined similarly to dynamical zeta functions, except that they additionally keep track of where geodesics lie in homology. For each geodesic γ we can associate an element $[\gamma] \in H_1(V, \mathbb{Z})$, the first homology group. Let $\chi : H_1(V, \mathbb{Z}) \to \mathbb{C}$ be a character (i.e., $\chi(h_1 h_2) = \chi(h_1)\chi(h_2)$).

Definition 83. We formally define a dynamical L-function for a negatively curved surface by

$$L_V(s, \chi) = \prod_{\gamma} \left(1 - \chi([\gamma]) e^{-sl(\gamma)}\right)^{-1} \text{ for } s \in \mathbb{C}.$$

Of course, this is analogous to the Dirichlet L-functions for primes, generalizing the Riemann zeta function.

In the particular case that V is a surface of constant curvature $\kappa = -1$ then this converges for $Re(s) > 1$ and has a meromorphic extension to $\mathbb{C}$. Moreover, the value $L(0, \chi)$ at $s = 0$ has a purely topological interpretation (in terms of the torsion).

Question 84. Are the corresponding results true for surfaces of variable negative curvature?

The concept of expressing interesting numerical quantities as special values of dynamical questions opens up the possibility of using the closed geodesics (or, more generally, closed orbits) for their numerical computation. We will review this approach in the next few subsections.

5.5 Dimension of Limit Sets

Given $p \geq 2$, fix $2p$ disjoint closed discs $D_1, \ldots, D_{2p}$ in the plane, and Möbius maps $g_1, \ldots, g_p$ such that each g_i maps the interior of D_i to the exterior of D_{p+i}.

Definition 85. The corresponding Schottky group Γ is defined as the free group generated by $g_1, \ldots, g_p$. The associated limit set Λ is the accumulation points of the points $g(0)$, where $g \in \Gamma$. In particular, Λ is a Cantor set contained the union of the interiors of the discs $D_1, \ldots, D_{2p}$. We define a map T on this union by $T|_{int(D_i)} = g_i$ and $T|_{int(D_{p+i})} = g_i^{-1}$.

The following result is wellknown.

Theorem 86 (Bowen, Ruelle). *Let Γ be a Schottky group with associated limit set Λ. The Hausdorff dimension of the limit set $dim(\Lambda)$ is a zero of the associated dynamical zeta function $\zeta(s)$. [24]*

This leads to the following.

Theorem 87. *Let Γ be a Schottky group, with associated limit set Λ and let $T : \Lambda \to \Lambda$ be the associated dynamical system. For each $N \geq 1$ we can explicitly define real numbers s_N, using only the derivatives $D_z T^n$ evaluated at periodic points $T^n(z) = z$, for $1 \leq n \leq N$, and associate $C > 0$ and $0 < \delta < 1$ such that*

$$|dim(\Lambda) - s_N| \leq C\delta^{N^{3/2}}.$$

[8]

A variant of this construction involves *reflection groups*, which are essentially Schottky groups with $D_i = D_{p+i}$ for all $i = 1, \ldots, p$.

Example 88 (McMullen). Consider three circles $C_0, C_1, C_2 \subset \mathbb{C}$ of equal radius, arranged symmetrically around S^1, each intersecting S^1 orthogonally, and meeting S^1 in an arc of length θ, where $0 < \theta < 2\pi/3$ (see Figure 2). Let $\Lambda_\theta \subset \mathbb{S}^1$ denote the limit set associated to the group Γ_θ of transformations given by reflection in these circles.

For example, with the value $\theta = \pi/6$ we show that the dimension of the limit set $\Lambda_{\pi/6}$ is

$$\dim(\Lambda_{\pi/6}) = 0.18398306124833918694118127344474173288\ldots$$

which is empirically accurate to the 38 decimal places given.

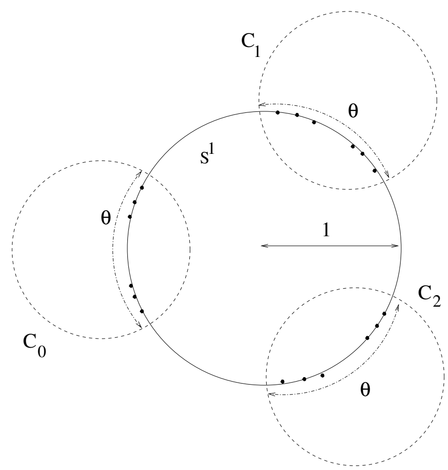


Figure 13: The limit set arising from the reflection in three circles

Remark 89. Equivalently, the smallest eigenvalue of the Laplacian on $\mathbb{H}^3/\Gamma$ is

$\lambda_0 = \dim(\Lambda_{\pi/6})(2-\dim(\Lambda_{\pi/6})) = 0\cdot 33411635567036824526131067983039328 95\cdots$

The main ingredient in the computation of $\dim(\Lambda)$ via the zeros of $\zeta(z)$ (or, equivalently, the determinant) is the following:

Lemma 90. *Let $det(I - z\mathcal{L}_s) = 1 + \sum_{N=1}^{\infty} d_N(s) z^N$ be the power series expansion of the determinant of the transfer operator $\mathcal{L}_s$. Then*

$$d_N(s) = \sum_{\substack{(n_1,\ldots,n_m)\\ n_1+\cdots+n_m=N}} \frac{(-1)^m}{m!} \prod_{l=1}^{m} \frac{1}{n_l} \sum_{\underline{i}\in Fix(n_l)} \frac{|D\phi_{\underline{i}}(z_{\underline{i}})|^s}{det(I - D\phi_{\underline{i}}(z_{\underline{i}}))},$$

where the summation is over all ordered m-tuples of positive integers whose sum is N.

We then choose s_N such that $\Delta_N(s_N) = 0$ where $\Delta_N(s) = 1 + \sum_{N=1}^{\infty} d_N(s) z^N$.

5.6 Integrals and Mahler Measures

As another application, the estimates on zeta functions can be used to integrate analytic functions $f : [0, 1] \to \mathbb{R}$ (with respect to Lebesgue measure)

Theorem 91. *We can approximate the integral*

$$\int_0^1 f(x)dx = m_n(f) + O(e^{-(\log 2-\epsilon)n^2}).$$

where $m_n(f)$ is explicitly given in terms of the values at f at the 2^n periodic points for the doubling map $Tx = 2x$ (mod 1). [9]

In particular, if $f(x)$ is a polynomial then one defines the Mahler measure by:

$$M(f) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log(|f(e^{i\theta})|)d\theta\right).$$

Generalizations of these integrals often come up in formulae the entropy of $\mathbb{Z}^d$ actions.

5.7 Lyapunov Exponents for Random Matrix Products

As a final illustration of the use of zeta functions in numerical estimates, we consider the case of Lyapunov exponents. We shall consider a simplified setting, and we begin with some notation.

1. Let $A_1, A_2 \in GL(2, \mathbb{R})$ be 2×2 non-singular matrices with real entries.
2. Let $\|\cdot\|$ be a norm on matrices (e.g., $\|A\|^2 = \mathrm{trace}(AA^T)$).

We recall the definition of the maximal Lyapunov exponent.

Definition 92. The (maximal) Lyapunov Exponent is defined to be

$$\lambda := \lim_{n\to\infty} \frac{1}{2^n} \underbrace{\frac{1}{n} \sum_{i_1,\cdots,i_n \in \{1,2\}} \log \|A_{i_1} \cdots A_{i_n}\|}_{=:\lambda_n}$$

Such Lyapunov exponents play an important role in various setting (e.g., skew product dynamical systems, hidden Markov processes, etc.). This leads naturally to the following question.

Question 93. How do we estimate λ?

We have the following result.

Theorem 94. *For each $n \geq 1$ one can find approximations ξ_n which converge to λ superexponentially, i.e., $\exists \beta > 0$ such that $|\lambda - \xi_n| = O\left(e^{-\beta n^2}\right)$. [17]*

For each $n \geq 1$, we can consider the finite products of matrices $A_{i_1} \cdots A_{i_n}$, where $i_1, \cdots, i_n \in \{1, 2\}$. The numbers ξ_n are explicitly defined in terms of the entries and eigenvalues of these 2^n matrices. The number of matrices we need to compute ξ_n grows exponentially, i.e.,

$$\sharp\{A_{i_1} \cdots A_{i_n} : i_1, \cdots, i_n \in \{1, 2\}\} = 2^n$$

and so it is particularly significant that the convergence in the theorem is superexponential.

Example 95. Let $p_1 = p_2 = \frac{1}{2}$ and choose

$$A_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$$

then with $n = 9$:

$$\lambda = \underbrace{1 \cdot 1433110351029492458432518536555882994025}_{=\xi_{n=9}} + O\left(10^{-40}\right).$$

6 Zeta Functions and Counting

One of the most useful applications of zeta functions is in counting problems.

6.1 Prime Number Theorem

Recall the following classical result proved independently by Hadamard and de Vallee Poussin.

Hadamard (1865-1963)

de la Vallée Poussin(1866-1962)

Theorem 96 (Prime number theorem). *The number $\pi(x)$ of primes numbers less than x satisfies*

$$\pi(x) \sim \frac{x}{\log x} \text{ as } x \to +\infty.$$

The standard proof uses the Riemann zeta function and two of its basic properties:

1. $\zeta(s)$ has a simple pole at $s = 1$;
2. $\zeta(s)$ otherwise has a zero-free analytic extension to a neighbourhood of $Re(s) = 1$.

Thereafter, the traditional proofs of the prime number theorem use either the residue theorem or the Ikehara-Wiener Tauberian theorem.

A knowledge of the Riemann hypothesis would improve the asymptotic in the Prime Number Theorem to the following:

Conjecture 97 (Riemann Hypothesis: alternative formulation). We have

$$\pi(x) = \int_2^x \frac{1}{\log u} du + O\left(x^{1/2} \log x\right).$$

This, of course, contains the original Prime Number Theorem since the leading term satisfies $\int_2^x \frac{1}{\log u} du \sim \frac{x}{\log x}$. In the absence of the Riemann Conjecture the original proof of Hadamard and de la Vallee Poussin still shows that there were no zeros in a region $\{s = \sigma + it : \sigma > 1 - C/\log|t| \text{ and } |t| \geq 1\}$. This corresponds to an error term of

$$\pi(x) = \int_2^x \frac{1}{\log u} du + O\left(xe^{-a\sqrt{\log|x|}}\right).$$

6.2 The Prime Geodesic Theorem for $\kappa = -1$

There is a corresponding result for closed geodesics on negatively curved surfaces with curvature $\kappa = -1$, due to Huber.

Theorem 98 (Prime geodesic theorem for $\kappa = -1$). *Let V be a compact surface with $\kappa = -1$. The number $N(x)$ of closed geodesics with $e^{l(\gamma)} \leq x$ less than x satisfies*

$$N(x) \sim \frac{x}{\log x} \text{ as } x \to +\infty.$$

Equivalently, we have that

$$Card\{\gamma : l(\gamma) \leq T\} \sim \frac{e^T}{T} \text{ as } T \to +\infty.$$

This follows from an analysis of the Selberg Zeta function Z(s). In particular, since a partial analogue of the Riemann Hypothesis holds in this case, one sees that a partial analogue of the conjectured error term for primes holds:

Theorem 99. *There exists $\epsilon > 0$ such that*

$$\pi(x) = \int_2^x \frac{1}{\log u} du + O\left(x^{1-\epsilon}\right)$$

The value of $\epsilon > 0$ is related to the least distance of the zeros for $Z(s)$ from the line $Re(s) = 1$. This is determined by the smallest non-zero eigenvalue of the Laplacian. By a result of Schoen-Wolpert-Yau, this is comparable size to the length of the shortest geodesic dividing the surface into two pieces. It also has a dynamical analogue, as the rate of mixing of the associated geodesic flow.

6.3 Prime Geodesic Theorem for $\kappa < 0$

We can ask whether these results are still true in the case of manifolds of variable negative curvature. Indeed, analogues of these results do hold and it is possible

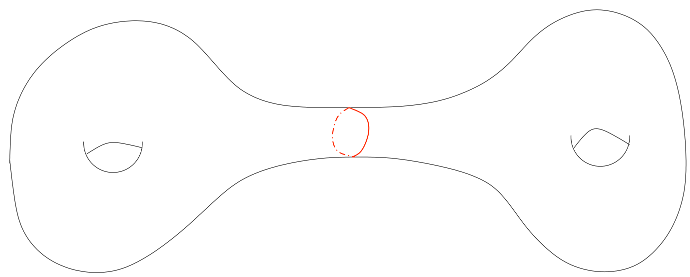


Figure 14: When $\kappa = -1$ then the value ϵ can be chosen to be comparable to the length $l(\gamma_0)$ of the shortest closed geodesic γ_0

to give proofs using zeta function methods, but in this broader context there is no analogue of the Selberg trace function approach and one has to use the transfer operator method (with analytic functions replaced by C^1 functions, say).

Let V be a compact surface with negative curvature $\kappa < 0$. We can again define a zeta function by $\zeta_V(s) = \prod_\gamma \left(1 - e^{-sl(\gamma)}\right)^{-1}$. The following is easy to show.

Theorem 100. *The zeta function $\zeta_V(s)$ converges on the half-plane $Re(s) > h$ where h is the (topological) entropy.*

We recall the elegant definition of entropy due to Anthony Manning. Let $\widetilde{V}$ be the universal cover of V. Lift the metric from V to $\widetilde{V}$ and let $B_{\widetilde{V}}(x_0, R)$ be a ball of radius $R > 0$ about any fixed reference point $x_0 \in \widetilde{V}$.

Definition 101. We can then write:

$$h = \lim_{R \to +\infty} \frac{1}{R} \log\left(\mathrm{vol}\left(B_{\widetilde{V}}(x_0, R)\right)\right)$$

For example, if the curvature $\kappa < 0$ is constant then $h = \sqrt{|\kappa|}$.

By analogy with the constant curvature case we have the following results on the domain of the zeta function.

Theorem 102. *There exists $h > 0$ such that:*

1. *$\zeta_V(s)$ converges for $Re(s) > h$;*
2. *$\zeta_V(s)$ has a simple pole at $s = h$;*
3. *$\zeta_V(s)$ otherwise has a non-zero analytic extension to a neighbourhood of $Re(s) = h$*

The same basic proof as for the Prime Number Theorem now gives the Prime Geodesic Theorem.

Theorem 103 (Prime geodesic theorem for $\kappa < 0$). *Let V be a compact surface with $\kappa < 0$. The number $N(x)$ of closed geodesics with $e^{hl(\gamma)} \leq x$ less than x satisfies*

$$N(x) \sim \frac{x}{\log x} \text{ as } x \to +\infty.$$

Equivalently, we have that

$$Card\{\gamma : l(\gamma) \leq T\} \sim \frac{e^{hT}}{hT} \text{ as } T \to +\infty.$$

Definition 104. There exists $h > 0$ such that $\zeta_V(s)$ is analytic for $Re(s) > h$ and $s = h$ is a simple pole. (The value h is simply the *topological entropy* of the flow ϕ.)

In the case of the zeta function $\zeta_V(s)$ for more general surfaces of variable negative curvature we have typically a weaker result for than for surfaces of constant negative curvature.

Theorem 105 (after Dolgopyat). *The exists $\epsilon > 0$ such that $\zeta_V(s)$ has an analytic zero-free extension to $Re(s) > h - \epsilon$*

The biggest difference from the case of constant curvature is that we have no useful estimate on $\epsilon > 0$. This has the (expected) following consequence.

Corollary 106. *We can estimate that the number $N(x)$ of closed geodesics γ with $l(\gamma) \leq x$ by*

$$N(x) = \int_2^{e^{hx}} \frac{1}{\log u} du + O(e^{(h-\epsilon)x})$$

[18]

Remark 107. Recall that the Teichmüller flow on the moduli space of flat metrics is a generalization of the geodesic flow on the modular surface. Athreya-Bufetov-Eskin-Mirzakhani proved the asymptotic formula in this case.

6.4 Counting Sums of Squares

Let us now recall a result from number theory which will also prove to have a geometric analogue. Recall that there are asymptotic estimates on the natural numbers which are also sums of two squares $1, 2, 4, 5, 8, \cdots, u^2 + v^2, \cdots$ (where $u, v \in \mathbb{N}$) less than x. For example, we can write:

Definition 108. Let $S(x) = \text{Card}\{u^2 + v^2 \leq x\}$ be the number of such sums of squares less than x.

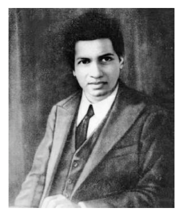
Theorem 109 (Ramamnujan-Landau). *We have*

$$S(x) \sim \frac{x}{(\log x)^{\frac{1}{2}}} \text{ as } x \to +\infty$$

$$\begin{aligned}1 &= 0^2 + 1^2\\ 2 &= 1^2 + 1^2\\ 4 &= 0^2 + 2^2\\ 5 &= 1^2 + 2^2\\ 8 &= 2^2 + 2^2\\ &\vdots\end{aligned}$$

The theorem was originally proved by Landau in 1908, although it was independently stated in Ramanujan's first famous letter to Hardy from 16 January 1913.


Landau (1877-1938)


Ramanujan (1887-1920)

There is a very interesting account of this in the excellent book of Hardy: "Ramanujan: Twelve lectures on subjects suggested by his life and work". The basic idea of Landau's proof is that in place of the zeta function, one uses another complex function (generating function) given by

$$\eta(s) = \sum_{n=1}^{\infty} b_n n^{-s} \text{ where } b_n = \begin{cases} 1 & \text{if } n = u^2 + v^2 \textit{ is} \text{ a sum of squares} \\ 0 & \text{if } n \textit{ is not} \text{ a sum of squares} \end{cases}$$

This converges for $Re(s) > 1$. But this has an algebraic pole at $s = 1$:

Lemma 110. *We can write*

$$\eta(s) = \frac{C}{\sqrt{s-1}} + A(s)$$

where $A(s)$ is analytic in a neighbourhood of $Re(s) \geq 1$.

Using complex analysis one can write

$$S(x) = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} \eta(s) \frac{x^s}{s} ds \text{ for } c > 1$$

and the asymptotic formula comes from moving the line of integration to the left. One can actually get more detailed asymptotic formulae.

Theorem 111 (Expansion for sums of squares). *There exist constants $b_0, b_1, b_2 \cdots$ such that*

$$S(T) = \frac{T}{\sqrt{\log T}} \left(b_0 + \frac{b_1}{\log T} + \frac{b_2}{(\log T)^2} + \cdots \right)$$

as $T \to +\infty$.

6.5 Closed Geodesics in Homology Classes

Philips and Sarnak considered the problem of additionally restricting to geodesics in a given homology class. In particular, they gave an asymptotic formulae for the number $N_0(T)$ of such closed geodesics on a surface of constant negative curvature whose length is at most T.

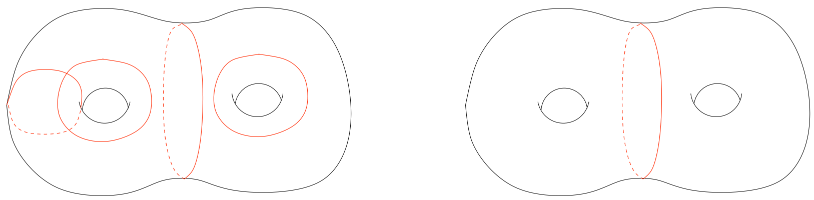


Figure 15: We can restrict the counting to those closed geodesics which are, say, null in homology as shown in the second figure

More precisely, they established the following result.

Theorem 112 (Phillips-Sarnak). *For a surface of negative curvature $\kappa = -1$, and genus $g > 1$, there exists c_0 such that*

$$N_0(T) \sim c_0 \frac{e^T}{T^{g+1}} \text{ as } T \to +\infty$$

Moreover, there exist constants c_0, c_1, c_2 such that

$$N(a,T) = \frac{e^{hT}}{T^{g+1}} \left(c_0 + c_1/T + c_2/T^2 + \cdots\right) \text{ as } T \to +\infty$$

[16]

Recall that the asymptotic estimate for counting closed geodesics without the homology restrictions gave $\pi(T) \sim \frac{e^{hT}}{hT}$, which is necessarily larger than $N_0(T)$.

The similar form of the asymptotic arises for the same reason, when on defining the appropriate zeta function using only those geodesics null in homology it leads to a singularity of the form $(s-h)^{-1/2}$. As one would expect the result extends to surfaces of variable negative curvature, using a similar approach to that used for $\zeta_V(s)$ in order to study the corresponding zeta function in this case.

Theorem 113 ([10], [19]). *For a surface of (variable) negative curvature and genus $g > 1$ there exist constants c_0, c_1, c_2 such that*

$$N_0(T) = \frac{e^{hT}}{T^{g+1}} (c_0 + c_1/T + c_2/T^2 + \ldots) \text{ as } T \to +\infty.$$

7 Circle Problem

We begin with another famous result in number theory. We can consider estimates on the number $N(r)$ of pairs $(n,m) \in \mathbb{Z}^2$ such that $m^2 + n^2 \leq r^2$. It is easy to see that $N(r) \sim \pi r^2$. In fact, there is a better estimate due to Gauss.

Lemma 114 (Gauss). $N(r) = \pi r^2 + O(r)$

The following remains an open problem.

Conjecture 115 (Circle problem). For each $\epsilon > 0$ we have that $N(r) = \pi r^2 + O(r^{1/2+\epsilon})$

In the case of the analogue of the circle problem for spaces of negative curvature it is a problem to even find the main term in the corresponding asymptotic.

Question 116. What happens if we replace $\mathbb{R}^2$ by the Poincaré disk $\mathbb{D}^2$, and we replace $\mathbb{Z}^2$ by a discrete (Fuchsian) group of isometries?

We address this question in the next subsection.

7.1 Hyperbolic Circle Problem

Let $\mathbb{D}^2 = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk with the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

of constant curvature $\kappa = -1$. This is equivalent to the Poincaré upper half plane $\mathbb{H}^2$, but this model has the advantage that the circle counting pictures look more natural.

Consider the action $g : \mathbb{D}^2 \to \mathbb{D}^2$ defined by $g(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ for g lying in a discrete group

$$\Gamma < \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1 \right\}$$

Let us assume that Γ is cocompact, i.e., the quotient $\mathbb{D}^2/\Gamma$ is compact.

Consider the orbit $\Gamma 0$ orbit of the fixed reference point $0 \in \mathbb{D}^2$. The following is the analogue of the circle problem for negative curvature.

Question 117. What are the asymptotic estimates for the counting function $N(T) = \mathrm{Card}\{g \in \Gamma : d(0, g0) \leq T\}$ as $T \to +\infty$?

More generally, we can consider the orbit of a fixed reference point $x \in \mathbb{D}^2$. We want to find asymptotic estimates for the counting function

$$N(T) = \mathrm{Card}\{g \in \Gamma : d(x, gx) \leq T\}$$

Assume that Γ is cocompact, i.e., the quotient surface $V = \mathbb{D}^2/\Gamma$ is compact.

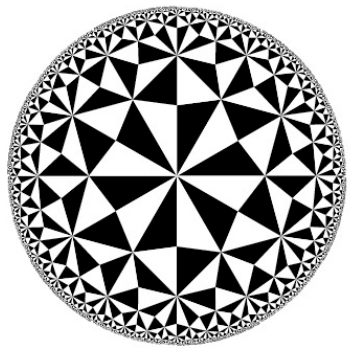


Figure 16: The Poincaré disk. The triangles all have the same size in the Poincaré metric, but seem smaller as they approach the boundary in the Euclidean metric.

Theorem 118. *There exists $C > 0$. $N(R) \sim Ce^T$ as $T \to +\infty$.*

In fact there is an error term in this case too.

Theorem 119. *There exists $C > 0$ and $\epsilon > 0$. $N(R) = Ce^T + O(e^{(1-\epsilon)T})$ as $T \to +\infty$.*

The basic method of proof is the same as for counting primes and closed geodesics. In place of the zeta function one wants to study the Poincaré series.

Definition 120. We define the Poincaré series by

$$\eta(s) = \sum_{g \in \Gamma} e^{-sd(gx,x)}$$

where $s \in \mathbb{C}$. This converges to an analytic function for $Re(s) > 1$.

The Poincaré series can be shown to have the following analyticity properties:

Lemma 121. *The Poincaré series has a meromorphic extension to $\mathbb{C}$ with a simple pole at $s = 1$ and no poles other poles on the line $Re(s) = 1$.*

The proof of this result is usually based on the Selberg trace formula (and this is where we use that Γ is cocompact). In order to show how it leads to a proof of the theorem, let $L(T) = \text{Card}\{g \in \Gamma : \exp(d(x, gx)) \leq T\}$ for $T > 0$. We can then write

$$\eta(s) = \int_1^\infty t^{-s} dL(t)$$

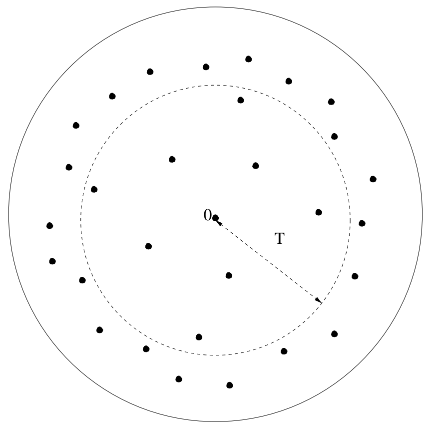


Figure 17: We want to count the points in the orbit $\Gamma 0$ of 0 which are at a distance at most T from 0.

and employ the following standard Tauberian Theorem

Lemma 122 (Ikehara-Wiener). *If there exists $C > 0$ such that*

$$\psi(s) = \int_1^\infty t^{-s} dL(t) - \frac{C}{s-1}$$

is analytic in a neighbourhood of $Re(s) \geq h$ then $L(T) \sim CT$ as $T \to +\infty$.

Finally, we can deduce from $L(T) \sim CT$ that $N(T) \sim Ce^T$ as $T \to +\infty$., which completes the proof.

7.2 Schottky Groups and the Hyperbolic Circle Problem

Assume that Γ is a Schottky group. In particular, we require that Γ is a free group, although we shall assume that $\mathbb{D}^2/\Gamma$ is non-compact. In this context the hyperbolic analogue of the circle counting problem takes a slightly different form.

Theorem 123. *There exists $C > 0$ and $\delta > 0$ such that*

$$N(T) := Card\{g \in \Gamma : d(0, g0) \leq T\} \sim Ce^{\delta T}$$

as $T \to +\infty$.

Here $0 < \delta < 2$ is the Hausdorff dimension of the limit set (i.e., the Euclidean accumulation points of the orbit $\Gamma 0$ on the unit circle). We can briefly sketch the proof in this case.

1st step of proof: We define the Poincaré series by

$$\eta(s) = \sum_{g \in \Gamma} e^{-sd(g0,0)}.$$

This converges to an analytic function for $Re(s) > \delta$. Moreover, we can establish the following result on the domain of $\eta(s)$

Lemma 124. *The Poincaré series has a meromorphic extension to $\mathbb{C}$ with:*

1. *a simple pole at $s = \delta$; and*
2. *no poles other poles on the line $Re(s) = \delta$.*

When $\mathbb{D}^2/\Gamma$ is compact then this result is based on the Selberg trace formula. However, when Γ is Schottky group a dynamical approach works better, as in step 3 (with a hint of automatic group theory, as in step 2)

2nd step of proof: Consider the model case of a Schottky group $\Gamma = \langle a, b\rangle$. In this case, each $g \in \Gamma - \{e\}$ can be written $g = g_1 g_2 \cdots g_n$, say, where $g_i \in \{a, b, a^{-1}, b^{-1}\}$ with $g_i \neq g_{i+1}^{-1}$.

Definition 125. Consider the space of infinite sequences

$$\Sigma_A = \{(x_n)_{n=0}^{\infty} : A(x_n, x_{n+1}) = 1 \text{ for } n \geq 0\}, \text{ where } A = \begin{pmatrix} 1&1&0&1\\ 1&1&1&0\\ 0&1&1&1\\ 1&0&1&1 \end{pmatrix}$$

and the shift map $\sigma : \Sigma_A \to \Sigma_A$ given by $(\sigma x)_n = x_{n+1}$.

We then have the following way to enumerate the displacements $d(g0, 0)$:

Lemma 126. *There exists a (Hölder) continuous function $f : \Sigma_A \to \mathbb{R}$ and $x' \in \Sigma_A$ such that there is a correspondence between the sequences $\{d(g0, 0) : g \in \Gamma\}$ and*

$$\left\{ \sum_{k=0}^{n-1} f(\sigma^k y) : \sigma^n y = x', n \geq 0 \right\}.$$

1st step of proof: Extending the Poincaré series comes from the following.

Corollary 127. *We can write that*

$$\eta(s) = \sum_{n=1}^{\infty} \sum_{\sigma^n y = x_0} \exp\left(-s \sum_{k=0}^{n-1} f(\sigma^k y) \right)$$

To derive the corollary, we define families of transfer operators $\mathcal{L}_s : C(\Sigma_A) \to C(\Sigma_A)$, $s \in \mathbb{C}$, by

$$\mathcal{L}_s w(x) = \sum_{\sigma y = x} e^{-sf(y)} w(y), \text{ where } w \in C(\Sigma_A),$$

and then we can formally write

$$\eta(s) = \sum_{n=1}^{\infty} \mathcal{L}_s^n 1(x).$$

Finally, the better spectral properties of the operators $\mathcal{L}_s$ on the smaller space of Hölder functions give the meromorphic extension, and the other properties follow by more careful analysis.

4th step of proof: The final step is done just by using the Tauberian theorem, by analogy with the previous counting results.

7.3 Apollonian Circle Packings

There is an interesting connection between the hyperbolic circle problem and a problem for Apollonian circle packings. Consider a circle packing where four circles are arranged so that each is tangent to the other three.

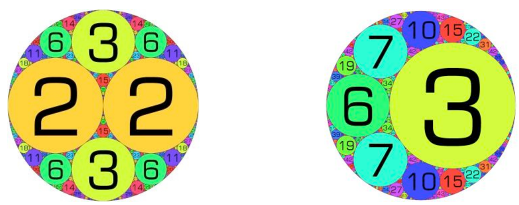
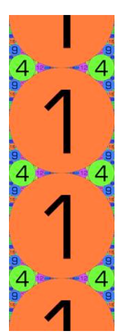


Figure 18: Two very different example of Apollonian circle packings. The numbers represent the curvatures of the circles, i.e., the reciprocal of their radii

Assume that the circles have radii r_1, r_2, r_3, r_4 and denote their curvatures by $c_i = \frac{1}{r_i}$.

Lemma 128 (Descartes). *The curvatures satisfy* $2(c_1^2 + c_2^2 + c_3^2 + c_4^2) = (c_1 + c_2 + c_3 + c_4)^2$

Fir the most familiar Apollonian circle packing we can consider the special case $c_1 = c_2 = 0$ and $c_3 = c_4 = 1$.



The curvatures of the circles are $0, 1, 4, 9, 12, \ldots$.

Question 129 (Lagarias et al). How do these numbers grow?

We want to count these curvatures with their multiplicity.

Definition 130. Let $C(T)$ be the number of circles whose curvatures are at most T.

There is the following asymptotic formula.

Theorem 131 (Kontorovich-Oh). *There exists $K > 0$ and $\delta > 0$ such that*

$$C(T) \sim KT^{\delta} \text{ as } T \to +\infty$$

Remark 132. For the standard Apollonian circle packing:

1. The value $\delta = 1 \cdot 30568 \cdots$ is the dimension of the limit set; and

2. The value $K = 0 \cdot 0458 \cdots$ can be estimated too

The proof is dynamical and comes from reformulating the problem in terms of a discrete Kleinian group acting on three dimensional hyperbolic space. We briefly recall idea of the proof:

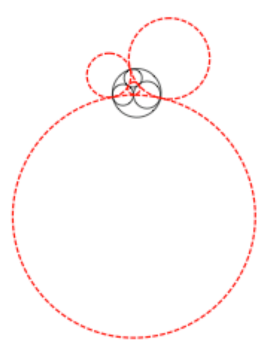


Figure 19: The black circles generate the Apollonian circle packing. Reflecting in the complimentary red circles generates the Schottky group.

- Given a family of tangent circles (in black) leading to a circle packing, we associate a new family of tangent circles (red).

- One then defines isometries of 3-dimensional hyperbolic space $\mathbb{H}^3$ by reflecting in the associated geodesic planes (i.e., hemispheres in upper half-space).

- Let Γ be the Schottky group (for $\mathbb{H}^3$). Then one can reduce the theorem to a counting problem for Γ.

Question 133. Can these results be generalized to other circle packings?

7.4 Variable Curvature

Let now assume that V is a surface of variable negative curvature. Let $\widetilde{V}$ be the universal covering space for V with the lifted metric d. The covering group $\Gamma = \pi_1(V)$ acts by isometries on $\widetilde{V}$ and $\widetilde{V}/\Gamma$. We want to find asymptotic estimates for the counting function

$$N(T) = \mathrm{Card}\{g \in \Gamma : d(x, gx) \leq T\}.$$

Let $h > 0$ be the topological entropy.

Theorem 134. *There exists $C > 0$ such that $N(R) \sim Ce^{hT}$ as $T \to +\infty$.*

One might expect the following result to be true:

Conjecture 135. There exists $C > 0$ and $\epsilon > 0$ such that $N(T) = Ce^T + O(e^{(1-\epsilon)T})$ as $T \to +\infty$.

The basic method of proof is the same as for counting primes and closed geodesics. In place of the zeta function one wants to study the Poincaré series.

Definition 136. We define

$$\eta(s) = \sum_{g \in \Gamma} e^{-sd(gx,x)}.$$

This converges to an analytic function for $Re(s) > h$.

Lemma 137. *The Poincaré series has a meromorphic extension to $\mathbb{C}$ with a simple pole at $s = 1$ and no other poles on the line $Re(s) = h$.*

Let $M(T) = \mathrm{Card}\{g \in \Gamma : \exp(hd(x, gx)) \leq T\}$. Then we can write

$$\eta(s) = \int_1^\infty t^{-s} dM(t)$$

and use the Tauberian Theorem to deduce that $M(T) \sim CT$ and thus that $N(T) \sim Ce^{hT}$ as $T \to +\infty$.

8 The Extension of $\zeta_V(s)$ for $\kappa < 0$

Consider a compact surface V of variable curvature $\kappa < 0$ and the associated geodesic flow $\phi_t : M \to M$ on the unit tangent bundle $M = SV$. In order to extend the zeta function $\zeta_V(s)$ to all $s \in \mathbb{C}$ we need to develop different machinery. Motivated by work of Gouezel-Liverani (and Baladi-Tsujii) for Anosov diffeomorphisms, we want to employ the basic philosophy: *We want to study simple operators on complicated Banach spaces.* More precisely, following Butterley and Liverani:

Definition 138. For each $s \in \mathbb{C}$ let $R(s) : C^0(M) \to C^0(M)$ be the operator defined by

$$R(s)w(x) = \int_0^\infty e^{-st}\mathcal{L}_t w(x)dt \text{ where } w \in C^0(M)$$

where $\mathcal{L}_t w(x) = w(\phi_t x)$ is simply composition with the flow.

We need to replace $C^0(M)$ by a "better" Banach space, for which there is less spectrum. Actually, we need a family of *complicated* Banach spaces. For each $k \geq 1$ we can associate a Banach space of distributions $\mathcal{B}_k$ such that the restriction $R(s) : \mathcal{B}_k \to \mathcal{B}_k$ has only isolated eigenvalues for $Re(s) > -k$.

We then want to use the operator $\mathcal{L}_t$ to extend $\zeta_{V(s)}$. We need to make sense of

$$\text{``}\det(I - R(s)) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n}\mathrm{tr}\left(R(s)^n\right)\right)''$$

for operators $R(s)$ which are not trace class.

Finally, we need to relate $\zeta_V(s)$ to "$\det(I - R(s))''$, which in fact means we have to do repeat the analysis all over again for Banach spaces of differential forms as well as functions in order to fix up the identities.

Theorem 139 (Giulietti-Liverani-P.). *For compact surfaces V with $\kappa < 0$:*

1. *The zeta function $\zeta_V(s)$ has a meromorphic extension to $\mathbb{C}$.*
2. *For each $k \geq 1$, the poles and zeros $s = s_n$ for $\zeta_V(s)$ in the region $Re(s) > -k$ correspond to 1 being an eigenvalue for $R(s)$.*

9 Appendix: Selberg Trace Formulae

In this survey, we are actually more interested in the dynamical approach to zeta functions, but for completeness we attempt to sketch the original approach using the Selberg trace formula.

9.1 The Laplacian

Assume that the surface V is compact then there are countably many distinct solutions $\Delta\phi_i + \lambda_i\phi_i = 0$ (with $\|\phi_i\|_2 = 1$ and $0 = \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \to +\infty$ are real valued, since the operator is self-adjoint). We define the Heat Kernel on V by

$$K(x, y, t) = \sum_n \phi_n(x)\phi_n(y)e^{-\lambda_n t}.$$

(This has a nice interpretation as the probability of a Brownian Motion path going from x to y in time $t > 0$.) In particular we can define the trace:

$$\mathrm{tr}(K(\cdot,\cdot,t)) := \int_M K(x, x, t)d\nu(x) = \sum_n e^{-\lambda_n t}.$$

9.2 Homogeneity and the Trace

We can consider the Poincaré distance between $\overline{x}, \overline{y} \in \mathbb{H}^2$ given by

$$d(\overline{x}, \overline{y}) = \cosh^{-1}\left(1 + \frac{|\overline{x} - \overline{y}|^2}{2Im(\overline{x}).Im(\overline{y})}\right).$$

The heat kernel is homogeneous, i.e., the heat kernel $\widetilde{K}(\cdot,\cdot,t)$ on $\mathbb{H}^2$ depends only on the relative distance $d(\overline{x}, \overline{y})$ apart of $\overline{x}, \overline{y} \in \mathbb{H}^2$. Let us now fix lifts $\overline{x}$ and $\overline{y}$ of x and y, respectively, in some fundamental domain F. We can then write

$$K(x, y.t) = \sum_{\gamma\in\Gamma} \widetilde{K}(\overline{x}, \gamma\overline{y}, t) = \sum_{\gamma\in\Gamma} \widetilde{K}(d(\overline{x}, \gamma\overline{y}), t)$$

and evaluate the trace

$$\mathrm{tr}(K(\cdot,\cdot,t)) := \int_F K(x, x, t)d\nu(x) = \int_F \widetilde{K}(d(\overline{x}, \overline{x}), t)d\nu(x) + \sum_{\gamma\in\Gamma-\{e\}} \int_F \widetilde{K}(d(\overline{x}, \gamma\overline{x}), t)d\nu(x).$$

Lemma 140. *There is a correspondence between Γ and $\{kp^nk^{-1} : n \in \mathbb{Z}, p, k \in \Gamma/\Gamma_p\}$ where p runs through the non-conjugate primitive elements of Γ and $g \in \Gamma$.*

Since ν is preserved by translation by elements of the group, we can write

$$\sum_{\gamma\in\Gamma-\{e\}} \int_F \widetilde{K}(d(\overline{x}, \gamma\overline{x}), t)d\nu(x) = \sum_{n=1}^{\infty}\sum_p \underbrace{\left(\sum_{k\in\Gamma/\Gamma_p} \int_F \widetilde{K}(d(k^{-1}\overline{x}, p^n k^{-1}\overline{x}), t)d\nu(x)\right)}_{=\int_{F_p} \widetilde{K}(d(\overline{x}, p^n\overline{x}), t)d\nu(x)}. \tag{1}$$

where F_p is a fundamental domain for Γ_p. Without loss of generality, we can take the fundamental domain $z = x + iy$ where $1 \leq y \leq m^2$ (where $l(g) = 2\log m$) and the integral can be explicitly evaluated

$$\frac{l(p)}{\sqrt{\cosh(l(p^n)) - 1}} \int_{\cosh l(p^n)}^{\infty} \frac{\widetilde{K}(s)ds}{\sqrt{s - \cosh l(p^n)}}. \tag{2}$$

([12], page 232).

9.3 Explicit Formulae and Trace Formula

The heat kernel $\widetilde{K}(\overline{x}, \overline{y}, t)$ on $\mathbb{H}^2$ has the following standard explicit form, which we state without proof.

Lemma 141. *One can write*

$$\widetilde{K}(\overline{x}, \overline{y}, t) = \widetilde{K}(d(\overline{x}, \overline{y}), t) = \frac{\sqrt{2}}{(4\pi t)^{3/2}} e^{-t/4} \int_{d(\overline{x},\overline{y})}^{\infty} \frac{ue^{-u^2/4t}}{\sqrt{\cosh(u) - \cosh(d(\overline{x}, \overline{y}))}} du$$

and, in particular,

$$\widetilde{K}(0, t) = \frac{1}{(4\pi)^{3/2}} \int_0^{\infty} \frac{ue^{-u^2/4t}}{\sinh(u/2)} du.$$

We start with the first term in (1):

$$\int_M \widetilde{K}(0, t) d\nu(x) = \frac{\nu(M) e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{se^{-s^2/4t}}{\sinh(s/2)} ds.$$

The connection with the original problem comes via the standard result: The length of a closed geodesic in the conjugacy class of g is given by $l(g) = \inf_{x \in H} d(xg, x)$. Moreover, $l(g^n) = |n| l(g)$.

Lemma 142. *There is a bijection between nontrivial conjugacy class and closed geodesics on the surface. Moreover, $tr(g) = 2\cosh(l(\gamma))$.*

Furthermore, we can write

$$\sum_{\gamma \in \Gamma - \{e\}} \int_M \widetilde{K}(d(\overline{x}, \gamma\overline{x}), t) d\nu(x) = \sum_{[\gamma]} \int_M \widetilde{K}(d(\overline{x}, \gamma\overline{x}), t) d\nu(x)$$

which allows us to write the rest of the trace explicitly:

Corollary 143 (McKean). *The trace takes the form*

$$\frac{\nu(M) e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{se^{-s^2/4t}}{\sinh(s/2)} ds + \frac{e^{-t/4}}{(16\pi t)^{1/2}} \sum_{n=1}^{\infty} \sum_{\gamma} \frac{l(\gamma)}{\sinh(nl(\gamma)/2)} e^{(nl(\gamma))^2/4t}$$

where the sum if over γ if over conjugacy classes in Γ.

This is an explicit computation. Using Lemma 141 and (2) we have that

$$\int_{\cosh l(p^n)}^{\infty} \frac{\widetilde{K}(s)ds}{\sqrt{s-\cosh l(p^n)}} = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{\cosh l(p^n)}^{\infty} \int_{s}^{\infty} \frac{se^{-s^2/4t}}{\sqrt{\cosh(u)-\cosh(s)}} duds$$

which we can immediately evaluate as $\frac{e^{-t/4}}{\sqrt{8\pi t}} e^{-l(p^n)^2/4t}$, and we are done.

9.4 Return to the Selberg Zeta Function

The connection to the Selberg zeta function is actually via its logarithmic derivative $Z'(s)/Z(s)$.

Lemma 144. *We can write*

$$\frac{Z'(s)}{Z(s)} = (2s-1) \int_0^{\infty} e^{-s(s-1)t} \theta(t) dt$$

where

$$\theta(t) = \sum_{n=0}^{\infty} e^{-t\lambda_n} - \frac{\nu(M)e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} \frac{se^{-s^2/4t}}{\sinh(s/2)} ds$$

Proof. For $s > 1$ we can explicitly write

$$\frac{Z'(s)}{Z(s)} = \sum_p \sum_{k=0}^{\infty} \frac{l(p)e^{-(s+k)l(p)}}{1-e^{-(s+k)l(p)}} = \sum_p \sum_{k=0}^{\infty} \frac{l(p)}{\sinh(l(p^n)/2)} e^{-(s-1/2)l(p^n)}$$

□

In particular, we can write

$$\frac{1}{2s-1} \frac{Z'(s)}{Z(s)} = \int_0^{\infty} e^{-s(s-1)t} \left(\sum_{n=0}^{\infty} e^{-t\lambda_n} - \frac{2\nu(M)e^{-t/4}}{(4\pi t)^{3/2}} \int_0^{\infty} se^{-s^2/4t} \sum_{n=0}^{\infty} e^{-(n+1/2)s} ds \right) dt$$

This can formally be written as

$$\sum_{n=0}^{\infty} \left(\frac{1}{s(s-1)-\lambda_n} - \frac{\nu(M)}{4\pi} \frac{1}{s+n} \right)$$

where:

1. The solutions s to $s(s-1) = \lambda_n$ correspond to poles for $\frac{Z'(s)}{Z(s)}$; and
2. By Gauss-Bonnet $\nu(M) = 4\pi(g-1)$ and the solutions $s = -n$ correspond to poles for $\frac{Z'(s)}{Z(s)}$.

10 Final Comments

Of course there have been many important contributions to number theory using ergodic theory which I have not even touched upon, but which I felt it appropriate to commend to the reader.

1. The first of Khinchin's pearls was van de Waerden's theorem (on arithmetic progressions). This, and many deep generalizations, including the recent work of Green and Tao, have ergodic theoretic proofs in the spirit of Furstenberg, etc.

2. Margulis' proof of the Oppenheim Conjecture (on the values of indefinite quadratic forms)

3. The work of Einseidler, Katok and Lindenstrauss work on the Littlewood conjecture (on simultaneous diophantine approximations).

4. Benoist-Quint work on orbits of groups on homogeneous spaces.

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