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# Hybrid Nonnegative and Compartmental Dynamical Systems

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Nonnegative and compartmental dynamical systems are governed by conservation laws and are comprised of homogeneous compartments which exchange variable nonnegative quantities of material via intercompartmental flow laws. These systems typically possess hierarchical (and possibly hybrid) structures and are remarkably effective in capturing the phenomenological features of many biological and physiological dynamical systems. In this paper we develop several results on stability and dissipativity of hybrid nonnegative and compartmental dynamical systems. Specifically, using *linear* Lyapunov functions we develop sufficient conditions for Lyapunov and asymptotic stability for hybrid nonnegative dynamical systems. In addition, using *linear* and *nonlinear* storage functions with *linear* hybrid supply rates we develop *new* notions of dissipativity theory for hybrid nonnegative dynamical systems. Finally, these results are used to develop general stability criteria for feedback interconnections of hybrid nonnegative dynamical systems.

**Key words:** Nonnegative systems; Compartmental models; Hybrid systems; Impulsive systems; Stability theory; Dissipativity theory

## 1 INTRODUCTION

Nonnegative systems [1–5] are essential in capturing the phenomenological features of a wide range of dynamical systems involving dynamic states whose values are nonnegative. A subclass of nonnegative dynamical systems are compartmental systems [6–15]. These systems are derived from mass and energy balance considerations and are comprised of homogeneous interconnected macroscopic subsystems or compartments which exchange variable quantities of material via intercompartmetnal flow laws. Since biological and physiological systems have numerous input–output properties related to conservation, dissipation, and transport of mass and energy, nonnegative and compartmental systems are remarkably effective in describing the essential features of these dynamical systems. The range of applications of nonnegative and compartmental systems is not limited to biological and medical systems. Their usage includes demographic systems, epidemic systems [13, 16], ecological systems [17], economic systems [18], telecommunication systems [19], transportation systems, power systems, and large scale systems [20, 21], to cite but a few examples.

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Complex biological and physiological systems typically possess a multiechelon hierarchical hybrid structure characterized by continuous-time dynamics at the lower-level units and logical decision-making units at the higher-level of the hierarchy. The logical decision making units serve to coordinate and reconcile the (sometimes competing) actions of the lower-level units. Due to their multiechelon hierarchical structure, hybrid dynamical systems are capable of simultaneously exhibiting continuous-time dynamics, discrete-time dynamics, logic commands, discrete-events, and resetting events. Hence, hybrid dynamical systems involve an *interacting* countable collection of dynamical systems wherein control actions are not independent of one another and yet not all control actions are of equal precedence. For example, in physiological systems the blood pressure and blood flow to different tissues of the human body are controlled to provide sufficient oxygen to the cells of each organ. Certain organs such as the kidneys normally require higher blood flows than is necessary to satisfy basic oxygen needs. However, during stress (such as hemorrhage) when perfusion pressure falls, perfusion of certain regions (*e.g.*, brain and heart) takes precedence over perfusion of other regions and hierarchical controls (overriding controls) shut down flow to these other regions. The mathematical descriptions of many hybrid dynamical systems can be characterized by impulsive differential equations [22–25]. Impulsive dynamical systems can be viewed as a subclass of hybrid systems and consist of three elements; namely, a continuous-time differential equation, which governs the motion of the dynamical system between impulsive or resetting events; a difference equation, which governs the way the system states are instantaneously changed when a resetting event occurs; and a criterion for determining when the states of the system are to be reset.

In this paper we develop several basic mathematical results on stability and dissipativity of hybrid nonnegative and compartmental dynamical systems. Specifically, using *linear* Lyapunov functions we develop sufficient conditions for Lyapunov stability and asymptotic stability for hybrid nonnegative dynamical systems. The consideration of a linear Lyapunov function leads to a *new* set of Lyapunov-like equations for examining the stability of linear impulsive nonnegative systems. The motivation for using a linear Lyapunov function follows from the fact that the state of a nonnegative dynamical system is nonnegative and hence a linear Lyapunov function is a valid Lyapunov function candidate.

Next, using *linear* and *nonlinear* storage functions with *linear* hybrid supply rates we develop *new* notions of classical dissipativity theory [26, 27] and exponential dissipativity theory [28] for hybrid nonnegative dynamical systems. The overall approach provides a new interpretation of a mass balance for hybrid nonnegative systems with linear hybrid supply rates and linear and nonlinear storage functions. Specifically, we show that dissipativity of a hybrid nonnegative dynamical system involving a linear storage function and a linear hybrid supply rate implies that the system mass transport (resp., change in system mass) is equal to the supplied system flux (resp., mass) over the continuous-time dynamics (resp., the resetting instants) minus the expelled system flux (resp., mass) over the continuous-time dynamics (resp., the resetting instants). In addition, we develop *new* Kalman–Yakubovich–Popov equations for hybrid nonnegative systems for characterizing dissipativity with linear and nonlinear storage functions and linear hybrid supply rates.

Finally, using concepts of dissipativity and exponential dissipativity for hybrid nonnegative dynamical systems, we develop feedback interconnection stability results for nonlinear nonnegative impulsive systems. Specifically, general stability criteria are given for Lyapunov and asymptotic stability of feedback hybrid nonnegative systems. These results can be viewed as a generalization of the positivity and the small gain theorems [29] to hybrid nonnegative systems with linear supply rates involving net input–output system flux.

The contents of the paper are as follows. In Section 2 we establish definitions, notation, and introduce the notion of impulsive nonnegative dynamical systems. Furthermore, we present Lyapunov, asymptotic, and invariant set stability theorems for impulsive nonnegative dynamical systems. Then, in Section 3, we specialize the results of Section 2 to show that nonlinear hybrid compartmental dynamical systems are a special case of hybrid nonnegative dynamical systems. In Section 4 we extend the notion of dissipativity theory to hybrid nonnegative dynamical systems with linear and nonlinear storage functions and linear hybrid supply rates. In addition, we develop new Kalman–Yakubovich–Popov equations in terms of linear and nonlinear storage functions and linear hybrid supply rates for characterizing dissipativeness for hybrid nonnegative dynamical systems. Furthermore, a generalized hybrid mass balance interpretation involving the system's stored or, accumulated mass, expelled mass over the continuous-time dynamics, and the expelled mass at the resetting instants is given. In Section 5 we specialize the results of Section 4 to linear impulsive nonnegative dynamical systems. In Section 6, we use the results of Section 4 to state and prove feedback interconnection stability results for dissipative hybrid nonnegative dynamical systems. Finally, we draw conclusions in Section 7.

## 2 STABILITY THEORY FOR NONLINEAR HYBRID NONNEGATIVE DYNAMICAL SYSTEMS

In this section we provide sufficient conditions for stability of state-dependent impulsive nonnegative dynamical systems; that is, state-dependent impulsive dynamical systems [30] whose solutions remain in the nonnegative orthant for nonnegative initial conditions. First however, we establish notation and definitions that are necessary for developing the main results of this paper. Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}^n$ denote the set of $n \times 1$ column vectors, let $(\,)^{\mathrm{T}}$ denote transpose, let $\mathcal{N}$ denote the set of nonnegative integers, and let $I_n$ denote the $n \times n$ identity matrix. Furthermore, let $\bar{\mathcal{S}}$ and $\partial\mathcal{S}$ denote the closure and the boundary of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. We write $\|\cdot\|$ for the Euclidean vector norm, $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, for the open ball centered at $\alpha$ with radius $\varepsilon$, $V'(x)$ for the Fréchet derivative of $V$ at $x$, $\mathcal{N}(M)$ for the null space of $M$, and $M \geq 0$ (resp., $M > 0$) to denote the fact that the Hermitian matrix $M$ is nonnegative (resp., positive) definite. For $x \in \mathbb{R}^n$ we write $x \geq\geq 0$ (resp., $x \gg 0$) to indicate that every component of $x$ is nonnegative (resp., positive). In this case we say that $x$ is *nonnegative* or *positive*, respectively. Likewise $A \in \mathbb{R}^{n\times m}$ is *nonnegative* or *positive*[1] if every entry of $A$ is nonnegative or positive, respectively, which is written as $A \geq\geq 0$ or $A \gg 0$, respectively. Let $\bar{\mathbb{R}}^n_+$ and $\mathbb{R}^n_+$ denote the nonnegative and positive orthants of $\mathbb{R}^n$; that is, if $x \in \mathbb{R}^n$, then $x \in \bar{\mathbb{R}}^n_+$ and $x \in \mathbb{R}^n_+$ are equivalent, respectively, to $x \geq\geq 0$ and $x \gg 0$. The following definition introduces the notion of $Z$-, $M$-, and essentially nonnegative matrices.

DEFINITION 2.1 [18] *Let $A \in \mathbb{R}^{n\times n}$. $A$ is a* Z-matrix *if $A_{(i,j)} \leq 0$, $i,j = 1,\ldots,n$, $i \neq j$. $A$ is an* M-matrix *(resp., a* nonsingular M-matrix*) if $A$ is a Z-matrix and all the principal minors of $A$ are nonnegative (resp., positive). $A$ is* essentially nonnegative *if $-A$ is a Z-matrix; that is, $A_{(i,j)} \geq 0, i,j = 1,\ldots,n, i \neq j$.*

The following definitions introduce the notions of essentially nonnegative and nonnegative vector fields.

[1] In this paper it is important to distinguish between a square nonnegative (resp., positive) matrix and a nonnegative-definite (resp., positive-definite) matrix.

DEFINITION 2.2 *Let $f_c = [f_{c_1}, \ldots, f_{c_n}]^{\mathrm{T}}: \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\bar{\mathbb{R}}_+^n$. Then $f_c$ is* essentially nonnegative *if $f_{c_i}(x) \geq 0$, for all $i = 1, \ldots, n$, and $x \in \bar{\mathbb{R}}_+^n$ such that $x_i = 0$, where $x_i$ denotes the ith entry of $x$.*

DEFINITION 2.3 *Let $f_d = [f_{d_1}, \ldots, f_{d_n}]^{\mathrm{T}}: \mathcal{D} \to \mathbb{R}^n$, where $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\bar{\mathbb{R}}_+^n$. Then $f_d$ is* nonnegative *if $f_{d_i}(x) \geq 0$, for all $i = 1, \ldots, n$, and $x \in \bar{\mathbb{R}}_+^n$.*

Note that if $f_c(x) = A_c x$, where $A_c \in \mathbb{R}^{n\times n}$, then $f_c$ is essentially nonnegative if and only if $A_c$ is essentially nonnegative. Similarly, if $f_d(x) = (A_d - I_n)x$, where $A_d \in \mathbb{R}^{n\times n}$, then $x + f_d(x)$ is nonnegative for all $x \in \bar{\mathbb{R}}_+^n$ if and only if $A_d$ is nonnegative.

In the first part of this paper we consider nonlinear state-dependent impulsive dynamical systems of the form

$$\dot{x}(t) = f_c(x(t)), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{1}$$

$$\Delta x(t) = f_d(x(t)), \quad x(t) \in \mathcal{Z}_x, \tag{2}$$

where $t \geq 0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open subset of $\mathbb{R}^n$ that contains $\bar{\mathbb{R}}_+^n$ with $0 \in \mathcal{D}$, $\Delta x(t) \stackrel{\Delta}{=} x(t^+) - x(t)$, $f_c: \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_c(0) = 0$, $f_d: \mathcal{D} \to \mathbb{R}^n$ is continuous, and $\mathcal{Z}_x \subset \mathcal{D}$ is the *resetting set*. We refer to the differential Eq. (1) as the *continuous-time dynamics*, and we refer to the difference Eq. (2) as the *resetting law*. Note that since the resetting set $\mathcal{Z}_x$ is a subset of the state space $\bar{\mathbb{R}}_+^n$ and is independent of time, state-dependent impulsive dynamical systems are time-invariant. For a particular trajectory $x(t)$, we let $\tau_k(x_0)$ denote the $k$th instant of time at which $x(t)$ intersects $\mathcal{Z}_x$, and we call the times $\tau_k(x_0)$ the *resetting times*. Thus the trajectory of the system (1), (2) from the initial condition $x(0) = x_0$ is given by $s(t, x_0)$ for $0 < t \leq \tau_1(x_0)$. If and when the trajectory reaches a state $x_1 \stackrel{\Delta}{=} x(\tau_1(x_0))$ satisfying $x_1 \in \mathcal{Z}_x$, then the state is instantaneously transferred to $x_1^+ \stackrel{\Delta}{=} x_1 + f_d(x_1)$ according to the resetting law (2). The trajectory $x(t)$, $\tau_1(x_0) < t \leq \tau_2(x_0)$, is then given by $s(t - \tau_1(x_0), x_1^+)$, and so on. Note that the solution $x(t)$ of (1), (2) is left-continuous; that is, it is continuous everywhere except at the resetting times $\tau_k(x_0)$, and

$$x_k \stackrel{\Delta}{=} x(\tau_k(x_0)) = \lim_{\varepsilon \to 0^+} x(\tau_k(x_0) - \varepsilon), \tag{3}$$

$$x_k^+ \stackrel{\Delta}{=} x(\tau_k(x_0)) + f_d(x(\tau_k(x_0))), \tag{4}$$

for $k = 1, 2, \ldots$.

We make the following additional assumptions:

A1. If $x(t) \in \overline{\mathcal{Z}_x} \backslash \mathcal{Z}_x$, then there exists $\varepsilon > 0$ such that, for all $0 < \delta < \varepsilon$, $s(\delta, x(t)) \notin \mathcal{Z}_x$.
A2. If $x \in \mathcal{Z}_x$, then $x + f_d(x) \notin \mathcal{Z}_x$.

Assumption A1 ensures that if a trajectory reaches the closure of $\mathcal{Z}_x$ at a point that does not belong to $\mathcal{Z}_x$, then the trajectory must be directed away from $\mathcal{Z}_x$; that is, a trajectory cannot enter $\mathcal{Z}_x$ through a point that belongs to the closure of $\mathcal{Z}_x$ but not to $\mathcal{Z}_x$. Furthermore, A2 ensures that when a trajectory intersects the resetting set $\mathcal{Z}_x$, it instantaneously exits $\mathcal{Z}_x$. Finally, we note that if $x_0 \in \mathcal{Z}_x$, then the system initially resets to $x_0^+ = x_0 + f_d(x_0) \notin \mathcal{Z}_x$ which serves as the initial condition for the continuous dynamics (1). It follows from A1 and A2 that $\partial\mathcal{Z}_x \cap \mathcal{Z}_x$ is closed and hence the resetting times $\tau_k(x_0)$ are well defined and distinct. Furthermore, it follows from A2 that if $x^* \in \bar{\mathbb{R}}_+^n$ satisfies $f_d(x^*) = 0$, then $x^* \notin \mathcal{Z}_x$. To see this, suppose $x^* \in \mathcal{Z}_x$. Then $x^* + f_d(x^*) = x^* \in \mathcal{Z}_x$, contradicting A2. In particular, we note $0 \notin \mathcal{Z}_x$. For further insights on Assumptions A1 and A2 the interested reader is referred to [30, 31].

Next, we present a result which shows that $\bar{\mathbb{R}}_+^n$ is an invariant set for (1), (2) if $f_c\colon \mathcal{D} \to \mathbb{R}^n$ is essentially nonnegative and $f_d\colon \mathcal{D} \to \mathbb{R}^n$ is such that $x + f_d(x)$ is nonnegative for all $x \in \bar{\mathbb{R}}_+^n$.

PROPOSITION 2.1 *Suppose $\bar{\mathbb{R}}_+^n \subset \mathcal{D}$. If $f_c\colon \mathcal{D} \to \mathbb{R}^n$ is essentially nonnegative and $f_d\colon \mathcal{Z}_x \to \mathbb{R}^n$ is such that $x + f_d(x)$ is nonnegative, then $\bar{\mathbb{R}}_+^n$ is an invariant set with respect to* (1), (2).

*Proof* Consider the continuous-time dynamical system given by

$$\dot{x}_c(t) = f_c(x_c(t)), \quad x_c(0) = x_{c0}, \quad t \geq 0. \tag{5}$$

Now, it follows from Theorem 3.1 of Ref. [32] (see also Proposition 6.1 of [4]) that since $f_c\colon \mathcal{D} \to \mathbb{R}^n$ is essentially nonnegative, $\bar{\mathbb{R}}_+^n$ is an invariant set with respect to (5); that is, if $x_{c0} \in \bar{\mathbb{R}}_+^n$ then $x_c(t) \in \bar{\mathbb{R}}_+^n$, $t \geq 0$. Now, since with $x_{c0} = x_0$, $x(t) = x_c(t)$, $0 \leq t \leq \tau_1(x_0)$, it follows that $x(t) \in \bar{\mathbb{R}}_+^n$, $0 \leq t \leq \tau_1(x_0)$. Next, since $f_d\colon \mathcal{Z}_x \to \mathbb{R}^n$ is such that $x + f_d(x)$ is nonnegative it follows that $x_1^+ = x(\tau_1(x_0)) + f_d(x(\tau_1(x_0))) \in \bar{\mathbb{R}}_+^n$. Now, since $s(t, x_0) = s(t - \tau_1(x_0), x_1^+)$, $\tau_1(x_0) < t \leq \tau_2(x_0)$, with $x_{c0} = x_1^+$, it follows that $x(t) = x_c(t - \tau_1(x_0)) \in \bar{\mathbb{R}}_+^n$, $\tau_1(x_0) < t \leq \tau_2(x_0)$, and hence $x_2^+ = x(\tau_2(x_0)) + f_d(x(\tau_2(x_0))) \in \bar{\mathbb{R}}_+^n$. Repeating this procedure for $\tau_i(x_0)$, $i = 3, 4, \ldots$, it follows that $\bar{\mathbb{R}}_+^n$ is an invariant set with respect to (1), (2). ■

*Remark 2.1* It is important to note that unlike continuous-time nonnegative systems [4] and discrete-time nonnegative systems [5], Proposition 2.1 provides only sufficient conditions assuring that $\bar{\mathbb{R}}_+^n$ is an invariant set with respect to (1), (2). To see this, let $\mathcal{Z}_x = \partial\bar{\mathbb{R}}_+^n$ and assume $x + f_d(x)$, $x \in \mathcal{Z}_x$, is nonnegative. Then, $\bar{\mathbb{R}}_+^n$ remains invariant with respect to (1), (2) irrespective of whether $f_c(\cdot)$ is essentially nonnegative or not.

Next, we specialize Proposition 2.1 to linear state-dependent impulsive dynamical systems of the form

$$\dot{x}(t) = A_c x(t), \qquad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{6}$$

$$\Delta x(t) = (A_d - I_n)x(t), \qquad x(t) \in \mathcal{Z}_x, \tag{7}$$

where $t \geq 0$, $x(t) \in \bar{\mathbb{R}}_+^n$, $A_c \in \mathbb{R}^{n\times n}$ is essentially nonnegative, $A_d \in \mathbb{R}^{n\times n}$ is nonnegative, and $\mathcal{Z}_x \subset \bar{\mathbb{R}}_+^n$. Note that in this case A2 implies that if $x \in \mathcal{Z}_x$, then $A_d x \notin \mathcal{Z}_x$.

PROPOSITION 2.2 *Let $A_c \in \mathbb{R}^{n\times n}$ and $A_d \in \mathbb{R}^{n\times n}$. If $A_c$ is essentially nonnegative and $A_d$ is nonnegative, then $\bar{\mathbb{R}}_+^n$ is an invariant set with respect to* (6), (7).

*Proof* The proof is a direct consequence of Proposition 2.1 with $f_c(x) = A_c x$ and $f_d(x) = (A_d - I_n)x$. ■

The following definition introduces several types of stability corresponding to the equilibrium solution $x(t) \equiv x_e$ of (1), (2) whose solutions remain in the nonnegative orthant $\bar{\mathbb{R}}_+^n$.

DEFINITION 2.4 *Let $\bar{\mathbb{R}}_+^n$ be invariant with respect to* (1), (2) *and let $x_e \in \bar{\mathbb{R}}_+^n$. Then, the equilibrium solution $x(t) \equiv x_e$ of the hybrid nonnegative dynamical system* (1), (2) *is* Lyapunov stable *if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e) \cap \bar{\mathbb{R}}_+^n$, then $x(t) \in \mathcal{B}_\varepsilon(x_e) \cap \bar{\mathbb{R}}_+^n$, $t \geq 0$. The equilibrium solution $x(t) \equiv x_e$ of the hybrid nonnegative dynamical system* (1), (2) *is* asymptotically stable *if it is Lyapunov stable and there exists $\delta > 0$ such that if $x_0 \in \mathcal{B}_\delta(x_e) \cap \bar{\mathbb{R}}_+^n$, then $\lim_{t\to\infty} x(t) = x_e$. Finally, the equilibrium solution*

$x(t) \equiv x_e$ *of the hybrid nonnegative dynamical system* (1), (2) *is* globally asymptotically stable *if the previous statement holds for all* $x_0 \in \bar{\mathbb{R}}_+^n$.

Next, we present several key results on stability of nonlinear hybrid nonnegative dynamical systems. We note that standard Lyapunov stability theorems [30] and invariant set theorems [30, 33] for nonlinear hybrid dynamical systems can be used directly with the required sufficient conditions verified on $\bar{\mathbb{R}}_+^n$.

THEOREM 2.1 *Suppose there exists a continuously differentiable function* $V: \bar{\mathbb{R}}_+^n \to [0, \infty)$ *satisfying* $V(x_e) = 0$, $V(x) > 0$, $x \neq x_e$, *and*

$$V'(x)f_c(x) \leq 0, \quad x \notin \mathcal{Z}_x, \tag{8}$$

$$V(x + f_d(x)) \leq V(x), \quad x \in \mathcal{Z}_x. \tag{9}$$

*Then the equilibrium solution* $x(t) \equiv x_e$ *of the hybrid nonnegative dynamical system* (1), (2) *is Lyapunov stable. Furthermore, if the inequality* (8) *is strict for all* $x \neq x_e$, *then the equilibrium solution* $x(t) \equiv x_e$ *of the hybrid nonnegative dynamical system* (1), (2) *is asymptotically stable. Finally, if* $V(x) \to \infty$, *as* $\|x\| \to \infty$, *then the above results are global.*

*Proof* The proof is identical to the proof of Theorem 2 of [30] with $\mathcal{D} = \bar{\mathbb{R}}_+^n$ and $\mathcal{Z}_x \subset \bar{\mathbb{R}}_+^n$. ■

Next, we present a generalized Krasovskii-LaSalle invariant set stability theorem for nonlinear hybrid dynamical systems. The following key assumption is needed for the statement of this result.

ASSUMPTION 2.1 [30, 33] *Consider the impulsive nonnegative dynamical system* $\mathcal{G}$ *given by* (1), (2) *and let* $s(t, x_0)$, $t \geq 0$, *denote the solution to* (1), (2) *with initial condition* $x_0$. *Then for every* $x_0 \in \mathcal{D}$, *there exists a dense subset* $\mathcal{T}_{x_0} \subseteq [0, \infty)$ *such that* $[0, \infty) \backslash \mathcal{T}_{x_0}$ *is (finitely or infinitely) countable and for every* $\varepsilon > 0$ *and* $t \in \mathcal{T}_{x_0}$, *there exists* $\delta(\varepsilon, x_0, t) > 0$ *such that if* $\|x_0 - y\| < \delta(\varepsilon, x_0, t)$, $y \in \mathcal{D}$, *then* $\|s(t, x_0) - s(t, y)\| < \varepsilon$.

Assumption 2.1 is a generalization of the standard continuous dependence property for dynamical systems with continuous flows to dynamical systems with left-continuous flows and insures continuous dependence over a dense subset of $[0, \infty)$. Henceforth, we assume that $f_c(\cdot)$, $f_d(\cdot)$, and $\mathcal{Z}_x$ are such that Assumption 2.1 holds. Necessary and sufficient conditions that guarantee that the nonlinear impulsive dynamical system $\mathcal{G}$ given by (1), (2) satisfies Assumption 2.1 are given in Ref. [33]. For further discussion on Assumption 2.1 see Refs. [30, 33].

THEOREM 2.2 *Consider the hybrid nonnegative dynamical system* $\mathcal{G}$ *given by* (1), (2), *assume* $\mathcal{D}_c \subset \bar{\mathbb{R}}_+^n$ *is a compact positively invariant set with respect to* (1), (2), *and assume that there exists a continuously differentiable function* $V: \mathcal{D}_c \to \mathbb{R}$ *such that*

$$V'(x)f_c(x) \leq 0, \qquad x \in \mathcal{D}_c, \quad x \notin \mathcal{Z}_x, \tag{10}$$

$$V(x + f_d(x)) \leq V(x), \qquad x \in \mathcal{D}_c, \quad x \in \mathcal{Z}_x. \tag{11}$$

*Let* $\mathcal{R} \triangleq \{x \in \mathcal{D}_c : x \notin \mathcal{Z}_x, V'(x)f_c(x) = 0\} \cup \{x \in \mathcal{D}_c : x \in \mathcal{Z}_x, V(x + f_d(x)) = V(x)\}$ *and let* $\mathcal{M}$ *denote the largest invariant set contained in* $\mathcal{R}$. *If* $x_0 \in \mathcal{D}_c$, *then* $x(t) \to \mathcal{M}$ as $t \to \infty$.

*Proof* The proof is a direct consequence of Theorem 3 of Ref. [30] with $\mathcal{D} = \bar{\mathbb{R}}_+^n$. ■

Finally, we give sufficient conditions for Lyapunov stability and asymptotic stability for linear hybrid nonnegative dynamical systems using *linear* Lyapunov functions.

THEOREM 2.3 *Consider the linear hybrid dynamical system given by* (6), (7) *where* $A_c \in \mathbb{R}^{n\times n}$ *is essentially nonnegative and* $A_d \in \mathbb{R}^{n\times n}$ *is nonnegative. Then the following statements hold*:

(i) *If there exists vectors* $p, r_c, r_d \in \mathbb{R}^n$ *such that* $p \gg 0$, $r_c \geq\geq 0$, *and* $r_d \geq\geq 0$, *satisfy*

$$0 = x^{\mathrm{T}}(A_c^{\mathrm{T}}p + r_c), \quad x \notin \mathcal{Z}_x, \tag{12}$$

$$0 = x^{\mathrm{T}}(A_d^{\mathrm{T}}p - p + r_d), \quad x \in \mathcal{Z}_x, \tag{13}$$

*then the zero solution* $x(t) \equiv 0$ *to* (6), (7) *is Lyapunov stable.*

(ii) *If there exist vectors* $p, r_c, r_d \in \mathbb{R}^n$ *such that* $p \gg 0$, $r_c \gg 0$, *and* $r_d \geq\geq 0$ *satisfy* (12), (13), *then the zero solution* $x(t) \equiv 0$ *to* (6), (7) *is asymptotically stable.*

*Proof* The result is a direct consequence of Theorem 2.1 with $V(x) = p^{\mathrm{T}}x, f_c(x) = A_c x$, and $f_d(x) = (A_d - I_n)x$. Specifically, in this case, $V'(x)f_c(x) = p^{\mathrm{T}}A_c x = -r_c^{\mathrm{T}}x \leq 0$, $x \notin \mathcal{Z}_x$, and $V(x + f_d(x)) - V(x) = p^{\mathrm{T}}A_d x - p^{\mathrm{T}}x = -r_d^{\mathrm{T}}x \leq 0$, $x \in \mathcal{Z}_x$, so that all the conditions of Theorem 2.1 are satisfied which proves Lyapunov stability. In the case where $r_c \gg 0$ it follows that $V'(x)f_c(x) = p^{\mathrm{T}}A_c x = -r_c^{\mathrm{T}}x < 0$, $x \notin \mathcal{Z}_x$, which proves asymptotic stability. ■

*Remark 2.2* For asymptotic stability, conditions (12) and (13) are implied by $p \gg 0$, $A_c^{\mathrm{T}}p \ll 0$, and $(A_d - I)^{\mathrm{T}}p \leq\leq 0$ which can be solved using a linear matrix inequality (LMI) feasibility problem [34]. Specifically, for a given $r_c \in \mathbb{R}^n$ and $r_d \in \mathbb{R}^n$, note that there exists $p \in \mathbb{R}^n$ such that

$$0 = A_c^{\mathrm{T}}p + r_c, \tag{14}$$

$$0 = A_d^{\mathrm{T}}p - p + r_d, \tag{15}$$

if and only if rank $[A \;\, r] =$ rank $A$, where

$$A \triangleq \begin{bmatrix} A_c^{\mathrm{T}} \\ (A_d - I)^{\mathrm{T}} \end{bmatrix}, \qquad r \triangleq \begin{bmatrix} r_c \\ r_d \end{bmatrix}. \tag{16}$$

Now, there exist $p \gg 0$, $r_c \gg 0$, and $r_d \geq\geq 0$ such that (12), (13) are satisfied if and only if $p \gg 0$ and $-Ap \geq\geq 0$.

## 3 HYBRID COMPARTMENTAL DYNAMICAL SYSTEMS

In this section, we specialize the results of Section 2 to hybrid compartmental dynamical systems. Specifically, we show that nonlinear hybrid compartmental dynamical systems are a special case of hybrid nonnegative dynamical systems. To see this, let $x_i(t)$, $i = 1, \ldots, n$, denote the mass (and hence a nonnegative quantity) of the $i$th subsystem of the hybrid compartmental system shown in Figure 1, let $a_{cii}(x) \geq 0$, $x \notin \mathcal{Z}_x$, denote the rate of flow of material loss of the $i$th continuous-time subsystem, let $w_{ci}(t) \geq 0$, $t \geq 0$, $i = 1, \ldots, n$, denote the rate of mass inflow supplied to the $i$th continuous-time subsystem, and let $\phi_{cij}(x(t))$,

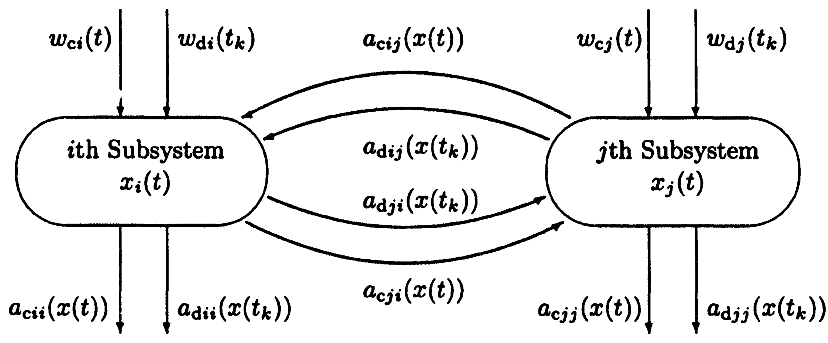


FIGURE 1 Nonlinear hybrid compartmental interconnected subsystem model.

$t \geq 0$, $i \neq j$, $i,j = 1, \ldots, n$, denote the net mass flow (or flux) from the $j$th continuous-time subsystem to the $i$th continuous-time subsystem given by $\phi_{cij}(x(t)) = a_{cij}(x(t)) - a_{cji}(x(t))$, where the rate of material flows are such that $a_{cij}(x) \geq 0$, $x \notin \mathcal{Z}_x$, $i \neq j$, $i,j = 1, \ldots, n$. Similarly, for the resetting dynamics, let $a_{dii}(x) \geq 0$, $x \in \mathcal{Z}_x$, denote the material loss of the $i$th discrete-time subsystem, let $w_{di}(t_k) \geq 0$, $i = 1, \ldots, n$, denote the mass inflow supplied to the $i$th discrete-time subsystem, and let $\phi_{dij}(t_k)$, $i \neq j$, $i,j = 1, \ldots, n$, denote the net mass exchange from the $j$th discrete-time subsystem to the $i$th discrete-time subsystem given by $\phi_{dij}(x(t_k)) = a_{dij}(x(t_k)) - a_{dji}(x(t_k))$, where $t_k = \tau_k(x_0)$ and the material flows are such that $a_{dij}(x) \geq 0$, $x \in \mathcal{Z}_x$, $i \neq j$, $i,j = 1, \ldots, n$. Hence, a mass balance for the whole hybrid compartmental system yields

$$\dot{x}_i(t) = -a_{cii}(x(t)) + \sum_{j=1, i \neq j}^{n} \phi_{cij}(x(t)) + w_{ci}(t), \quad x(t) \notin \mathcal{Z}_x, \quad i = 1, \ldots, n, \tag{17}$$

$$\Delta x_i(t) = -a_{dii}(x(t)) + \sum_{j=1, i \neq j}^{n} \phi_{dij}(x(t)) + w_{di}(t), \quad x(t) \in \mathcal{Z}_x, \quad i = 1, \ldots, n, \tag{18}$$

or, equivalently,

$$\dot{x}(t) = f_c(x(t)) + w_c(t), \qquad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{19}$$

$$\Delta x(t) = f_d(x(t)) + w_d(t), \qquad x(t) \in \mathcal{Z}_x, \tag{20}$$

where $x(t) \triangleq [x_1(t), \ldots, x_n(t)]^{\mathrm{T}}$, $w_c(t) \triangleq [w_{c1}(t), \ldots, w_{cn}(t)]^{\mathrm{T}}$, $w_d(t) \triangleq [w_{d1}(t), \ldots, w_{dn}(t)]^{\mathrm{T}}$, and for $i,j = 1, \ldots, n$,

$$f_{c_i}(x) = -a_{cii}(x) + \sum_{j=1, i \neq j}^{n} [a_{cij}(x) - a_{cji}(x)], \tag{21}$$

$$f_{d_i}(x) = -a_{dii}(x) + \sum_{j=1, i \neq j}^{n} [a_{dij}(x) - a_{dji}(x)]. \tag{22}$$

Since all mass flows as well as compartment sizes are nonnegative, it follows that for all $i = 1, \ldots, n$, $f_{ci}(x) \geq 0$ for all $x \notin \mathcal{Z}_x$ whenever $x_i = 0$ and whatever the values of $x_j$, $j \neq i$, and $x_i + f_{di}(x) \geq 0$ for all $x \in \mathcal{Z}_x$. The above physical constraints are implied by $a_{cij}(x) \geq 0$, $a_{cii}(x) \geq 0$, $x \notin \mathcal{Z}_x$, $a_{dij}(x) \geq 0$, $a_{dii}(x) \geq 0$, $x \in \mathcal{Z}_x$, $w_{ci} \geq 0$, $w_{di} \geq 0$, for all $i,j = 1, \ldots, n$, and if $x_i = 0$, then $a_{cii}(x) = 0$ and $a_{cji}(x) = 0$ for all $i,j = 1, \ldots, n$, so that $\dot{x}_i \geq 0$. In this case, $f_c(x)$, $x \notin \mathcal{Z}_x$, is essentially nonnegative and $x + f_d(x) \geq 0$, $x \in \mathcal{Z}_x$, and hence the hybrid compartmental model given by (17), (18) is a hybrid nonnegative dynamical system. Taking the total mass of the compartmental system $V(x) = e^{\mathrm{T}}x = \sum_{i=1}^{n} x_i$, where $e^{\mathrm{T}} \triangleq [1, 1, \ldots, 1]$, as a Lyapunov function for the undisturbed ($w_c(t) \equiv 0$ and $w_d(t_k) \equiv 0$) system (17), (18) and assuming $a_{cij}(0) = 0$, $i, j = 1, \ldots, n$, it follows that

$$\begin{aligned}\dot{V}(x) &= \sum_{i=1}^{n} \dot{x}_i \\ &= -\sum_{i=1}^{n} a_{cii}(x) + \sum_{i=1}^{n} \sum_{j=1, i\neq j}^{n} [a_{cij}(x) - a_{cji}(x)] \\ &= -\sum_{i=1}^{n} a_{cii}(x) \\ &\leq 0, \quad x \notin \mathcal{Z}_x,\end{aligned}$$

and

$$\begin{aligned}\Delta V(x) &= \sum_{i=1}^{n} \Delta x_i \\ &= -\sum_{i=1}^{n} a_{dii}(x) + \sum_{i=1}^{n} \sum_{j=1, i\neq j}^{n} [a_{dij}(x) - a_{dji}(x)] \\ &= -\sum_{t=1}^{n} a_{dii}(x) \\ &\leq 0, \quad x \in \mathcal{Z}_x,\end{aligned}$$

which, by Theorem 2.1, shows that the zero solution $x(t) \equiv 0$ of the nonlinear hybrid compartmental system given by (17), (18) is Lyapunov stable. If (17), (18) with $w_c(t) \equiv 0$ and $w_d(t_k) \equiv 0$ has losses (outflows) from all compartments over the continuous-time dynamics, then $a_{cii}(x) > 0$, $x \notin \mathcal{Z}_x$, $x \neq 0$, and by Theorem 2.1 the zero solution $x(t) \equiv 0$ to (17), (18) is asymptotically stable.

It is interesting to note that in the linear case $a_{cii}(x) = a_{cii}x_i$, $\phi_{cij}(x) = a_{cij}x_j - a_{cji}x_i$, $a_{dii}(x) = a_{dii}x_i$, and $\phi_{dij}(x) = a_{dji}x_j - a_{dij}x_i$, where $a_{cij} \geq 0$, and $a_{dij} \geq 0$, $i, j = 1, \ldots, n$, so that (19), (20) become

$$\dot{x}(t) = A_c x(t) + w_c(t), \quad x(0) = x_0, \quad x(t) \notin \mathcal{Z}_x, \tag{23}$$

$$\Delta x(t) = (A_d - I_n)x(t) + w_d(t), \quad x(t) \in \mathcal{Z}_x, \tag{24}$$

where for $i, j = 1, \ldots, n$,

$$A_{c(i,j)} = \begin{cases} -\sum_{l=1}^{n} a_{cli}, & i = j \\ a_{cij}, & i \neq j, \end{cases} \tag{25}$$

$$A_{d(i,j)} = \begin{cases} 1 - \sum_{l=1}^{n} a_{dli}, & i = j \\ a_{dij}, & i \neq j. \end{cases} \tag{26}$$

Note that since at any given instant of time compartmental mass can only be transported, stored, or discharged but not created and the maximum amount of mass that can be transported and/or discharged cannot exceed the mass in a compartment, it follows that $1 \geq \sum_{l=1}^{n} a_{dli}$. Thus $A_c$ is an essentially nonnegative matrix and $A_d$ is a nonnegative matrix and hence the hybrid compartmental model given by (19), (20) is a hybrid nonnegative dynamical system.

The hybrid compartmental system (17), (18) with no inflows; that is, $w_{ci}(t) \equiv 0$ and $w_{di}(t_k) \equiv 0, i = 1, \ldots, n$, is said to be *inflow-closed*. Alternatively, if (17), (18) possesses no losses (outflows) it is said to be *outflow-closed*. A hybrid compartmental system is said to be *closed* if it is inflow-closed and outflow-closed. Note that for a closed-system $\dot{V}(x) = 0$, $x \notin \mathcal{Z}_x$, and $\Delta V(x) = 0, x \in \mathcal{Z}_x$ which shows that the total mass inside a closed system is conserved. Alternatively, in the case where $a_{cii}(x) \neq 0, x \notin \mathcal{Z}_x$, $a_{dii}(x) \neq 0, x \in \mathcal{Z}_x$, $w_{ci}(t) \neq 0$, and $w_{di}(t_k) \neq 0$, $i = 1, \ldots, n$, it follows that (17), (18) can be equivalently written as

$$\dot{x}(t) = [J_{cn}(x(t)) - D_c(x(t))]\left(\frac{\partial V}{\partial x}(x(t))\right)^{\mathrm{T}} + w_c(t), \quad x(t) \notin \mathcal{Z}_x, \tag{27}$$

$$\Delta x(t) = [J_{dn}(x(t)) - D_d(x(t))]\left(\frac{\partial V}{\partial x}(x(t))\right)^{\mathrm{T}} + w_d(t), \quad x(t) \in \mathcal{Z}_x, \tag{28}$$

where $J_{cn}(x)$ and $J_{dn}(x)$ are skew-symmetric matrix functions with $J_{cn(i,i)}(x) = 0, J_{dn(i,i)}(x) = 0, J_{cn(i,j)}(x) = a_{cij}(x) - a_{cji}(x)$, and $J_{dn(i,j)}(x) = a_{dij}(x) - a_{dji}(x), i \neq j, D_c(x) = \mathrm{diag}[a_{c11}(x), a_{c22}(x), \ldots, a_{cnn}(x)] \geq\geq 0$, and $D_d(x) = \mathrm{diag}[a_{d11}(x), a_{d22}(x), \ldots, a_{dnn}(x)] \geq\geq 0, x \in \overline{\mathbb{R}}_+^n$. Hence, a linear hybrid compartmental system is a *hybrid port-controlled Hamiltonian system* with a Hamiltonian $\mathcal{H}(x) = V(x) = e^{\mathrm{T}}x$ representing the total mass in the system, $D_c(x)$ representing the outflow dissipation over the continnuous-time dynamics, $D_d(x)$ representing the outflow dissipation at the resetting instants, $w_c(t)$ representing the supplied flux to the system over the continuous-time dynamics, and $w_d(t_k)$ representing the supplied mass to the system at the resetting instants. This observation shows that hybrid compartmental systems are conservative systems. This will be further elaborated on in the following sections.

## 4 DISSIPATIVITY THEORY FOR HYBRID NONNEGATIVE DYNAMICAL SYSTEMS

In this section we extend the notion of dissipativity to nonlinear impulsive nonnegative dynamical systems. Specifically, we consider nonlinear impulsive dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f_c(x(t)) + G_c(x(t))u_c(t), \quad x(0) = x_0, \quad (x(t), u_c(t)) \notin \mathcal{Z}, \tag{29}$$

$$\Delta x(t) = f_d(x(t)) + G_d(x(t))u_d(t), \quad (x(t), u_c(t)) \in \mathcal{Z}, \tag{30}$$

$$y_c(t) = h_c(x(t)) + J_c(x(t))u_c(t), \quad (x(t), u_c(t)) \notin \mathcal{Z}, \tag{31}$$

$$y_d(t) = h_d(x(t)) + J_d(x(t))u_d(t), \quad (x(t), u_c(t)) \in \mathcal{Z}, \tag{32}$$

where $t \geq 0, x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}, \Delta x(t) \triangleq x(t^+) - x(t), u_c(t) \in U_c \subseteq \mathbb{R}^{m_c}, u_d(t_k) \in U_d \subseteq \mathbb{R}^{m_d}$, $t_k$ denotes the $k$th instant of time at which $(x(t), u_c(t))$ intersects $\mathcal{Z} \subset \mathcal{D} \times U_c$ for a particular trajectory $x(t)$ and input $u_c(t), y_c(t) \in Y_c \subseteq \mathbb{R}^{l_c}$, $y_d(t_k) \in Y_d \subseteq \mathbb{R}^{l_d}, f_c: \mathcal{D} \to \mathbb{R}^n$ is Lipschitz continuous and satisfies $f_c(0) = 0, G_c: \mathcal{D} \to \mathbb{R}^{n \times m_c}, f_d: \mathcal{D} \to \mathbb{R}^n$ is continuous, $G_d: \mathcal{D} \to \mathbb{R}^{n \times m_d}, h_c: \mathcal{D} \to \mathbb{R}^{l_c}$ and satisfies $h_c(0) = 0$, $J_c: \mathcal{D} \to \mathbb{R}^{l_c \times m_c}, h_d: \mathcal{D} \to \mathbb{R}^{l_d}, J_d: \mathcal{D} \to \mathbb{R}^{l_d \times m_d}$, and $\mathcal{Z} \subset \mathcal{D} \times U_c$. Here, we assume that $u_c(\cdot)$ and $u_d(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions such that $(u_c(t), u_d(t_k)) \in U_c \times U_d$ for all $t \geq 0$ and $k \in \mathcal{N}_{[0,t)} \triangleq \{k: 0 \leq t_k < t\}$, where the constraint set $U_c \times U_d$ is given with $(0, 0) \in U_c \times U_d$. Furthermore, we assume that the set $\mathcal{Z} \triangleq \{(x, u_c): \chi(x, u_c) = 0\}$, where $\chi: \mathcal{D} \times U_c \to \mathbb{R}$. In addition, we assume that the system functions $f_c(\cdot), f_d(\cdot), G_c(\cdot), G_d(\cdot), h_c(\cdot), h_d(\cdot), J_c(\cdot)$, and $J_d(\cdot)$ are continuously differentiable mappings. Finally, for the nonlinear dynamical system (29) we assume that the required properties for the existence and uniqueness of solutions are satisfied such that (29) has a unique solution for all $t \in \mathbb{R}$ [22, 24].

Next, we provide definitions and several results concerning dynamical systems of the form (29)–(32) with nonnegative inputs and nonnegative outputs.

DEFINITION 4.1 *The nonlinear dynamical system $\mathcal{G}$ given by* (29)–(32) *with* $x(0) = 0$ *is* input–output[2] nonnegative *if the hybrid output* $(y_c(t), y_d(t_k)), t \geq 0, k \in \mathcal{N}$, *is nonnegative for every nonnegative hybrid input* $(u_c(t), u_d(t_k)), t \geq 0, k \in \mathcal{N}$.

DEFINITION 4.2 *The nonlinear dynamical system $\mathcal{G}$ given by* (29)–(32) *is* nonnegative *if for every* $x(0) \in \overline{\mathbb{R}}_+^n$ *and nonnegative hybrid input* $(u_c(t), u_d(t_k)), t \geq 0, k \in \mathcal{N}$, *the solution* $x(t), t \geq 0$, *to* (29), (30) *and the hybrid output* $(y_c(t), y_d(t_k)), t \geq 0, k \in \mathcal{N}$, *are nonnegative.*

PROPOSITION 4.1 *Consider the nonlinear dynamical system $\mathcal{G}$ given by* (29)–(32). *If* $f_c: \mathcal{D} \to \mathbb{R}^n$ *is essentially nonnegative,* $f_d: \mathcal{D} \to \mathbb{R}^n$ *is such that* $x + f_d(x)$ *is nonnegative for all* $x \in \overline{\mathbb{R}}_+^n$, $G_c(x) \geq\geq 0$, $G_d(x) \geq\geq 0$, $h_c(x) \geq\geq 0$, $h_d(x) \geq\geq 0$, $J_c(x) \geq\geq 0$, *and* $J_d(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$, *then $\mathcal{G}$ is nonnegative.*

*Proof* The proof is similar to the proof of Proposition 2.1 and hence is omitted. ∎

For the impulsive dynamical system $\mathcal{G}$ given by (29)–(32) a function $(s_c(u_c, y_c), s_d(u_d, y_d))$, where $s_c: U_c \times Y_c \to \mathbb{R}$ and $s_d: U_d \times Y_d \to \mathbb{R}$ are such that $s_c(0,0) = 0$ and $s_d(0,0) = 0$, is called a *hybrid supply rate* if $s_c(u_c, y_c)$ is locally integrable; that is, for all input–output pairs $u_c(t) \in U_c, y_c(t) \in Y_c, s_c(\cdot,\cdot)$ satisfies $\int_t^{\hat{t}} |s_c(u_c(s), y_c(s))| ds < \infty, t, \hat{t} \geq 0$. Note that since all input–output pairs $u_d(t_k) \in U_d, y_d(t_k) \in Y_d$, are defined for discrete instants, $s_d(\cdot,\cdot)$ satisfies $\sum_{k \in \mathcal{N}_{[t,\hat{t})}} |s_d(u_d(t_k), y_d(t_k))| < \infty$, where $k \in \mathcal{N}_{[t,\hat{t})} \triangleq \{k: t \leq t_k < \hat{t}\}$. The following definition introduces the notion of dissipativity and exponential dissipativity for a nonlinear hybrid nonnegative dynamical system.

DEFINITION 4.3 *The impulsive dynamical system $\mathcal{G}$ given by* (29)–(32) *is* exponentially dissipative (*resp.*, dissipative) *with respect to the hybrid supply rate* $(s_c, s_d)$ *if there exists a continuous, nonnegative-definite function* $V_s: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$ *called a storage function and a scalar* $\varepsilon > 0$ (*resp.*, $\varepsilon = 0$) *such that* $V_s(0) = 0$ *and the dissipation inequality*

$$\begin{aligned} e^{\varepsilon T} V_s(x(T)) \leq e^{\varepsilon t_0} V_s(x(t_0)) &+ \int_{t_0}^{T} e^{\varepsilon t} s_c(u_c(t), y_c(t))\, \mathrm{d}t \\ &+ \sum_{k \in \mathcal{N}_{[t_0,t)}} e^{\varepsilon t_k} s_d(u_d(t_k), y_d(t_k)), \quad T \geq t_0, \end{aligned} \tag{33}$$

*is satisfied for all* $T \geq t_0$. *The impulsive dynamical system given by* (29)–(32) *is* lossless *with respect to the hybrid supply rate* $(s_c, s_d)$ *if the dissipation inequality* (33) *is satisfied as an equality with* $\varepsilon = 0$ *for all* $T \geq t_0$.

The following result gives necessary and sufficient conditions for dissipativity over an interval $t \in (t_k, t_{k+1}]$ involving the consecutive resetting times $t_k$ and $t_{k+1}$. First, however, the following definition is required.

[2] The outputs here refer to measured outputs or observations and may have nothing to do with material outflows of the nonnegative compartmental system.

DEFINITION 4.4 *An impulsive dynamical system $\mathcal{G}$ given by* (29)–(32) *is* zero-state observable *if* $(u_c(t), u_d(t_k)) \equiv (0,0), (y_c(t), y_d(t_k)) \equiv (0,0), k \in \mathcal{N}$, *implies* $x(t) \equiv 0$. *An impulsive dynamical system $\mathcal{G}$ given by* (29)–(32) *is* strongly zero-state observable *if* $u_c(t) \equiv 0, y_c(t) \equiv 0$ *implies* $x(t) \equiv 0$. *An impulsive dynamical system $\mathcal{G}$ is* completely reachable *if for all* $(t_0, x_i) \in \mathbb{R} \times \mathcal{D}$, *there exist a finite time* $t_i \le t_0$, *square integrable inputs* $u_c(t)$ *defined on* $[t_i, t_0]$, *and inputs* $u_d(t_k)$ *defined on* $k \in \mathcal{N}_{[t_i,t_0)}$, *such that the state* $x(t)$, $t \ge t_i$, *can be driven from* $x(t_i) = 0$ *to* $x(t_0) = x_i$. *Finally, an impulsive system $\mathcal{G}$ is* minimal *if it is zero-state observable and completely reachable.*

THEOREM 4.1 *Assume $\mathcal{G}$ is completely reachable. Then $\mathcal{G}$ is dissipative with respect to the hybrid supply rate $(s_c, s_d)$ if and only if there exists a continuous, nonnegative-definite function $V_s: \bar{\mathbb{R}}^n_+ \to \bar{\mathbb{R}}_+$ such that, for all $k \in \mathcal{N}$,*

$$V_s(x(\hat{t})) - V_s(x(t)) \le \int_t^{\hat{t}} s_c(u_c(s), y_c(s))\mathrm{d}s, \quad t_k < t \le \hat{t} \le t_{k+1}, \tag{34}$$

$$V_s(x(t_k) + f_d(x(t_k)) + G_d(x(t_k))u_d(t_k)) - V_s(x(t_k)) \le s_d(u_d(t_k), y_d(t_k)). \tag{35}$$

*Furthermore, $\mathcal{G}$ is exponentially dissipative with respect to the hybrid supply rate $(s_c, s_d)$ if and only if there exists a continuous, nonnegative-definite function $V_s: \bar{\mathbb{R}}^n_+ \to \bar{\mathbb{R}}_+$ such that*

$$e^{\varepsilon\hat{t}} V_s(x(\hat{t})) - e^{\varepsilon t} V_s(x(t)) \le \int_t^{\hat{t}} e^{\varepsilon s} s_c(u_c(s), y_c(s))\,\mathrm{d}s, \quad t_k < t \le \hat{t} \le t_{k+1}, \tag{36}$$

$$V_s(x(t_k) + f_d(x(t_k)) + G_d(x(t_k))u_d(t_k)) - V_s(x(t_k)) \le s_d(u_d(t_k), y_d(t_k)). \tag{37}$$

*Finally, $\mathcal{G}$ is lossless with respect to the hybrid supply rate $(s_c, s_d)$ if and only if there exists a continuous, nonnegative-definite function $V_s: \bar{\mathbb{R}}^n_+ \to \bar{\mathbb{R}}_+$ such that (34) and (35) are satisfied as equalities.*

Proof The proof is identical to the proof of Theorem 6 of Ref. [30]. ■

*Remark 4.1* If $V_s(\cdot)$ is continuously differentiable then an equivalent statement for dissipativeness of the impulsive dynamical system $\mathcal{G}$ with respect to the hybrid supply rate $(s_c, s_d)$ is

$$\dot{V}_s(x(t)) \le s_c(u_c(t), y_c(t)), \quad t_k < t \le t_{k+1}, \tag{38}$$

$$\Delta V_s(x(t_k)) \le s_d(u_d(t_k), y_d(t_k)), \quad k \in \mathcal{N}, \tag{39}$$

where $\dot{V}_s(\cdot)$ denotes the total derivative of $V_s(x(t))$ along the state trajectories $x(t)$, $t \in (t_k, t_{k+1}]$, of the impulsive dynamical system (29)–(32) and $\Delta V_s(x(t_k)) \triangleq V_s(x(t_k^+)) - V_s(x(t_k)) = V_s(x(t_k) + f_d(x(t_k)) + G_d(x(t_k))u_d(t_k)) - V_s(x(t_k))$, $k \in \mathcal{N}$, denotes the difference of the storage function $V_s(x)$ at the resetting times $t_k$, $k \in \mathcal{N}$, of the impulsive dynamical system (29)–(32). Furthermore, an equivalent statement for exponential dissipativeness of the impulsive dynamical system $\mathcal{G}$ with respect to the supply rate $(s_c, s_d)$ is given by

$$\dot{V}_s(x(t)) + \varepsilon V_s(x(t)) \le s_c(u_c(t), y_c(t)), \quad t_k < t \le t_{k+1}, \tag{40}$$

and (39).

The following result presents Kalman–Yakubovich–Popov conditions for hybrid nonnegative dynamical systems with linear hybrid supply rates of the form $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$, where $q_c \in \mathbb{R}^{l_c}$, $q_c \neq 0$, $r_c \in \mathbb{R}^{m_c}$, $r_c \neq 0$, $q_d \in \mathbb{R}^{l_d}$, $q_d \neq 0$, $r_d \in \mathbb{R}^{m_d}$, and $r_d \neq 0$. For the remainder of the section we assume that $U_c = \overline{\mathbb{R}}_+^{m_c}$, $U_d = \overline{\mathbb{R}}_+^{m_d}$, and $\mathcal{Z} = \mathcal{Z}_x \times \overline{\mathbb{R}}^{m_c}$ so that resetting occurs only when $x(t)$ intersects $\mathcal{Z}_x$.

THEOREM 4.2 *Let $q_c \in \mathbb{R}^{l_c}$, $r_c \in \mathbb{R}^{m_c}$, $q_d \in \mathbb{R}^{l_d}$, and $r_d \in \mathbb{R}^{m_d}$. Consider the nonlinear hybrid dynamical system $\mathcal{G}$ given by* (29)–(32) *where $f_c: \mathcal{D} \to \mathbb{R}^n$ is essentially nonnegative, $f_d: \mathcal{Z}_x \to \mathbb{R}^n$ is such that $x + f_d(x)$ nonnegative, $G_c(x) \geq\geq 0$, $G_d(x) \geq\geq 0$, $h_c(x) \geq\geq 0$, $h_d(x) \geq\geq 0$, $J_c(x) \geq\geq 0$, and $J_d(x) \geq\geq 0$, $x \in \overline{\mathbb{R}}_+^n$. If there exist functions $V_s: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\ell_c: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\ell_d: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+$, $\mathcal{W}_c: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^{m_c}$, $\mathcal{W}_d: \overline{\mathbb{R}}_+^n \to \overline{\mathbb{R}}_+^{m_d}$, and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_s(\cdot)$ is continuously differentiable, nonnegative definite, $V_s(0) = 0$,*

$$V_s(x + f_d(x) + G_d(x)u_d) = V_s(x + f_d(x)) + V_s'(x + f_d(x))G_d(x)u_d, \quad x \in \mathcal{Z}_x, \quad u_d \in \overline{\mathbb{R}}_+^{m_d}, \tag{41}$$

*and*

$$0 = V_s'(x)f_c(x) + \varepsilon V_s(x) - q_c^{\mathrm{T}} h_c(x) + \ell_c(x), \quad x \notin \mathcal{Z}_x, \tag{42}$$
$$0 = V_s'(x)G_c(x) - q_c^{\mathrm{T}} J_c(x) - r_c^{\mathrm{T}} + \mathcal{W}_c^{\mathrm{T}}(x), \quad x \notin \mathcal{Z}_x, \tag{43}$$
$$0 = V_s(x + f_d(x)) - V_s(x) - q_d^{\mathrm{T}} h_d(x) + \ell_d(x), \quad x \in \mathcal{Z}_x, \tag{44}$$
$$0 = V_s'(x + f_d(x))G_d(x) - q_d^{\mathrm{T}} J_d(x) - r_d^{\mathrm{T}} + \mathcal{W}_d^{\mathrm{T}}(x), \quad x \in \mathcal{Z}_x, \tag{45}$$

*then the nonlinear impulsive system $\mathcal{G}$ given by* (29)–(32) *is exponentially dissipative (resp., dissipative) with respect to the linear hybrid supply rate $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$.*

*Proof* For any admissible input $u_c(\cdot)$, $t, \hat{t} \in \mathbb{R}$, $t_k < t \leq \hat{t} \leq t_{k+1}$, and $k \in \mathcal{N}$, it follows from (42), (43) that for all $x \notin \mathcal{Z}_x$ and $u_c \in \overline{\mathbb{R}}_+^{m_c}$,

$$\begin{aligned}
\dot{V}_s(x) + \varepsilon V_s(x) &= V_s'(x)(f_c(x) + G_c(x)u_c) + \varepsilon V_s(x) \\
&= q_c^{\mathrm{T}} h_c(x) - \ell_c(x) + q_c^{\mathrm{T}} J_c(x)u_c + r_c^{\mathrm{T}} u_c - \mathcal{W}_c^{\mathrm{T}}(x)u_c \\
&= q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c - \ell_c(x) - \mathcal{W}_c^{\mathrm{T}}(x)u_c \\
&\leq q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c \\
&= s_c(u_c, y_c).
\end{aligned} \tag{46}$$

Next, it follows from (44), (45), and the structural storage function constraint (41) that for all $x \in \mathcal{Z}_x$ and $u_d \in \overline{\mathbb{R}}_+^{m_d}$,

$$\begin{aligned}
\Delta V_s(x) &= V_s(x + f_d(x) + G_d(x)u_d) - V_s(x) \\
&= V_s(x + f_d(x)) - V_s(x) + V_s'(x + f_d(x))G_d(x)u_d \\
&= q_d^{\mathrm{T}} h_d(x) - \ell_d(x) + q_d^{\mathrm{T}} J_d(x)u_d + r_d^{\mathrm{T}} u_d - \mathcal{W}_d^{\mathrm{T}}(x)u_d \\
&= s_d(u_d, y_d) - \ell_d(x) - \mathcal{W}_d^{\mathrm{T}}(x)u_d \\
&\leq s_d(u_d, y_d).
\end{aligned} \tag{47}$$

Now, using (46) and (47) the result follows from Theorem 4.1. ■

*Remark 4.2* The structural constraint (41) on the system storage function is similar to the structural constraint invoked in standard nonlinear discrete-time dissipativity theory [35,36] and hybrid dissipativity theory [30]. However, since $V_s: \bar{\mathbb{R}}_+^n \to \bar{\mathbb{R}}_+$, we can take a first-order Taylor expansion in (41) as opposed to the second-order Taylor expansion given in Refs. [30, 35, 36].

*Remark 4.3* As in standard dissipativity theory with quadratic supply rates [37], the concepts of linear supply rates and linear storage functions provide a generalized mass balance interpretation. Specifically, using (42)–(45), it follows that for $\hat{t} \geq t \geq 0$ and $k \in \mathcal{N}_{[t,\hat{t})}$

$$\begin{aligned}\int_t^{\hat{t}} [q_c^{\mathrm{T}} y_c(s) + r_c^{\mathrm{T}} u_c(s)]\,\mathrm{d}s + \sum_{k\in\mathcal{N}_{[t,\hat{t})}} [q_d^{\mathrm{T}} y_d(t_k) + r_d^{\mathrm{T}} u_d(t_k)] \\ = V_s(x(\hat{t})) - V_s(x(t)) + \int_t^{\hat{t}} [\ell_c^{\mathrm{T}}(x(s))x(s) + \mathcal{W}_c^{\mathrm{T}}(x(s))u_c(s)]\,\mathrm{d}s \\ + \sum_{k\in\mathcal{N}_{[t,\hat{t})}} [\ell_d^{\mathrm{T}}(x(t_k))x(t_k) + \mathcal{W}_d^{\mathrm{T}}(x(t_k))u_d(t_k)], \end{aligned}\tag{48}$$

which can be interpreted as a generalized mass balance equation where $V_s(x(\hat{t})) - V_s(x(t))$ is the stored mass of the nonlinear hybrid dynamical system, the second path-dependent term on the right corresponds to the expelled mass of the nonnegative system over the continuous-time dynamics, and the third discrete term on the right correspond to the expelled mass at the resetting instants. Equivalently, it follows from Theorem 4.1 that (48) can be rewritten as

$$\dot{V}_s(x(t)) = q_c^{\mathrm{T}} y_c(t) + r_c^{\mathrm{T}} u_c(t) - [\ell_c^{\mathrm{T}}(x(t))x(t) + \mathcal{W}_c^{\mathrm{T}}(x(t))u_c(t)], \quad t_k < t < t_{k+1}, \tag{49}$$

$$\Delta V_s(x(t_k)) = q_d^{\mathrm{T}} y_d(t_k) + r_d^{\mathrm{T}} u_d(t_k) - [\ell_d^{\mathrm{T}}(x(t_k))x(t_k) + \mathcal{W}_d^{\mathrm{T}}(x(t_k))u_d(t_k)], \quad k \in \mathcal{N}, \tag{50}$$

which yields a set of generalized mass conservation equations. Specifically, (49) and (50) show that the system mass transport (resp., change in system mass) over the interval $t \in (t_k, t_{k+1}]$ (resp., the resetting instants) is equal to the supplied system flux (resp., mass) minus the expelled system flux (resp., mass).

*Remark 4.4* Note that if a nonnegative dynamical system $\mathcal{G}$ is dissipative with respect to the linear hybrid supply rate $(q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$ with a continuously differentiable, positive-definite storage function and if $q_c \leq\leq 0, q_d \leq\leq 0$, and $(u_c(t), u_d(t_k)) \equiv (0,0)$, it follows that $\dot{V}_s(x(t)) \leq q_c^{\mathrm{T}} y_c(t) \leq 0, t \geq 0$, and $\Delta V_s(x(t_k)) \leq q_d^{\mathrm{T}} y_d(t_k) \leq 0, k \in \mathcal{N}$. Hence, the undisturbed $((u_c(t), u_d(t_k)) \equiv (0,0))$ system $\mathcal{G}$ is Lyapunov stable. Furthermore, if a nonnegative dynamical system $\mathcal{G}$ is exponentially dissipative with respect to the hybrid linear supply rate $(q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c; q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$ with a continuously differentiable, positive-definite storage function and $q_c \leq\leq 0$, $q_d \leq\leq 0$, and $(u_c(t), u_d(t_k)) \equiv (0,0)$, it follows that $\dot{V}_s(x(t)) \leq -\varepsilon V_s(x(t)) + q_c^{\mathrm{T}} y_c(t) < 0, x(t) \neq 0, t \geq 0$, where $\varepsilon > 0$, and $\Delta V_s(x(t_k)) \leq q_d^{\mathrm{T}} y_d(t_k) \leq 0, k \in \mathcal{N}$. Hence, the undisturbed $((u_c(t), u_d(t_k)) \equiv (0,0))$ system $\mathcal{G}$ is asymptotically stable.

Next, we provide necessary and sufficient conditions for the case where $\mathcal{G}$ given by (29)–(32) is lossless with respect to the linear hybrid supply rate of the form $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$.

THEOREM 4.3 *Let $q_c \in \mathbb{R}^{l_c}, r_c \in \mathbb{R}^{m_c}, q_d \in \mathbb{R}^{l_d}$, and $r_d \in \mathbb{R}^{m_d}$. Consider the nonlinear hybrid dynamical system $\mathcal{G}$ given by (29)–(32) where $f_c: \mathcal{D} \to \mathbb{R}^n$ is essentially nonnegative, $f_d: \mathcal{Z}_x \to \mathbb{R}^n$ is such that $x + f_d(x)$ is nonnegative, $G_c(x) \geq\geq 0, G_d(x) \geq\geq 0, h_c(x) \geq\geq 0, h_d(x) \geq\geq 0, J_c(x) \geq\geq 0$, and $J_d(x) \geq\geq 0, x \in \bar{\mathbb{R}}^n_+$. Then $\mathcal{G}$ is lossless with respect to the supply rate $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$ if and only if there exists a function $V_s: \bar{\mathbb{R}}^n_+ \to \bar{\mathbb{R}}_+$ such that $V_s(\cdot)$ is continuously differentiable, nonnegative definite, $V_s(0) = 0$, and for all $x \in \mathcal{Z}_x, u_d \in \bar{\mathbb{R}}^{m_d}_+$, (41) holds, and*

$$0 = V_s'(x) f_c(x) - q_c^{\mathrm{T}} h_c(x), \quad x \notin \mathcal{Z}_x, \tag{51}$$

$$0 = V_s'(x) G_c(x) - q_c^{\mathrm{T}} J_c(x) - r_c^{\mathrm{T}}, \quad x \notin \mathcal{Z}_x, \tag{52}$$

$$0 = V_s(x + f_d(x)) - V_s(x) - q_d^{\mathrm{T}} h_d(x), \quad x \in \mathcal{Z}_x, \tag{53}$$

$$0 = V_s'(x + f_d(x)) G_d(x) - q_d^{\mathrm{T}} J_d(x) - r_d^{\mathrm{T}}, \quad x \in \mathcal{Z}_x, \tag{54}$$

*Proof* Sufficiency follows as in the proof of Theorem 4.2. To show necessity, suppose that the nonlinear impulsive dynamical system $\mathcal{G}$ is lossless with respect to the linear supply rate $(s_c, s_d)$. Then, it follows that for all $k \in \mathcal{N}$,

$$V_s(x(\hat{t})) - V_s(x(t)) = \int_t^{\hat{t}} s_c(u_c(s), y_c(s))\, \mathrm{d}s, \quad t_k < t \leq \hat{t} \leq t_{k+1}, \tag{55}$$

and

$$V_s(x(t_k) + f_d(x(t_k)) + G_d(x(t_k)) u_d(t_k)) = V_s(x(t_k)) + s_d(u_d(t_k), y_d(t_k)). \tag{56}$$

Now, dividing (55) by $\hat{t} - t^+$ and letting $\hat{t} \to t^+$, (55) is equivalent to

$$\dot{V}_s(x(t)) = V_s'(x(t))[f_c(x(t)) + G_c(x(t)) u_c(t)] = s_c(u_c(t), y_c(t)), \quad t_k < t \leq t_{k+1}. \tag{57}$$

Next, with $t = 0$, it follows from (57) that

$$V_s'(x_0)[f_c(x_0) + G_c(x_0) u_c(0)] = s_c(u_c(0), y_c(0)), \quad x_0 \notin \mathcal{Z}_x, \;\; u_c(0) \in \bar{\mathbb{R}}^{m_c}_+. \tag{58}$$

Since $x_0 \notin \mathcal{Z}_x$ is arbitrary, it follows that

$$\begin{aligned} V_s'(x)[f_c(x) + G_c(x) u_c] &= q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c \\ &= q_c^{\mathrm{T}} h_c(x) + (r_c^{\mathrm{T}} + q_c^{\mathrm{T}} J_c(x)) u_c, \quad x \notin \mathcal{Z}_x, \;\; u_c \in \bar{\mathbb{R}}^{m_c}_+. \end{aligned}$$

Now, setting $u_c = 0$ yields (51) which further yields (52). Next, it follows from (56) with $k = 1$ that

$$V_s(x(t_1) + f_d(x(t_1)) + G_d(x(t_1)) u_d(t_1)) = V_s(x(t_1)) + s_d(u_d(t_1), y_d(t_1)). \tag{59}$$

Now, since the continuous-time dynamics (29) are Lipschitz, it follows that for arbitrary $x \in \mathcal{Z}_x$ there exists $x_0 \notin \mathcal{Z}_x$ such that $x(t_1) = x$. Hence, it follows from (59) that

$$\begin{aligned} V_s(x + f_d(x) + G_d(x) u_d) &= V_s(x) + q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d \\ &= V_s(x) + q_d^{\mathrm{T}} h_d(x) + (r_d^{\mathrm{T}} + q_d^{\mathrm{T}} J_d(x)) u_d, \quad x \in \mathcal{Z}_x, u_d \in \bar{\mathbb{R}}^{m_d}_+. \end{aligned} \tag{60}$$

Since the right-hand-side of (60) is linear in $u_d$ it follows that $V_s(x+f_d(x)+G_d(x)u_d)$ is linear in $u_d$ and hence there exists $P_{1u_d}: \bar{\mathbb{R}}_+^n \to \mathbb{R}^{1\times m_d}$ such that

$$V_s(x+f_d(x)+G_d(x)u_d) = V_s(x+f_d(x)) + P_{1u_d}(x)u_d. \tag{61}$$

Since $V_s(\cdot)$ is continuously differentiable, applying Taylor series expansion on (61) about $u_d = 0$ yields

$$P_{1u_d}(x) = \left.\frac{\partial V_s(x+f_d(x)+G_d(x))u_d}{\partial u_d}\right|_{u_d=0} = V_s'(x+f_d(x))G_d(x). \tag{62}$$

Now, using (61) and equating coefficients of equal powers in (60) yields (53), (54). ■

Next, we provide a key definition for hybrid nonnegative dynamical systems which are dissipative with respect to very special supply rate.

DEFINITION 4.5 *A hybrid nonnegative dynamical system $\mathcal{G}$ of the form* (29)–(32) *is* nonaccumulative (*resp.*, exponentially nonaccumulative) *if $\mathcal{G}$ is dissipative* (*resp.*, *exponentially dissipative*) *with respect to the supply rate* $(s_c(u_c, y_c), s_d(u_d, y_d)) = (e^{\mathrm{T}}u_c - e^{\mathrm{T}}y_c, e^{\mathrm{T}}u_d - e^{\mathrm{T}}y_d)$.

If $\mathcal{G}$ is nonaccumulative, then it follows that

$$\dot{V}_s(x(t)) \leq e^{\mathrm{T}}u_c(t) - e^{\mathrm{T}}y_c(t), \quad t_k < t \leq t_{k+1}, \tag{63}$$

$$\Delta V_s(x(t_k)) \leq e^{\mathrm{T}}u_d(t_k) - e^{\mathrm{T}}y_d(t_k), \quad k \in \mathcal{N}. \tag{64}$$

If the components $u_{ci}(\cdot), i = 1, \ldots, m_c$, and $u_{di}(\cdot), i = 1, \ldots, m_d$, of $u_c(\cdot)$ and $u_d(\cdot)$, respectively, denote flux and mass inputs of the hybrid system $\mathcal{G}$ and the components $y_{ci}(\cdot), i = 1, \ldots, l_c$, and $y_{di}(\cdot), i = 1, \ldots, l_d$, of $y_c(\cdot)$ and $y_d(\cdot)$, respectively, denote flux and mass outputs of the hybrid system $\mathcal{G}$, then nonaccumulativity implies that the system mass transport (resp., change in system mass) is always less than or equal to the difference between the system flux (resp., mass) input and system flux (resp., mass) output.

Next, we show that all hybrid compartmental systems with measured outputs corresponding to material outflows are nonaccumulative. Specifically, consider (27), (28) with storage function $V_s(x) = e^{\mathrm{T}}x$ and hybrid outputs $y_c = D_c(x)(\partial V/\partial x)^{\mathrm{T}} = [a_{c11}(x), a_{c22}(x), \ldots, a_{cnn}(x)]^{\mathrm{T}}$ and $y_d = D_d(x)(\partial V/\partial x)^{\mathrm{T}} = [a_{d11}(x), a_{d22}(x), \ldots, a_{dnn}(x)]^{\mathrm{T}}$. Now, it follows that

$$\begin{aligned}\dot{V}_s(x) &= e^{\mathrm{T}}\left[[J_{cn}(x) - D_c(x)]\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} + w_c\right] \\ &= e^{\mathrm{T}}w_c - e^{\mathrm{T}}y_c + e^{\mathrm{T}}J_{cn}(x)e \\ &= e^{\mathrm{T}}w_c - e^{\mathrm{T}}y_c, \quad x \notin \mathcal{Z}_x,\end{aligned} \tag{65}$$

and

$$\begin{aligned}\Delta V_s(x) &= e^{\mathrm{T}}\left[[J_{dn}(x) - D_d(x)]\left(\frac{\partial V}{\partial x}\right)^{\mathrm{T}} + w_d\right] \\ &= e^{\mathrm{T}}w_d - e^{\mathrm{T}}y_d + e^{\mathrm{T}}J_{dn}(x)e \\ &= e^{\mathrm{T}}w_d - e^{\mathrm{T}}y_d, \quad x \in \mathcal{Z}_x,\end{aligned} \tag{66}$$

which shows that all hybrid compartmental systems are lossless with respect to the linear supply rate $(s_c, s_d) = (e^{\mathrm{T}}w_c - e^{\mathrm{T}}y_c, e^{\mathrm{T}}w_d - e^{\mathrm{T}}y_d)$. Alternatively, if the hybrid outputs $y_c$ and $y_d$ correspond to a partial observation of the material outflows, then it can easily be shown that the nonlinear hybrid compartmental system is dissipative with respect to the supply rate $(s_c, s_d) = (e^{\mathrm{T}}w_c - e^{\mathrm{T}}y_c, e^{\mathrm{T}}w_d - e^{\mathrm{T}}y_d)$.

## 5 SPECIALIZATION TO LINEAR IMPULSIVE DYNAMICAL SYSTEMS

In this section we specialize the results of Section 4 to the case of linear impulsive dynamical systems. Specifically, setting $f_c(x) = A_c x$, $G_c(x) = B_c$, $h_c(x) = C_c x$, $J_c(x) = D_c$, $f_d(x) = (A_d - I_n)x$, $G_d(x) = B_d$, $h_d(x) = C_d x$, and $J_d(x) = D_d$, the nonnegative state-dependent impulsive dynamical system given by (29)–(32) specializes to

$$\dot{x}(t) = A_c x(t) + B_c u_c(t), \quad x(t) \notin \mathcal{Z}_x, \tag{67}$$

$$\Delta x(t) = (A_d - I_n)x(t) + B_d u_d(t), \quad x(t) \in \mathcal{Z}_x, \tag{68}$$

$$y_c(t) = C_c x(t) + D_c u_c(t), \quad x(t) \notin \mathcal{Z}_x, \tag{69}$$

$$y_d(t) = C_d x(t) + D_d u_d(t), \quad x(t) \in \mathcal{Z}_x, \tag{70}$$

where $A_c \in \mathbb{R}^{n\times n}$ is essentially nonnegative, $B_c \in \mathbb{R}^{n\times m_c}$, $C_c \in \mathbb{R}^{l_c\times n}$, $D_c \in \mathbb{R}^{l_c\times m_c}$, $A_d \in \mathbb{R}^{n\times n}$ is nonnegative, $B_d \in \mathbb{R}^{n\times m_d}$, $C_d \in \mathbb{R}^{l_d\times n}$, and $D_d \in \mathbb{R}^{l_d\times m_d}$.

THEOREM 5.1 *Let $q_c \in \mathbb{R}^{l_c}, r_c \in \mathbb{R}^{m_c}, q_d \in \mathbb{R}^{l_d}$, and $r_d \in \mathbb{R}^{m_d}$. Consider the linear impulsive dynamical system $\mathcal{G}$ given by (67)–(70) and assume that $A_c$ is essentially nonnegative, $A_d$ is nonnegative, $B_c \geq\geq 0, B_d \geq\geq 0, C_c \geq\geq 0, C_d \geq\geq 0, D_c \geq\geq 0$, and $D_d \geq\geq 0$. If there exist vectors $p \in \overline{\mathbb{R}}_+^n, l_c \in \overline{\mathbb{R}}_+^n, l_d \in \overline{\mathbb{R}}_+^n, w_c \in \overline{\mathbb{R}}_+^{m_c}, w_d \in \overline{\mathbb{R}}_+^{m_d}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that*

$$0 = x^{\mathrm{T}}(A_c^{\mathrm{T}}p + \varepsilon p - C_c^{\mathrm{T}}q_c + l_c), \quad x \notin \mathcal{Z}_x, \tag{71}$$

$$0 = B_c^{\mathrm{T}}p - D_c^{\mathrm{T}}q_c - r_c + w_c, \tag{72}$$

$$0 = x^{\mathrm{T}}(A_d^{\mathrm{T}}p - p - C_d^{\mathrm{T}}q_d + l_d), \quad x \in \mathcal{Z}_x, \tag{73}$$

$$0 = B_d^{\mathrm{T}}p - D_d^{\mathrm{T}}q_d - r_d + w_d, \tag{74}$$

*then the linear impulsive dynamical system $\mathcal{G}$ given by (67)–(70) is exponentially dissipative (resp., dissipative) with respect to the linear supply rate $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}}y_c + r_c^{\mathrm{T}}u_c, q_d^{\mathrm{T}}y_d + r_d^{\mathrm{T}}u_d)$.*

*Proof* The proof follows from Theorem 4.2 with $f_c(x) = A_c x$, $G_c(x) = B_c$, $h_c(x) = C_c x$, $J_c(x) = D_c$, $f_d(x) = (A_d - I_n)x$, $G_d(x) = B_d$, $h_d(x) = C_d x$, $J_d(x) = D_d$, $V_s(x) = p^{\mathrm{T}}x$, $\ell_c(x) = l_c^{\mathrm{T}}x$, $\ell_d(x) = l_d^{\mathrm{T}}x$, $\mathcal{W}_c(x) = w_c$, $\mathcal{W}_d(x) = w_d$. ■

*Remark 5.1* For a given $l_c \in \mathbb{R}^n, w_c \in \mathbb{R}^{m_c}, l_d \in \mathbb{R}^n$, and $w_d \in \mathbb{R}^{m_d}$, note that if rank $[M \quad y] =$ rank $M$, where

$$M \triangleq \begin{bmatrix} A_c^{\mathrm{T}} + \varepsilon I \\ B_c \\ A_d^{\mathrm{T}} - I \\ B_d^{\mathrm{T}} \end{bmatrix}, \qquad y \triangleq \begin{bmatrix} C_c^{\mathrm{T}}q_c - l_c \\ D_c^{\mathrm{T}}q_c + r_c - w_c \\ C_d^{\mathrm{T}}q_d - l_d \\ D_d^{\mathrm{T}}q_d + r_d - w_d \end{bmatrix}, \tag{75}$$

then there exist $p \in \mathbb{R}^n$ such that (71)–(74) are satisfied. Now, if there exists $p \in \mathbb{R}^n$ such that inequalities

$$p \geq\geq 0 \tag{76}$$

$$z - Mp \geq\geq 0, \tag{77}$$

where

$$z \triangleq \begin{bmatrix} C_c^{\mathrm{T}} q_c \\ D_c^{\mathrm{T}} q_c + r_c \\ C_d^{\mathrm{T}} q_d \\ D_d^{\mathrm{T}} q_d + r_d \end{bmatrix}, \tag{78}$$

are satisfied, then there exists $l_c \geq\geq 0, w_c \geq\geq 0, l_d \geq\geq 0$, and $w_d \geq\geq 0$ such that (71)–(74) hold. Equations (76), (77) comprise a set of $3n + m_c + m_d$ linear inequalities with $p_i, i = 1, \ldots, n$, variables and hence the feasibility of $p \geq\geq 0$ such that (76), (77) hold can be checked by standard linear matrix inequality (LMI) techniques [34].

Next, we provide sufficient conditions for the case where $\mathcal{G}$ given by (67)–(70) is lossless with respect to the linear supply rate $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$.

THEOREM 5.1 *Let $q_c \in \mathbb{R}^{l_c}$, $r_c \in \mathbb{R}^{m_c}$, $q_d \in \mathbb{R}^{l_d}$, and $r_d \in \mathbb{R}^{m_d}$. Consider the linear impulsive dynamical system $\mathcal{G}$ given by* (67)–(70) *and assume that $A_c$ is essentially nonnegative, $A_d$ is nonnegative, $B_c \geq\geq 0, B_d \geq\geq 0, C_c \geq\geq 0, C_d \geq\geq 0, D_c \geq\geq 0$, and $D_d \geq\geq 0$. If there exist $p \in \overline{\mathbb{R}}_+^n$ such that*

$$0 = x^{\mathrm{T}}(A_c^{\mathrm{T}} p - C_c^{\mathrm{T}} q_c), \quad x \notin \mathcal{Z}_x, \tag{79}$$

$$0 = B_c^{\mathrm{T}} p - D_c^{T} q_c - r_c, \tag{80}$$

$$0 = x^{\mathrm{T}}(A_d^{\mathrm{T}} p - p - C_d^{\mathrm{T}} q_d), \quad x \in \mathcal{Z}_x, \tag{81}$$

$$0 = B_d^{\mathrm{T}} p - D_d^{\mathrm{T}} q_d - r_d, \tag{82}$$

*then the linear impulsive dynamical system $\mathcal{G}$ given by* (67)–(70) *is lossless with respect to the linear supply rate $(s_c(u_c, y_c), s_d(u_d, y_d)) = (q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$.*

*Proof* The proof follows from Theorem 4.3 with $f_c(x) = A_c x$, $G_c(x) = B_c$, $h_c(x) = C_c x$, $J_c(x) = D_c$, $f_d(x) = (A_d - I_n)x$, $G_d(x) = B_d$, $h_d(x) = C_d x$, $J_d(x) = D_d$, and $V_s(x) = p^{\mathrm{T}} x$. ■

## 6 FEEDBACK INTERCONNECTIONS OF NONLINEAR HYBRID NONNEGATIVE DYNAMICAL SYSTEMS

In this section we consider stability of feedback interconnections of hybrid nonnegative dynamical systems. We begin by considering the nonlinear impulsive hybrid dynamical system $\mathcal{G}$ given by (29)-(32) with the nonlinear impulsive nonnegative feedback system $\mathcal{G}_c$ given by

$$\dot{x}_c(t) = f_{cc}(x_c(t)) + G_{cc}(x_c(t))u_{cc}(t), \quad x_c(0) = x_{c0}, \quad (x_c(t), u_{cc}(t)) \notin \mathcal{Z}_c, \tag{83}$$

$$\Delta x_c(t) = f_{dc}(x_c(t)) + G_{dc}(x_c(t))u_{dc}(t), \quad (x_c(t), u_{cc}(t)) \in \mathcal{Z}_c, \tag{84}$$

$$y_{cc}(t) = h_{cc}(x_c(t)), \quad (x_c(t), u_{cc}(t)) \notin \mathcal{Z}_c, \tag{85}$$

$$y_{dc}(t) = h_{dc}(x_c(t)), \quad (x_c(t), u_{cc}(t)) \in \mathcal{Z}_c, \tag{86}$$

where $t \geq 0, x_c(t) \in \overline{\mathbb{R}}_+^{n_c}$, $\Delta x_c(t) \triangleq x_c(t^+) - x_c(t)$, $u_{cc}(t) \in U_{cc} \subseteq \overline{\mathbb{R}}_+^{m_{cc}}$, $u_{dc}(t_k) \in U_{dc} \subseteq \overline{\mathbb{R}}_+^{m_{dc}}$, $t_k$ denotes the $k$th instant of time at which $(x_c(t), u_{cc}(t))$ intersects $\mathcal{Z}_c \subset \overline{\mathbb{R}}_+^{n_c} \times U_{cc}$ for a particular trajectory $x_c(t)$ and input $u_{cc}(t), y_{cc}(t) \in Y_{cc} \subseteq \overline{\mathbb{R}}_+^{l_{cc}}, y_{dc}(t_k) \in Y_{dc} \subseteq \overline{\mathbb{R}}_+^{l_{dc}}, f_{cc}: \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ is Lipschitz continuous and is essentially nonnegative, $G_{cc} : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c \times m_{cc}}$ and satisfies $G_{cc}(x_c) \geq\geq 0, x_c \in \overline{\mathbb{R}}_+^{n_c}, f_{dc} : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$ is continuous and is such that $x_c + f_{dc}(x_c)$ is nonnegative for all $x_c \in \overline{\mathbb{R}}_+^{n_c}$, $G_{dc} : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c \times m_{dc}}$ and satisfies $G_{dc}(x_c) \geq\geq 0, x_c \in \overline{\mathbb{R}}_+^{n_c}, h_{cc}$: $\mathbb{R}^{n_c} \to \mathbb{R}^{l_{cc}}$ and satisfies $h_{cc}(x_c) \geq\geq 0, x_c \in \overline{\mathbb{R}}_+^{n_c}, h_{dc} : \mathbb{R}^{n_c} \to \mathbb{R}^{l_{dc}}$ and satisfies $h_{dc}(x_c) \geq\geq 0$, $x_c \in \overline{\mathbb{R}}_+^{n_c}, m_{cc} = l_c, m_{dc} = l_d, l_{cc} = m_c, l_{dc} = m_d$, and $\mathcal{Z}_c \triangleq \mathcal{Z}_{cx_c} \times \mathcal{Z}_{cu_{cc}} \subset \overline{\mathbb{R}}_+^{n_c} \times U_{cc}$. Here, we assume that $u_{cc}(\cdot)$ and $u_{dc}(\cdot)$ are restricted to the class of *admissible* inputs consisting of measurable functions such that $(u_{cc}(t), u_{dc}(t_k)) \in U_{cc} \times U_{dc}$ for all $t \geq 0$ and $k \in \mathcal{N}_{[0,t)} \triangleq \{k : 0 \leq t_k < t\}$, where the constraint set $U_{cc} \times U_{dc}$ is given with $(0, 0) \in U_{cc} \times U_{dc}$. Furthermore, we assume that the set $\mathcal{Z}_c = \{(x_c, u_{cc}) : \mathcal{X}_c(x_c, u_{cc}) = 0\}$, where $\mathcal{X}_c : \overline{\mathbb{R}}_+^{n_c} \times U_{cc} \to \mathbb{R}$. In addition, we assume that the system functions $f_{cc}(\cdot), f_{dc}(\cdot), G_{cc}(\cdot), G_{dc}(\cdot), h_{cc}(\cdot)$, and $h_{dc}(\cdot)$ are continuously differentiable mappings. Finally, for the nonlinear dynamical system (83) we assume that the required properties for the existence and uniqueness of solutions are satisfied such that (83) has a unique solution for all $t \in \mathbb{R}$ [22, 24]. Note that with the positive feedback interconnection given by Figure 2, $(u_{cc}, u_{dc}) = (y_c, y_d)$ and $(y_{cc}, y_{dc}) = (u_c, u_d)$. Furthermore, even though the input-output pairs of the feedback interconnection shown on Figure 2 consist of two-vector inputs/two-vector outputs, at any given instant of time a single-vector input/single-vector output is active. Next, we define the closed-loop resetting set

$$\tilde{\mathcal{Z}}_{\tilde{x}} \triangleq \mathcal{Z}_x \times \mathcal{Z}_{cx_c} \cup \{(x, x_c) : (h_c(x) + J_c(x)h_{cc}(x_c), h_{cc}(x_c)) \in \mathcal{Z}_{cu_{cc}} \times \mathcal{Z}_{u_c}\}. \tag{87}$$

Note that since the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed, it follows that $\tilde{\mathcal{Z}}_{\tilde{x}}$ is well defined and depends on the closed-loop states $\tilde{x} \triangleq [x^{\mathrm{T}} x_c^{\mathrm{T}}]^{\mathrm{T}}$. As in Section 2, here we assume that the solution $s(t, \tilde{x}_o)$ to the dynamical system resulting from the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is such that Assumption 2.1 is satisfied.

The following theorem gives sufficient conditions for Lyapunov and asymptotic stability of the positive feedback interconnection given by Figure 2. For the statement of this result let $\mathcal{T}^c_{x_0,u_c}$ denote the set of resetting times of $\mathcal{G}$, let $\mathcal{T}_{x_0,u_c}$ denote the complement of $\mathcal{T}^c_{x_0,u_c}$; that is, $\mathcal{T}_{x_0,u_c} = [0,\infty)\backslash\mathcal{T}^c_{x_0,u_c}$, let $\mathcal{T}^c_{x_{c0},u_{cc}}$ denote the set of resetting times of $\mathcal{G}_c$ and let $\mathcal{T}_{x_{c0},u_{cc}}$ denote the complement of $\mathcal{T}^c_{x_{c0},u_{cc}}$; that is, $\mathcal{T}_{x_{c0},u_{cc}} = [0,\infty)\backslash\mathcal{T}^c_{x_{c0},u_{cc}}$.

THEOREM 6.1 *Let* $q_c \in \mathbb{R}^{l_c}, r_c \in \mathbb{R}^{m_c}, q_d \in \mathbb{R}^{l_d}, r_d \in \mathbb{R}^{m_d}, q_{cc} \in \mathbb{R}^{l_{cc}}, r_{cc} \in \mathbb{R}^{m_{cc}}, q_{dc} \in \mathbb{R}^{l_{dc}}$, $r_{dc} \in \mathbb{R}^{m_{dc}}$. *Consider the nonlinear impulsive nonnegative dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ given by* (29)–(32) *and* (83)–(86), *respectively. Assume $\mathcal{G}$ and $\mathcal{G}_c$ are dissipative with respect to the linear hybrid supply rates* $(q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$ *and* $(q_{cc}^{\mathrm{T}} y_{cc} + r_{cc}^{\mathrm{T}} u_{cc}, q_{dc}^{T} y_{dc} + r_{dc}^{\mathrm{T}} u_{dc})$

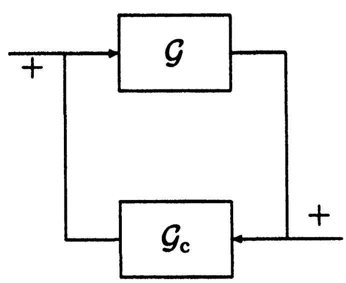


FIGURE 2 Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$.

*and with continuously differentiable, positive definite storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that $V_s(0) = 0$ and $V_{sc}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $q_c + \sigma r_{cc} \leq\leq 0, r_c + \sigma q_{cc} \leq\leq 0, q_d + \sigma r_{dc} \leq\leq 0$, and $r_d + \sigma q_{dc} \leq\leq 0$. Then the following statements hold:*

(i) *The positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is Lyapunov stable.*
(ii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable and $q_c + \sigma r_{cc} \ll 0$ and $r_c + \sigma q_{cc} \ll 0$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable.*
(iii) *If $\mathcal{G}$ is strongly zero-state observable, $\mathcal{G}_c$ is exponentially dissipative with respect to the linear hybrid supply rate $(q_{cc}^{\mathrm{T}} y_{cc} + r_{cc}^{\mathrm{T}} u_{cc}, q_{dc}^{\mathrm{T}} y_{dc} + r_{dc}^{\mathrm{T}} u_{dc})$, and rank $G_{cc}(0) = m_{cc}$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable.*
(iv) *If $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative with respect to linear hybrid supply rates $(q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c, q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d)$ and $(q_{cc}^{\mathrm{T}} y_{cc} + r_{cc}^{\mathrm{T}} u_{cc}, q_{dc}^{\mathrm{T}} y_{dc} + r_{dc}^{\mathrm{T}} u_{dc})$, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable.*

*Proof* Let $\tilde{\mathcal{T}}^c \triangleq \mathcal{T}^c_{x_0,u_c} \cup \mathcal{T}^c_{x_{c0},u_{cc}}$ and $t_k \in \tilde{\mathcal{T}}^c$, $k \in \mathcal{N}$. Note that it follows from Assumptions A1 and A2 that the resetting times $t_k = \tau_k(\tilde{x}_0)$ for the feedback system are well defined and distinct for every closed-loop trajectory. Furthermore, note that the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is defined by the closed-loop dynamics given by

$$\begin{bmatrix} \dot{x}(t) \\ \dot{x}_c(t) \end{bmatrix} = \begin{bmatrix} f_c(x(t)) + G_c(x(t))h_{cc}(x_c(t)) \\ f_{cc}(x_c(t)) + G_{cc}(x_c(t))h_c(x(t)) + G_{cc}(x_c(t))J_c(x(t))h_{cc}(x_c(t)) \end{bmatrix}, (x(t), x_c(t)) \notin \tilde{\mathcal{Z}}_{\tilde{x}} \tag{88}$$

$$\begin{bmatrix} \Delta x(t) \\ \Delta x_c(t) \end{bmatrix} = \begin{bmatrix} f_d(x(t)) + G_d(x(t))h_{dc}(x_c(t)) \\ f_{dc}(x_c(t)) + G_{dc}(x_c(t))h_d(x(t)) + G_{dc}(x_c(t))J_d(x(t))h_{dc}(x_c(t)) \end{bmatrix}, (x(t), x_c(t)) \in \tilde{\mathcal{Z}}_{\tilde{x}}, \tag{89}$$

which implies that

$$\tilde{f}_c(\tilde{x}) \triangleq \begin{bmatrix} f_c(x) + G_c(x)h_{cc}(x_c) \\ f_{cc}(x_c) + G_{cc}(x_c)h_c(x) + G_{cc}(x_c)J_c(x)h_{cc}(x_c) \end{bmatrix} \tag{90}$$

is essentially nonnegative and

$$\tilde{x} + \tilde{f}_d(\tilde{x}) \triangleq \begin{bmatrix} x + f_d(x) + G_d(x)h_{dc}(x_c) \\ x_c + f_{dc}(x_c) + G_{dc}(x_c)h_d(x) + G_{dc}(x_c)J_d(x)h_{dc}(x_c) \end{bmatrix} \tag{91}$$

is nonnegative. Hence, it follows from Proposition 2.1 that $\bar{\mathbb{R}}_+^n \times \bar{\mathbb{R}}_+^{n_c}$ is an invariant set with respect to the closed-loop system (88), (89), and thus $x(t) \geq\geq 0, x_c(t) \geq\geq 0, u_c(t) = y_{cc}(t) \geq\geq 0, u_d(t_k) = y_{dc}(t_k) \geq\geq 0, y_c(t) = u_{cc}(t) \geq\geq 0$, and $y_d(t_k) = u_{dc}(t_k) \geq\geq 0, t \geq 0$, $k \in \mathcal{N}$.

(i) Consider the Lyapunov function candidate $V(x, x_c) = V_s(x) + \sigma V_{sc}(x_c)$. Now, the corresponding Lyapunov derivative of $V(x, x_c)$ along the state trajectories $(x(t), x_c(t))$, $t \in (t_k, t_{k+1})$, is given by

$$\begin{aligned} \dot{V}(x(t), x_c(t)) &= \dot{V}_s(x(t)) + \sigma \dot{V}_{sc}(x_c(t)) \leq q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c + \sigma(q_{cc}^{\mathrm{T}} y_{cc} + r_{cc}^{\mathrm{T}} u_{cc}) \\ &\leq 0, \quad (x(t), x_c(t)) \notin \tilde{\mathcal{Z}}_{\tilde{x}}, \end{aligned} \tag{92}$$

and the Lyapunov difference of $V(x, x_c)$ at the resetting times $t_k$, $k \in \mathcal{N}$, is given by

$$\begin{aligned}\Delta V(x(t_k), x_c(t_k)) = \Delta V_s(x(t_k)) + \sigma \Delta V_{sc}(x_c(t_k)) &\le q_d^{\mathrm{T}} y_d + r_d^{\mathrm{T}} u_d \\ &+ \sigma(q_{dc}^{\mathrm{T}} y_{dc} + r_{dc}^{\mathrm{T}} u_{dc}) \le 0, \quad (x(t), x_c(t)) \in \tilde{\mathcal{Z}}_{\tilde{x}}.\end{aligned} \tag{93}$$

Now, Lyapunov stability of the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ follows as a direct consequence of Theorem 2.1.

(ii) With $V(x, x_c) = V_s(x) + \sigma V_{sc}(x_c)$, Lyapunov stability follows from (i). Furthermore, if $q_c + \sigma r_{cc} \ll 0$ and $r_c + \sigma q_{cc} \ll 0$, then, using the observability assumptions, it follows from (92), (93) that the largest invariant set contained in

$$\begin{aligned}\mathcal{R} \overset{\Delta}{=} &\{(x, x_c) \in \mathcal{D} \times \mathbb{R}^{n_c} : (x, x_c) \notin \tilde{\mathcal{Z}}_{\tilde{x}}, \dot{V}(x, x_c) = 0\} \\ &\cup \{(x, x_c) \in \mathcal{D} \times \mathbb{R}^{n_c} : (x, x_c) \in \tilde{\mathcal{Z}}_{\tilde{x}}, \Delta V(x, x_c) = 0\}\end{aligned} \tag{94}$$

is given by $\mathcal{M} = \{(0, 0)\}$. Hence, asymptotic stability of the closed-loop system follows from Theorem 2.2.

(iii) If $\mathcal{G}_c$ is exponentially dissipative it follows that for some scalar $\varepsilon_{cc} > 0$

$$\begin{aligned}\dot{V}(x(t), x_c(t)) &= \dot{V}_s(x(t)) + \sigma \dot{V}_{sc}(x_c(t)) \\ &\le -\sigma \varepsilon_{cc} V_{sc}(x_c(t)) + q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c + \sigma(q_{cc}^{\mathrm{T}} y_{cc} + r_{cc}^{\mathrm{T}} u_{cc}) \\ &\le -\sigma \varepsilon_{cc} V_{sc}(x_c(t)) \le 0, \quad (x(t), x_c(t)) \notin \tilde{\mathcal{Z}}_{\tilde{x}},\end{aligned} \tag{95}$$

and the Lyapunov difference $\Delta V(x(t_k), x_c(t_k))$, $k \in \mathcal{N}$, at the resetting times for the closed-loop system satisfies (93). Since $V_{sc}(x_c)$ is positive definite, note that $\dot{V}(x, x_c) = 0$ for all $(x, x_c) \notin \tilde{\mathcal{Z}}_{\tilde{x}}$ only if $x_c = 0$. Furthermore, since rank $G_{cc}(0) = m_{cc}$, it follows that on every invariant set $\mathcal{M}$ contained in $\mathcal{R}$ given by (94), $u_{cc}(t) = y_c(t) \equiv 0$ and hence $y_{cc}(t) = u_c(t) \equiv 0$ so that $\dot{x}(t) = f_c(x(t))$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that $\mathcal{R} = \{(0, 0)\} \cup \{(x, x_c) \in \overline{\mathbb{R}}_+^n \times \overline{\mathbb{R}}_+^{n_c} : (x, x_c) \in \tilde{\mathcal{Z}}_{\tilde{z}}, \Delta V(x, x_c) = 0\}$ contains no solution other than the trivial solution $(x(t), x_c(t)) \equiv (0, 0)$. Hence, it follows from Theorem 2.2 that the closed-loop system is asymptotically stable.

(iv) Finally, if $\mathcal{G}$ and $\mathcal{G}_c$ are exponentially dissipative it follows that

$$\begin{aligned}\dot{V}(x(t), x_c(t)) &= \dot{V}_s(x(t)) + \sigma \dot{V}_{sc}(x_c(t)) \\ &\le -\varepsilon_c V_s(x(t)) - \sigma \varepsilon_{cc} V_{sc}(x_c(t)) + q_c^{\mathrm{T}} y_c + r_c^{\mathrm{T}} u_c + \sigma(q_{cc}^{\mathrm{T}} y_{cc} + r_{cc}^{\mathrm{T}} u_{cc}) \\ &\le -\min\{\varepsilon_c, \varepsilon_{cc}\} V(x(t), x_c(t)), \quad (x(t), x_c(t)) \notin \tilde{\mathcal{Z}}_{\tilde{x}},\end{aligned} \tag{96}$$

and $\Delta V(x(t_k), x_c(t_k))$, $(x(t), x_c(t)) \in \tilde{\mathcal{Z}}_{\tilde{x}}$, $k \in \mathcal{N}$, satisfies (93). Now, Theorem 2.1 implies that the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable. ■

*Remark 6.1* Theorem 6.1 also holds for the more general architecture of the feedback system $\mathcal{G}_c$ wherein $y_{cc} = h_{cc}(x_c) + J_{cc}(x_c)u_{cc}$ and $y_{dc} = h_{dc}(x_c) + J_{dc}(x_c)u_{dc}$, where $J_{cc}$: $\mathbb{R}^{n_c} \to \mathbb{R}^{l_{cc} \times m_{cc}}$, $J_{dc}$: $\mathbb{R}^{n_c} \to \mathbb{R}^{l_{dc} \times m_{dc}}$, $J_{cc}(x_c) \ge\ge 0, x_c \notin \mathcal{Z}_c$, and $J_{dc}(x_c) \ge\ge 0, x_c \in \mathcal{Z}_c$. In this case however, we assume that the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed; that is, $\det[I_{m_c} + J_{cc}(x_c)J_c(x)] \neq 0, (x, x_c) \notin \tilde{\mathcal{Z}}_{\tilde{x}}$, and $\det[I_{m_d} + J_{dc}(x_c)J_d(x)] \neq 0$, $(x, x_c) \in \tilde{\mathcal{Z}}_{\tilde{x}}$.

The following corollary to Theorem 6.1 addresses linear hybrid supply rates of the form $(s_c(u_c, y_c), s_d(u_d, y_d)) = (e^{\mathrm{T}} u_c - e^{\mathrm{T}} y_c, e^{\mathrm{T}} u_d - e^{\mathrm{T}} y_d)$ and $(s_{cc}(u_{cc}, y_{dc}), s_{dc}(u_{dc}, y_{cc})) = (e^{\mathrm{T}} u_{cc} - e^{\mathrm{T}} y_{cc}, e^{\mathrm{T}} u_{dc} - e^{\mathrm{T}} y_{dc})$.

COROLLARY 6.1 *Consider the nonlinear impulsive nonnegative dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ given by* (29)–(32) *and* (83)–(86), *respectively. Assume $\mathcal{G}$ is nonaccumulative with a continuously differentiable, positive-definite storage function $V_s(\cdot)$ and $\mathcal{G}_c$ is exponentially nonaccumulative with a continuously differentiable, positive-definite storage function $V_{sc}(\cdot)$. Then the following statements hold*:

(i) *If $\mathcal{G}$ is strongly zero-state observable and rank $\mathcal{G}_{cc}(0) = m_{cc}$, then the positive feedback interconnection of $\mathcal{G}$* and *$\mathcal{G}_c$ is asymptotically stable.*
(ii) *If $\mathcal{G}$ is exponentially nonaccumulative, then the positive feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is asymptotically stable.*

Proof The proof is a direct consequence of (iii) and (iv) of Theorem 6.1 with $\sigma = 1$, $q_c = -r_{cc} = -e$, $q_d = -r_{dc} = -e$, $r_c = -q_{cc} = e$, and $r_d = -q_{dc} = e$. ■

## 7 CONCLUSION

Nonnegative and compartmental dynamical systems play a key role in understanding numerous processes in biological and physiological sciences. Such systems are composed of homogeneous interconnected compartments with conservation laws describing transfers, accumulations, and elimination between compartments and the environment. In this paper we developed stability and dissipativity results for state-dependent hybrid nonnegative and compartmental dynamical systems. In addition, using these results general stability criteria were obtained for Lyapunov and asymptotic stability of feedback interconnections of nonlinear hybrid nonnegative dynamical systems. Finally, since the theory of dissipative time-dependent hybrid dynamical systems [30] closely parallels that of dissipative state-dependent dynamical systems, many of the results of this paper can be easily extended to time-dependent hybrid nonnegative dynamical systems.

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