Mathematical Problems in Engineering
Volume 6, pp. 321–357
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Improvability of Assembly Systems I: Problem Formulation and Performance Evaluation*

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(Received 5 November 1999)

This work develops improvability theory for assembly systems. It consists of two parts. Part I includes the problem formulation and the analysis technique. Part II presents the so-called improvability indicators and a case study.

Improvability theory addresses the questions of improving performance in production systems with unreliable machines. We consider both constrained and unconstrained improvability. In the constrained case, the problem consists of determining if there exists a re-distribution of resources (inventory and workforce), which leads to an increase in the system's production rate. In the unconstrained case, the problem consists of identifying a machine and a buffer, which impede the system performance in the strongest manner.

The investigation of the improvability properties requires an expression for the system performance measures as functions of the machine and buffer parameters. This paper presents a method for evaluating these functions and illustrates their practical utility using a case study at an automotive components plant. Part II uses the method developed here to establish conditions of improvability and to describe additional results of the case study.

Keywords: Improvability; Assembly system; Bottleneck

* This work was supported by NSF Grant No. DMI 9531694.

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1 INTRODUCTION

Assembly system is a production system which includes the merging of parts produced. In most cases, it consists of two or more serial lines producing component parts, one or more assembly machines where the components are merged and, if necessary, additional machines for processing the assembly. In practice, the machines are not absolutely reliable and may experience random breakdowns. This leads to a reduction of the production rate ($\widetilde{PR}$), which is defined as the average number of parts produced by the last machine of the system per unit of time. In order to localize the negative effect of the machine breakdowns, each pair of consecutive machines is usually separated by a finite buffer, which is supposed to attenuate the perturbations. An example of such a system is shown in Fig. 1, where m_{ij}, $i=1,2$, $j=1,\dots,M_i$, are the component machines, m_{01} is the assembly machine, m_{0j}, $j=1,\dots,M_0$, are the additional processing machines, and b_{ij} and b_{0j} are the buffers. Systems of this structure are often encountered in the automotive industry, and they are the focus of this work.

The idea of *improvability* of production systems was introduced in [1] to facilitate the development of quantitative engineering methods for design of continuous improvement projects and requirements for automation. According to [1], a production system is improvable under constraints if its resources can be re-distributed so that the $\widetilde{PR}$ is

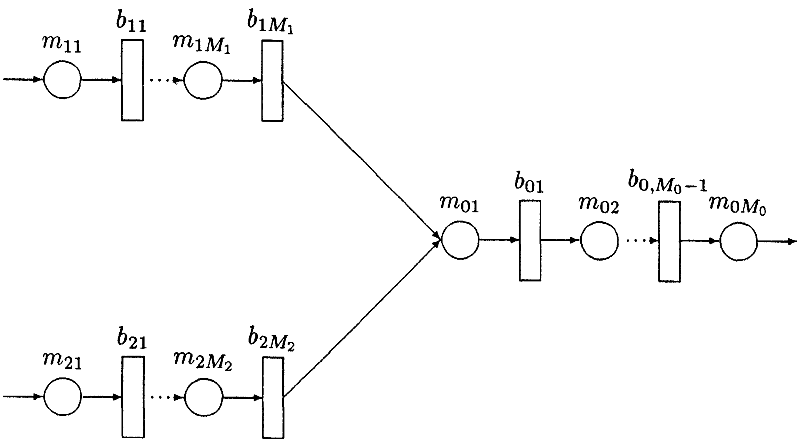

FIGURE 1 Assembly system.

increased. Unconstrained improvability addresses the question of constraints relaxation and, in particular, identification of a machine improvement of which leads to the largest improvement of the system as a whole; such a machine is referred to as the bottleneck.

The properties of improvability for *serial production lines* have been investigated in [1]. Roughly speaking, it was shown that a serial line is unimprovable with respect to workforce (WF) re-distribution if each of its buffers is on the average half full and with respect to work-in-process (WIP) re-distribution if each machine is blocked (by a full buffer after it) and starved (by an empty buffer in front of it) with equal frequency. A method for identifying the bottleneck machine, in terms of frequencies of machine blockages and starvations, has also been developed [2]. These results found applications in continuous improvement projects at several automotive plants and led to improvements in system performance [3–5].

Analysis of improvability properties of assembly systems was initiated in [6]. Only the simplest case of two component machines and one assembly machine (i.e., $M_1 = M_2 = M_0 = 1$) was addressed. Since in most applications more than a single machine is involved in the production of each component part, development of improvability theory for assembly systems, shown in Fig. 1, is desirable. The purpose of this research is to develop such a theory and demonstrate its applicability using a case study at an automotive component plant.

More precisely, the goal of this work is to characterize improvability properties of assembly systems in terms of variables that can be measured on the factory floor during the normal system operation. The variables involved are the machine average up- and down-time, starvation time, blockage time, and the average buffer occupancy. In terms of these variables, we derive simple rules that allow factory floor personnel to determine if the system is improvable and design projects leading to the improvement. These rules, referred to as *Indicators of Improvability*, are the main results of this research.

The derivation of the improvability indicators requires expressions for the system performance measures as functions of the machine and buffer parameters. Exact analytical evaluation of these functions is, unfortunately, impossible. Therefore, approximations are necessary. Such approximations have been developed in [7–9], under the Markovian assumption on the machine reliability, using a decomposition technique.

Unfortunately, no convergence conditions or analytical estimates of accuracy have been provided. Since these conditions and estimates are necessary for the development of improvability theory and since the reliability characteristics considered in this work are Bernoulli, rather than Markovian, this paper presents an aggregation-based method for performance analysis of assembly systems, provides a proof of its convergence and an estimate of accuracy. Part II of this work uses these results to establish the conditions of improvability.

While the results presented here address only the assembly system shown in Fig. 1, an extension to more complex structures, with more than two component lines and more than one assembly machines, is straightforward.

Finally, although the performance analysis technique developed in this paper is motivated by improvability considerations, it can be used independently – as a tool for throughput and inventory analysis of assembly systems in large volume manufacturing environment. To illustrate this point, the paper presents an industrial case study where the method developed is utilized.

The outline of the paper is as follows: Section 2 describes the assumptions on the assembly system considered in this work. Section 3 formulates the problems of improvability theory. Section 4 presents the idea of the approach to performance analysis used in this work. The aggregation procedure and its convergence properties are described in Section 5. Section 6 discusses the accuracy of the estimates obtained. Section 7 presents the case study. Finally, in Section 8 conclusions are formulated. The proofs are included in Appendices A and B.

2 THE MODEL

The following model is considered throughout this work

(i) The system consists of component machines, m_{ij}, $i=1,2$, $j=1,\ldots,M_i$, an assembly machine, m_{01}, additional processing machines, m_{0j}, $j=2,\ldots,M_0$, and buffers, b_{ij}, $i=1,2$, $j=1,\ldots,M_i$ and b_{0j}, $j=1,\ldots,M_0-1$, storing the parts produced by m_{ij}, respectively.

(ii) All the machines require a fixed, and identical, time to process a part. This time, T, is referred to as the *cycle time*. The time axis is slotted with the slot duration T.

(iii) Each machine is characterized by the probability, p_{ij}, $i=0,1,2$, $j=1,\dots,M_i$, to produce a part during a time slot, given that it is not blocked and not starved. This probability is referred to as the machine *production rate in isolation*.

(iv) Each buffer is characterized by its capacity, N_{ij}, $i=1,2$, $j=1,\dots,M_i$ and $N_{0j}, j=1,\dots,M_0-1$; the buffer capacity is assumed to be finite.

(v) Machine m_{ij} (except m_{01}) is starved during a time slot if buffer $b_{i,j-1}$ is empty at the beginning of the time slot. The assembly machine m_{01} is starved for parts, if at least one of the buffers b_{iM_i}, $i=1,2$, is empty at the beginning of the time slot. The first two component machines m_{i1}, $i=1,2$ are never starved.

(vi) Machine m_{ij} (except m_{iM_i}, $i=1,2$) is blocked during a time slot if buffer b_{ij} has N_{ij} parts at the beginning of this time slot and machine $m_{i,j+1}$ fails to take a part during this time slot. Machine m_{iM_i}, $i=1,2$, is blocked during a time slot if buffer b_{iM_i} is full and the assembly machine, m_{01}, fails to take parts from the buffers at the beginning of this time slot. Machine m_{0M_0} is never blocked.

A few remarks concerning this model are in order:

Remark 2.1 Assumption (ii) implies synchronous operation of the machines. It is introduced to simplify the analysis. Although somewhat restrictive, it is not unrealistic, especially for operations with automated material handling. On the other hand, the assumption that the processing time, T, is fixed does reflect the situation in most large volume manufacturing systems, with automated material handling or not. The often cited in the literature random processing time is not, in our opinion, appropriate in the context of these systems.

Remark 2.2 Assumption (iii) defines the Bernoulli statistics of machine reliability. It is appropriate for operations with down-time of the order of the cycle time T. This is often the case in modular assembly systems where operators use push-buttons to stop a module of the operational conveyor in order to complete the operation with the highest possible quality. The duration of this "breakdown" is of the order of the cycle time and, therefore, the probability to produce a part during the cycle time arises naturally. Another frequent perturbation is pallet jam on the operational conveyor; to correct for this problem, also a short period of time is required. In many automotive assembly lines, these are the predominant perturbations. In these situations, the Bernoulli model is appropriate. This model is not appropriate for machining lines, where the down-time is typically much longer than the cycle time. In these cases, the Markovian model, developed in the framework of assembly systems in [7–9], is an appropriate formalization.

Remark 2.3 Assumption (v) implies, in particular, that only one part of each type is required by the assembly machine. This assumption does not restrict generality and may be removed by an appropriate model modification.

Remark 2.4 Assumptions (iii), (v) and (vi) are formulated in terms of time-dependent failures, i.e., the machines can go down even when blocked or starved. Another possible model is operation-dependent failures, where no breakdown of starved or blocked machines is possible. Both models are practical, depending on the production system at hand: For automated palletized material handling, time-dependent model is more appropriate. In case of manual material handling, operation-dependent failures often take place.

Model (i)–(vi) is used below to introduce the problems of improvability theory.

3 IMPROVABILITY THEORY: PROBLEM FORMULATION

In this section, we formulate the problems of improvability theory, solution of which is the main goal of this research.

3.1 Constrained Improvability

Assume that the buffers capacity N_{ij} and machines efficiency p_{ij} are constrained as follows:

$$\sum_{i=1}^{M_1} N_{1i} + \sum_{i=1}^{M_2} N_{2i} + \sum_{i=1}^{M_0-1} N_{0i} = N^*, \tag{3.1}$$

$$\prod_{i=1}^{M_1} p_{1i} \prod_{i=1}^{M_2} p_{2i} \prod_{i=1}^{M_0} p_{0i} = p^*. \tag{3.2}$$

Expression (3.1) can be interpreted as WIP constraint: The total inventory in the system cannot exceed N^*. As in [1], expression (3.2) is interpreted as WF or machine efficiency (ME) constraint: The total work that can be carried out in the system is bounded by p^*, and a reassignment of the workforce or work among the operations leads to changes in p_{ij}'s compatible with (3.2).

Denote as p_i, N_i, p_i^* and N_i^*, $i=0,1,2$, vectors with components $[p_{i1},\dots,p_{iM_i}]$, $[N_{i1},\dots,N_{iM_i}]$ (for $i=0$, $N_0=[N_{01},\dots,N_{0,M_0-1}]$), $[p_{i1}^*,\dots,p_{iM_i}^*]$, and $[N_{i1}^*,\dots,N_{iM_i}^*]$ (for $i=0$, $N_0^*=[N_{01}^*,\dots,N_{0,M_0-1}^*]$), respectively.

DEFINITION 3.1 *The assembly system* (i)–(vi) *is* improvable with respect to WF *if there exist vectors* p_0^*, p_1^*, p_2^* *such that* $\prod_{i=1}^{M_1} p_{1i}^* \prod_{i=1}^{M_2} p_{2i}^* \prod_{i=1}^{M_0} p_{0i}^* = p^*$ *and*

$$\widetilde{PR}(p_1^*,p_2^*,p_0^*,N_1,N_2,N_0) > \widetilde{PR}(p_1,p_2,p_0,N_1,N_2,N_0),$$

where $\widetilde{PR}$ *denotes the production rate of the system, i.e., the average steady state number of parts produced by the last machine,* m_{0M_0} *per cycle time.*

DEFINITION 3.2 *The assembly system* (i)–(vi) *is* improvable with respect to WF and WIP *simultaneously if there exist vectors* p_0^*, p_1^*, p_2^* *and* N_1^*, N_2^*, N_0^*, *such that* $\prod_{i=1}^{M_1} p_{1i}^* \prod_{i=1}^{M_2} p_{2i}^* \prod_{i=1}^{M_0} p_{0i}^* = p^*$ *and* $\sum_{i=1}^{M_1} N_{1i}^* + \sum_{i=1}^{M_2} N_{2i}^* + \sum_{i=1}^{M_0-1} N_{0i}^* = N^*$ *and*

$$\widetilde{PR}(p_1^*,p_2^*,p_0^*,N_1^*,N_2^*,N_0^*) > \widetilde{PR}(p_1,p_2,p_0,N_1,N_2,N_0).$$

DEFINITION 3.3 *The assembly system* (i)–(vi) *is* improvable with respect to WIP *if there exist* N_1^*, N_2^* *and* N_0^* *such that* $\sum_{i=1}^{M_1} N_{1i}^* + \sum_{i=1}^{M_2} N_{2i}^* + \sum_{i=1}^{M_0-1} N_{0i}^* = N^*$ *and*

$$\widetilde{PR}(p_1,p_2,p_0,N_1^*,N_2^*,N_0^*) > \widetilde{PR}(p_1,p_2,p_0,N_1,N_2,N_0).$$

3.2 Unconstrained Improvability

When the system is no longer improvable under constraints or when the resource re-allocation, required by the improvability conditions, cannot be carried on the factory floor, a further increase in $\widetilde{PR}$ can be obtained only by the constraints relaxation, i.e., by increasing p^* or N^*. In practical terms, this amounts to the improvement of the machine isolation production rates (say, by improved preventive maintenance or by installing a more efficient machine) or by allocating additional in-process inventories to the system. The question arises: Which p_{ij} or N_{ij} should be increased so that the largest increase in $\widetilde{PR}$ is obtained? A formalization of this question is as follows:

DEFINITION 3.4 *Machine* m_{ij}, $i=0,1,2, j=1,\dots,M_i$ *is the* bottleneck machine (BN-M) *if*

$$\frac{\partial \widetilde{PR}}{\partial p_{ij}} > \frac{\partial \widetilde{PR}}{\partial p_{mn}}, \quad \forall mn \neq ij.$$

DEFINITION 3.5 *Buffer* b_{ij} *is the* bottleneck buffer (BN-B) *if*

$$\begin{aligned}&\widetilde{PR}(p_1,p_2,p_0,N_{11},N_{12},\dots,N_{ij}+1,\dots,N_{0M_0-1})\\&\quad > \widetilde{PR}(p_1,p_2,p_0,N_{11},N_{12},\dots,N_{mn}+1,\dots,N_{0M_0-1}), \quad \forall mn \neq ij.\end{aligned}$$

3.3 Potency of Material Handling System

Production lines consist of machines and buffers or, more generally, material handling system (MHS). In the framework of model (i)–(vi), the efficacy of each machine is characterized by its production rate in isolation, p_{ij}, $i=0,1,2, j=1,\dots,M_i$, and the efficacy of the production system from the point of view of the machines is defined by $\min_{ij} p_{ij}$. The efficacy of the MHS, however, does not seem to have a quantitative characterization. Indeed, buffer capacities, N_{ij}, $\forall ij \neq 0M_0$ do not, by themselves, define how efficient MHS is. Based on the bottleneck definition presented above, such a characterization is introduced as follows:

DEFINITION 3.6 *MHS is* weakly potent *if the machine with the smallest isolation production rate is the BN-M; otherwise, MHS is not potent. MHS is* potent *if it is weakly potent and, in addition, production rate of the system is close to that of the slowest machine in isolation. MHS is* strongly potent *if it is potent and this production is achieved using the smallest total buffer capacity* $N^* = \sum_{i=1}^{M_1} N_{1i} + \sum_{i=1}^{M_2} N_{2i} + \sum_{i=1}^{M_0-1} N_{0i}$.

Remark 3.1 The notion of MHS potency might be further quantified by introducing, say, 5% or 10% difference between the isolation production rate of the worst machine and the production rate of the system as a whole. Analogously, the notion of strong potency may he made more precise. We leave them, however, as they are, because the term "close" and "smallest" may have different meaning in various production systems and industries. Therefore, quantification of these terms is a function of a particular application.

3.4 Problems

The goal of this work is to derive conditions under which the assembly system (i)–(vi) is improvable in both constrained and unconstrained sense. As it was pointed out above, the conditions sought are to be formulated either in terms of the data available on the factory floor or in terms of the data that can be constructively calculated using the machine and buffer parameter vectors, p_i's and N_i's. These conditions are referred to as Indicators of Improvability. The problems, then, addressed in this work are:

PROBLEM 3.1 *Given model* (i)–(vi), *derive indicators of improvability with respect to WF, WIP, and WF and WIP simultaneously, which could be used based on either measured or calculated data.*

PROBLEM 3.2 *Given model* (i)–(vi), *derive indicators of improvability for bottleneck identification, which are based on real-time or calculated data.*

PROBLEM 3.3 *Given model* (i)–(vi), *determine the potency of MHS using the indicators of improvability mentioned above.*

Solutions of these problems are provided in Part II of this work, based on the performance evaluation technique developed below.

4 PERFORMANCE EVALUATION: IDEA OF THE APPROACH

The idea of the approximations used in this paper is as follows: Consider the *serial* production line consisting of the component machines $m_{11}, \ldots, m_{1M_1}$, the assembly machine m_{01}, the additional processing machines $m_{02}, \ldots, m_{0M_0}$, and the buffers separating the machines (see Fig. 1). A recursive procedure for evaluating the production rate of this line has been developed in [1]. This recursive procedure allows also for the evaluation of the steady state probabilities of buffer occupancy and probabilities of machine blockages and starvations. In order to use this procedure for the serial line at hand, assume that the isolation production rate of m_{01} is modified so as to account for the existence of the other component line. Specifically, introduce a fictitious assembly machine, denoted as m'_{01}, with the isolation production rate defined by p_{01} Prob{buffer b_{2M_2} is not empty}. If Prob{buffer b_{2M_2} is not empty} were known and if this probability were independent of the occupancy of buffer b_{1M_1}, the recursive procedure of [1] would result in the production rate of the assembly line (i)–(vi). Since this probability is unknown and the dependence does exist, we introduce iterations, as described below, and prove their convergence (Section 5). In constructing the iterations, we use several independence assumptions; they are justified numerically in Section 6 and, on this basis, the accuracy of the method developed is evaluated.

The iterations are introduced as follows: At the first step, assume that Prob{buffer b_{2M_2} is not empty} = 1. Then the serial production line $\{m_{11}, b_{11}, \ldots, m_{1M_1}, b_{1M_1}, m'_{01}, b_{01}, \ldots, m_{0M_0}\}$, which we refer to as the *upper line*, is defined completely and, using the recursive procedure of [1], calculate Prob{buffer b_{1M_1} is not empty}. Consider next what is referred to as the *lower line* $\{m_{21}, b_{21}, \ldots, m_{2M_2}, b_{2M_2}, m''_{01}, b_{01}, \ldots, m_{0M_0}\}$, where m''_{01} is another fictitious machine with the isolation production rate p_{01} Prob{buffer b_{1M_1} is not empty} and, again using the recursive procedure of [1], calculate Prob{buffer b_{2M_2} is not empty}. Use now this probability for the second iteration in analysis of the upper line and continue this process, alternating between the upper and the lower lines. As it is shown in Section 5, the iterations are convergent and result in the estimates of both Prob{buffer b_{iM_i} is not empty}, $i = 1, 2$, and $PR(p_1, p_2, p_0, N_1, N_2, N_0)$.

5 PERFORMANCE EVALUATION: RECURSIVE PROCEDURE AND ITS CONVERGENCE

In order to bring the notations in compliance with those used in the recursive procedure of [1], denote the upper line and the lower line, respectively, as

$$\begin{aligned}&\{m_{11}, b_{11}, m_{12}, \ldots, m_{1M_1}, b_{1M_1}, m'_{01}, b_{01}, m_{02}, \ldots, m_{0,M_0-1}, b_{0,M_0-1}, m_{0M_0}\}\\ &\quad = \{m'_1, b'_1, m'_2, \ldots, m'_{M_1}, b'_{M_1}, m'_{M_1+1}, b'_{M_1+1}, m'_{M_1+2}, \ldots,\\ &\qquad m'_{M_1+M_0-1}, b'_{M_1+M_0-1}, m'_{M_1+M_0}\}, \end{aligned} \tag{5.1}$$

where

$$m'_i = \begin{cases} m_{1i}, & i = 1, \ldots, M_1, \\ m'_{01}, & i = M_1 + 1, \\ m_{0,i-M_1}, & i = M_1 + 2, \ldots, M_1 + M_0, \end{cases}$$

$$b'_i = \begin{cases} b_{1i}, & i = 1, \ldots, M_1, \\ b_{0,i-M_1}, & i = M_1 + 1, \ldots, M_1 + M_0 - 1, \end{cases}$$

and

$$\begin{aligned}&\{m_{21}, b_{21}, m_{22}, \ldots, m_{2M_2}, b_{2M_2}, m''_{01}, b_{01}, m_{02}, \ldots, m_{0,M_0-1}, b_{0,M_0-1}, m_{0M_0}\}\\ &\quad = \{m''_1, b''_1, m''_2, \ldots, m''_{M_2}, b''_{M_2}, m''_{M_2+1}, b''_{M_2+1}, m''_{M_2+2}, \ldots,\\ &\qquad m''_{M_2+M_0-1}, b''_{M_2+M_0-1}, m''_{M_2+M_0}\}, \end{aligned} \tag{5.2}$$

where

$$m''_i = \begin{cases} m_{2i}, & i = 1, \ldots, M_2, \\ m''_{01}, & i = M_2 + 1, \\ m_{0,i-M_2}, & i = M_2 + 2, \ldots, M_2 + M_0, \end{cases}$$

$$b''_i = \begin{cases} b_{2i}, & i = 1, \ldots, M_2, \\ b_{0,i-M_2}, & i = M_2 + 1, \ldots, M_2 + M_0 - 1, \end{cases}$$

The isolation production rates of the machines and the capacity of the buffers for the upper and the lower lines are denoted as μ_i, ν_i, Γ_i, and Λ_i, respectively:

$$\mu_i = \begin{cases} p_{1i}, & i = 1, \ldots, M_1, \\ p_{0,i-M_1}, & i = M_1 + 2, \ldots, M_1 + M_0, \end{cases} \tag{5.3}$$

$$\Gamma_i = \begin{cases} N_{1i}, & i = 1, \ldots, M_1, \\ N_{0,i-M_1}, & i = M_1 + 1, \ldots, M_1 + M_0 - 1, \end{cases} \tag{5.4}$$

$$\nu_i = \begin{cases} p_{2i}, & i = 1, \ldots, M_2, \\ p_{0,i-M_2}, & i = M_2 + 2, \ldots, M_2 + M_0, \end{cases} \tag{5.5}$$

$$\Lambda_i = \begin{cases} N_{2i}, & i = 1, \ldots, M_2, \\ N_{0,i-M_2}, & i = M_2 + 1, \ldots, M_2 + M_0 - 1, \end{cases} \tag{5.6}$$

where, obviously,

$$\mu_{M_1+j} = \nu_{M_2+j}, \quad j = 2, \ldots, M_0 \tag{5.7}$$

and

$$\Gamma_{M_1+j} = \Lambda_{M_2+j}, \quad j = 1, \ldots, M_0 - 1. \tag{5.8}$$

Denote as $X_{M_1'}(0)$ the estimate of the steady state probability that buffer b'_{M_1} is empty and as $X_{M_2''}(0)$ the estimate of the steady state probability that buffer b''_{M_2} is empty. Then, in compliance with the approach described above, define the isolation production rates of m'_{01} and m''_{01} as

$$\mu_{M_1+1} = p_{01}[1 - X_{M_2''}(0)], \tag{5.9}$$

$$\nu_{M_2+1} = p_{01}[1 - X_{M_1'}(0)]. \tag{5.10}$$

At the first step of the iterations, assume that $X_{M_2''}(0) = 0$, i.e., Prob{buffer b_{2M_2} is not empty} = 1 and $\mu_{M_1+1} = p_{01}$. Then, the upper line $\{m'_1, b'_1, m'_2, \ldots, m'_{M_1}, b'_{M_1}, m'_{M_1+1}, b'_{M_1+1}, m'_{M_1+2}, \ldots, m'_{M_1+M_0-1}, b'_{M_1+M_0-1}, m'_{M_1+M_0}\}$ is defined and the recursive procedure of [1] can be written as follows:

$$\begin{aligned} \mu_i^b(k+1) &= \mu_i[1 - Q(\mu_{i+1}^b(k+1), \mu_i^f(k), \Gamma_i)], \quad 1 \le i \le M_1 + M_0 - 1, \\ \mu_i^f(k+1) &= \mu_i[1 - Q(\mu_{i-1}^f(k+1), \mu_i^b(k+1), \Gamma_{i-1})], \quad 2 \le i \le M_1 + M_0, \\ \mu_1^f(k) &= \mu_1, \quad \mu_{M_1+M_0}^b(k) = \mu_{M_1+M_0}, \qquad k = 0, 1, 2, 3, \ldots, \end{aligned} \tag{5.11}$$

with initial conditions

$$\mu_i^f(0) = \mu_i, \quad i = 1, \ldots, M_1 + M_0,$$

where

$$Q(x,y,N) = \begin{cases} \dfrac{(1-x)(1-\alpha)}{1-(x/y)\alpha^N}, & x \neq y,\ \alpha = \dfrac{x(1-y)}{y(1-x)}, \\ \dfrac{1-x}{N+1-x}, & x = y. \end{cases} \tag{5.12}$$

According to [1], this procedure is convergent to

$$\mu_i^f := \lim_{k\to\infty} \mu_i^f(k), \quad \mu_i^b := \lim_{k\to\infty} \mu_i^b(k), \qquad i = 1,\ldots,M_1+M_0, \tag{5.13}$$

and the steady state probability that buffer b'_{M_1} is empty is evaluated as

$$X_{M'_1}(0) = Q(\mu^f_{M_1}, \mu^b_{M_1+1}, \Gamma_{M_1}). \tag{5.14}$$

This completes the first iteration for the upper line.

The first iteration for the lower line $\{m''_1, b''_1, m''_2, \ldots, m''_{M_2}, b''_{M_2}, m''_{M_2+1}, b''_{M_2+1}, m''_{M_2+2}, \ldots, m''_{M_2+M_0-1}, b''_{M_2+M_0-1}, m''_{M_2+M_0}\}$ begins by defining the isolation production rate of m''_{M_2+1}:

$$\nu_{M_2+1} = p_{01}[1 - X_{M'_1}(0)], \tag{5.15}$$

where $X_{M'_1}(0)$ is evaluated above. Then, the lower line is also analyzed using the recursive procedure of [1]:

$$\begin{aligned} \nu_i^b(k+1) &= \nu_i[1 - Q(\nu^b_{i+1}(k+1), \nu_i^f(k), \Lambda_i)], \quad 1 \le i \le M_2 + M_0 - 1, \\ \nu_i^f(k+1) &= \nu_i[1 - Q(\nu^f_{i-1}(k+1), \nu_i^b(k+1), \Lambda_{i-1})], \quad 2 \le i \le M_2 + M_0, \\ \nu_1^f(k) &= \nu_1, \quad \nu^b_{M_2+M_0}(k) = \nu_{M_2+M_0}, \qquad k = 0,1,2,3,\ldots, \end{aligned} \tag{5.16}$$

with initial conditions

$$\nu_i^f(0) = \nu_i, \quad i = 1,\ldots,M_2+M_0,$$

where function $Q(x, y, N)$ is defined in (5.12). The limits of this procedure are denoted as

$$\lim_{k\to\infty} \nu_i^f(k) =: \nu_i^f, \quad \lim_{k\to\infty} \nu_i^b(k) =: \nu_i^b, \qquad i = 1, \ldots, M_2 + M_0 \quad (5.17)$$

and $X_{M_2''}(0)$, i.e., Prob{buffer b_{2M_2} is not empty}, can be evaluated as

$$X_{M_2''}(0) = Q(\nu_{M_2}^f, \nu_{M_2+1}^b, \Lambda_{M_2}). \quad (5.18)$$

This leads to the second iteration for the upper line, and the process is repeated again. Formally, the iterations between the upper and the lower lines can be represented as follows:

$$\begin{aligned}
\mu_{M_1+1}(s+1) &= p_{01}[1 - X_{M_2''}(0, s)],\\
\mu_i^b(s+1) &= \mu_i(s+1)[1 - Q(\mu_{i+1}^b(s+1), \mu_i^f(s+1), \Gamma_i)],\\
&\quad 1 \le i \le M_1 + M_0 - 1,\\
\mu_i^f(s+1) &= \mu_i(s+1)[1 - Q(\mu_{i-1}^f(s+1), \mu_i^b(s+1), \Gamma_{i-1})],\\
&\quad 2 \le i \le M_1 + M_0,\\
X_{M_1'}(0, s+1) &= Q(\mu_{M_1}^f(s+1), \mu_{M_1+1}^b(s+1), \Gamma_{M_1}),\\
\nu_{M_2+1}(s+1) &= p_{01}[1 - X_{M_1'}(0, s+1)],\\
\nu_i^b(s+1) &= \nu_i(s+1)[1 - Q(\nu_{i+1}^b(s+1), \nu_i^f(s+1), \Lambda_i)], \qquad (5.19)\\
&\quad 1 \le i \le M_2 + M_0 - 1,\\
\nu_i^f(s+1) &= \nu_i(s+1)[1 - Q(\nu_{i-1}^f(s+1), \nu_i^b(s+1), \Lambda_{i-1})],\\
&\quad 2 \le i \le M_2 + M_0,\\
X_{M_2''}(0, s+1) &= Q(\nu_{M_2}^f(s+1), \nu_{M_2+1}^b(s+1), \Lambda_{M_2}),\\
\mu_i(s+1) &= \mu_i, \quad i = 1, \ldots, M_1, M_1 + 2, \ldots, M_1 + M_0,\\
\nu_i(s+1) &= \nu_i, \quad i = 1, \ldots, M_2, M_2 + 2, \ldots, M_2 + M_0,\\
&\quad s = 0, 1, 2, 3, \ldots,
\end{aligned}$$

with the initial conditions

$$X_{M_2''}(0, 0) = 0,$$

where $\mu_i^f(s+1)$ and $\mu_i^b(s+1)$, $i = 1, \ldots, M_1 + M_0$, are given by (5.13), $\nu_i^f(s+1)$ and $\nu_i^b(s+1)$, $i = 1, \ldots, M_2 + M_0$, are given by (5.17).

THEOREM 5.1 *Recursive procedure* (5.19) *is convergent, i.e., the following limits exist:*

$$
\begin{aligned}
&\lim_{s\to\infty} \mu_{M_1+1}(s) =: \mu_{M_1+1}, \\
&\quad \lim_{s\to\infty} \mu_i^f(s) =: \mu_i^f, \quad i = 1, \dots, M_1 + M_0, \\
&\quad \lim_{s\to\infty} \mu_i^b(s) =: \mu_i^b, \quad i = 1, \dots, M_1 + M_0, \\
&\lim_{s\to\infty} \nu_{M_2+1}(s) =: \nu_{M_2+1}, \\
&\quad \lim_{s\to\infty} \nu_i^f(s) =: \nu_i^f, \quad i = 1, \dots, M_2 + M_0, \\
&\quad \lim_{s\to\infty} \nu_i^b(s) =: \nu_i^b, \quad i = 1, \dots, M_2 + M_0.
\end{aligned}
\tag{5.20}
$$

Moreover,

$$
\begin{aligned}
\mu_{M_1+i}^f &= \nu_{M_2+i}^f, \quad i = 1, \dots, M_0, \\
\mu_{M_1+i}^b &= \nu_{M_2+i}^b, \quad i = 2, \dots, M_0.
\end{aligned}
$$

Proof See Appendix A.

Parameters μ_i^f defined in (5.20) (respectively, ν_i^f) represent the aggregation of the first i machines of the upper (respectively, lower) line into a single machine. Analogously, μ_i^b (respectively, ν_i^b) represent the aggregation of the last M_1+M_0-i+1 (respectively, M_2+M_0-i+1) machines of the upper (respectively, lower) line into a single machine. Therefore, $\mu_{M_1+M_0}^f$ (respectively, $\nu_{M_2+M_0}^f$) can be used to define an estimate of the system production rate, PR, as follows:

$$
PR = \mu_{M_1+M_0}^f = \nu_{M_2+M_0}^f, \tag{5.21}
$$

where the second equality is due to the last statement of Theorem 5.1.

Recursive procedure (5.19) can be used to evaluate other performance measures as well. Indeed, let $E[\tilde{h}_{ij}]$, $\forall ij \neq 0M_0$, denote the average steady state occupancy of buffer b_{ij}. Then, the estimates can be introduced as follows:

$$E[h_{1j}]=\sum_{k=0}^{N_{1j}} kQ(\mu_j^f,\mu_{j+1}^b,N_{1j})\frac{1}{1-\mu_{j+1}^b}\left(\frac{\mu_j^f(1-\mu_{j+1}^b)}{\mu_{j+1}^b(1-\mu_j^f)}\right)^k,\quad j=1,\ldots,M_1,$$

$$E[h_{2j}]=\sum_{k=0}^{N_{2j}} kQ(\nu_j^f,\nu_{j+1}^b,N_{2j})\frac{1}{1-\nu_{j+1}^b}\left(\frac{\nu_j^f(1-\nu_{j+1}^b)}{\nu_{j+1}^b(1-\nu_j^f)}\right)^k,\quad j=1,\ldots,M_2,$$

$$E[h_{0j}]=\sum_{k=0}^{N_{0j}} kQ(\mu_{M_1+j}^f,\mu_{M_1+j+1}^b,N_{0j})\frac{1}{1-\mu_{M_1+j+1}^b}\left(\frac{\mu_{M_1+j}^f(1-\mu_{M_1+j+1}^b)}{\mu_{M_1+j+1}^b(1-\mu_{M_1+j}^f)}\right)^k$$

$$=\sum_{k=0}^{N_{0j}} kQ(\nu_{M_2+j}^f,\nu_{M_2+j+1}^b,N_{0j})\frac{1}{1-\nu_{M_2+j+1}^b}\left(\frac{\nu_{M_2+j}^f(1-\nu_{M_2+j+1}^b)}{\nu_{M_2+j+1}^b(1-\nu_{M_2+j}^f)}\right)^k,$$

$$j=1,\ldots,M_0-1. \tag{5.22}$$

Also, if the probability of machine manufacturing starvations and blockages are defined as

$$\begin{aligned}
\widetilde{ms}_{ij} &= \text{Prob}(\{m_{ij}\text{ is up during a time slot}\}\cap\\
&\quad \{b_{i,j-1}\text{ is empty at the beginning of this slot}\}),\\
&\quad\quad \forall ij\neq 11,21,01,\\
\widetilde{ms}_{01_i} &= \text{Prob}(\{m_{01}\text{ is up during a time slot}\}\cap\\
&\quad \{b_{iM_i}\text{ is empty at the beginning of this slot}\}),\quad i=1,2,\\
\widetilde{mb}_{ij} &= \text{Prob}(\{m_{ij}\text{ is up during a time slot}\}\cap\\
&\quad \{b_{ij}\text{ is full at the beginning of this slot}\}\cap\\
&\quad \{\text{the immediate downstream machine of } m_{ij}\text{ fails to take}\\
&\quad\ \text{a part from } b_{ij}\text{ at the beginning of this slot}\}),\\
&\quad\quad\quad \forall ij\neq 0M_0,
\end{aligned} \tag{5.23}$$

then, the respective estimates are (see [10] for details):

$$
\begin{aligned}
ms_{1j} &= \mu_j Q(\mu_{j-1}^f, \mu_j^b, \Gamma_{j-1}), \quad j = 2, \dots, M_1, \\
ms_{2j} &= \nu_j Q(\nu_{j-1}^f, \nu_j^b, \Lambda_{j-1}), \quad j = 2, \dots, M_2, \\
ms_{01_1} &= p_{01} Q(\mu_{M_1}^f, \mu_{M_1+1}^b, \Gamma_{M_1}), \\
ms_{01_2} &= p_{01} Q(\nu_{M_2}^f, \nu_{M_2+1}^b, \Lambda_{M_2}), \\
ms_{0j} &= \mu_{M_1+j} Q(\mu_{M_1+j-1}^f, \mu_{M_1+j}^b, \Gamma_{M_1+j-1}), \\
&= \nu_{M_2+j} Q(\nu_{M_2+j-1}^f, \nu_{M_2+j}^b, \Lambda_{M_2+j-1}), \quad j = 2, \dots, M_0, \\
mb_{1j} &= \mu_j Q(\mu_{j+1}^b, \mu_j^f, \Gamma_j), \quad j = 1, \dots, M_1, \\
mb_{2j} &= \nu_j Q(\nu_{j+1}^b, \nu_j^f, \Lambda_j), \quad j = 1, \dots, M_2, \\
mb_{0j} &= \mu_{M_1+j} Q(\mu_{M_1+j+1}^b, \mu_{M_1+j}^f, \Gamma_{M_1+j}) \\
&= \nu_{M_2+j} Q(\nu_{M_2+j+1}^b, \nu_{M_2+j}^f, \Lambda_{M_2+j}), \quad j = 1, \dots, M_0 - 1.
\end{aligned} \tag{5.24}
$$

Expressions (5.19)–(5.24) are used in Part II of this work as a basis for development of Improvability Theory. They also could be used independently as a tool for evaluating assembly systems performance (see Section 7).

6 PERFORMANCE EVALUATION: ACCURACY OF THE ESTIMATES

Under assumptions (i)–(vi), the dynamics of the assembly system are described by an irreducible, ergodic Markov chain with the states $(k_1', \dots, k_{M_1+M_0-1}', k_1'', \dots, k_{M_2}'')$, $k_i' = 0, \dots, \Gamma_i$, $i = 1, \dots, M_1 + M_0 - 1$, and $k_i'' = 0, \dots, \Lambda_i$, $i = 1, \dots, M_2$ (or the states $(k_1', \dots, k_{M_1}', k_1'', \dots, k_{M_2+M_0-1}'')$, $k_i' = 0, \dots, \Gamma_i$, $i = 1, \dots, M_1$, and $k_i'' = 0, \dots, \Lambda_i$, $i = 1, \dots, M_2 + M_0 - 1$). In terms of $\tilde{X}_{i'}(0)$ (or $\tilde{X}_{i''}(0)$), the production rate of the system is

$$
\widetilde{PR} = \mu_{M_1+M_0}(1 - \tilde{X}_{(M_1+M_0-1)'}(0)) \quad (\text{or } \nu_{M_2+M_0}(1 - \tilde{X}_{(M_2+M_0-1)''}(0))). \tag{6.1}
$$

To describe the relationship between the exact production rate $\widetilde{PR}$ (6.1) and its estimate (5.21), introduce the joint stationary probability $\tilde{X}_{i',\ldots,j',m'',\ldots,n''}(k_i',\ldots,k_j',k_m'',\ldots,k_n'')$ that consecutive buffers $i,i+1,\ldots,j$, $1\leq i\leq j\leq M_1+M_0-1$, contain k_i', $k_{i+1}',\ldots,k_j'$ parts, respectively, and consecutive buffers $m,m+1,\ldots,n$, $1\leq m\leq n\leq M_2+M_0-1$, contain $k_m'',k_{m+1}'',\ldots,k_n''$ parts, respectively. In general, of course, the joint probability distribution $\tilde{X}_{i',\ldots,j',m'',\ldots,n''}(k_i',\ldots,k_j',k_m'',\ldots,k_n'')$ is not equal to the product of its marginals, $\tilde{X}_{i'}(k_i')$ and $\tilde{X}_{i+1',\ldots,j',m'',\ldots,n''}(k_{i+1}',\ldots,k_j',k_m'',\ldots,k_n'')$. However, for certain values of $k_i',\ldots,k_j',k_m'',\ldots,k_n''$, related to blockages and starvations, they are indeed close. Specifically, define

$$
\begin{aligned}
\epsilon_{i',j',M_2''} &= \max_{k_{i+1}'} \big|\tilde{X}_{i',\ldots,j',M_2''}(0,k_{i+1}',\Gamma_{i+2},\ldots,\Gamma_j,0) \\
&\quad - \tilde{X}_{i',\ldots,j'}(0,k_{i+1}',\Gamma_{i+2},\ldots,\Gamma_j)\tilde{X}_{M_2''}(0)\big|, \\
\epsilon_{M_1',m'',n''} &= \max_{k_{m+1}''} \big|\tilde{X}_{M_1',m'',\ldots,n''}(0,0,k_{m+1}'',\Lambda_{m+2},\ldots,\Lambda_n) \\
&\quad - \tilde{X}_{M_1'}(0)\tilde{X}_{m'',\ldots,n''}(0,k_{m+1}'',\Lambda_{m+2},\ldots,\Lambda_n)\big|, \\
\theta_{i',j'}(b') &= \big|\tilde{X}_{i',\ldots,j'}(0,b',\Gamma_{i+2},\ldots,\Gamma_j) \\
&\quad - \tilde{X}_{i'}(0)\tilde{X}_{(i+1)',\ldots,j'}(b',\Gamma_{i+2},\ldots,\Gamma_j)\big|, \\
\theta^{i',j'}(a') &= \big|\tilde{X}_{i',\ldots,j'}(a',\Gamma_{i+1},\ldots,\Gamma_j) - \tilde{X}_{i'}(a')\tilde{X}_{(i+1)',\ldots,j'}(\Gamma_{i+1},\ldots,\Gamma_j)\big|, \\
\theta_{m'',n''}(b'') &= \big|\tilde{X}_{m'',\ldots,n''}(0,b'',\Lambda_{m+2},\ldots,\Lambda_n) \\
&\quad - \tilde{X}_{m''}(0)\tilde{X}_{(m+1)'',\ldots,n''}(b'',\Lambda_{m+2},\ldots,\Lambda_n)\big|, \\
\theta_{m'',n''}(a'') &= \big|\tilde{X}_{m'',\ldots,n''}(a'',\Lambda_{m+1},\ldots,\Lambda_n) \\
&\quad - \tilde{X}_{m''}(a'')\tilde{X}_{(m+1)'',\ldots,n''}(\Lambda_{m+1},\ldots,\Lambda_n)\big|, \\
\delta_1 &= \max_{i',j'} \max_{a',b'} \{\theta_{i',j'}(b'),\theta^{i',j'}(a'),\epsilon_{i',j',M_2''}\}, \\
\delta_2 &= \max_{m'',n''} \max_{a'',b''} \{\theta_{m'',n''}(b''),\theta^{m'',n''}(a''),\epsilon_{M_1',m'',n''}\}, \\
\delta &= \max(\delta_1,\delta_2).
\end{aligned}
\tag{6.2}
$$

Remark 6.1 Roughly speaking, parameters ϵ account for the coupling of the upper and lower lines and parameters θ account for the coupling of the machines within each line.

NUMERICAL FACT 6.1 *For the assembly system defined by assumption* (i)–(vi),

$$\delta \ll 1.$$

Numerical Justification Justification is carried out using numerical analysis of the Markov chain defined by assumption (i)–(vi). Specifically, the transition matrix has been constructed and iterated upon until the stationary probability distribution $\tilde{X}_{i',\dots,j',m'',\dots,n''}(k'_i,\dots,k'_j, k''_m,\dots,k''_n)$, has been reached. Then, the small parameters ϵ_{i',j',M''_2}, $\epsilon_{M'_1,m'',n''}$, $\theta_{i',j'}(b')$, $\theta^{i',j'}(a')$, $\theta_{m'',n''}(b'')$ and $\theta^{m'',n''}(a'')$ in (6.2) are calculated. Finally, the value of δ is evaluated. In every case analyzed, it was found that $\delta \ll 1$. Several typical examples are shown in Table I and the machines and buffer parameters of these examples are given in Table II. Based on this analysis, we conclude that Numerical Fact 6.1 holds.

Using the small parameter δ, the accuracy of estimates (5.21), (5.22) and (5.24) can be evaluated as follows:

THEOREM 6.1 *Under assumptions* (i)–(vi), *the following holds:*

(a) $\widetilde{PR} = PR + \mathcal{O}(\delta)$,
(b) $E[\tilde{h}_{ij}] = E[h_{ij}] + \mathcal{O}(\delta), \forall ij \neq 0M_0$,
(c) $\widetilde{ms}_{ij} = ms_{ij} + \mathcal{O}(\delta), \forall ij \neq 01, \widetilde{mb}_{ij} = mb_{ij} + \mathcal{O}(\delta), \forall ij$,
(d) $\widetilde{ms}_{01_{(k)}} = ms_{01_{(k)}} + \mathcal{O}(\delta), k = 1, 2$.

Proof See Appendix B.

TABLE I Illustration of δ and the estimation error

Cases	δ	$\widetilde{PR}$	PR	$(\lvert\widetilde{PR} - PR\rvert/\widetilde{PR}) \cdot 100$ (%)
I	0.0449	0.4605	0.4423	3.96
II	0.0491	0.3894	0.3747	3.77
III	0.0075	0.6267	0.6246	0.35
IV	0.0202	0.6848	0.6911	0.92

TABLE II Parameters of cases in Table I

Cases	N_{11}	N_{12}	N_{21}	N_{22}	N_{01}	N_{02}	p_{11}	p_{12}	p_{21}	p_{22}	p_{01}	p_{02}	p_{03}
I	1	1	1	1	1	1	0.8	0.8	0.8	0.8	0.8	0.8	0.8
II	1	1	1	1	1	1	0.8	0.6	0.8	0.8	0.9	0.62	0.8
III	2	2	2	2	2	2	0.8	0.8	0.8	0.8	0.8	0.8	0.8
IV	2	2	2	2	2	2	0.8	0.85	0.8	0.85	0.9	0.85	0.8

The accuracy of estimates (5.21), (5.22) and (5.24) has also been investigated numerically, using the direct Markov chain analysis. A few typical examples are shown in Tables I and III–V. In all cases analyzed, the estimates resulted in sufficiently high precision. In all examples, convergence took place in just a few iterations, and computational time was insignificant (a fraction of a second, using Pentium 133 MHz processor).

TABLE III Illustration of accuracy of $E[h_{ij}]$

Cases	$E[\tilde{h}_{11}]$ $E[h_{11}]$	$E[\tilde{h}_{12}]$ $E[h_{12}]$	$E[\tilde{h}_{21}]$ $E[h_{21}]$	$E[\tilde{h}_{22}]$ $E[h_{22}]$	$E[\tilde{h}_{01}]$ $E[h_{01}]$	$E[\tilde{h}_{02}]$ $E[h_{02}]$	$\max_{i,j}(\lvert E[\tilde{h}_{ij}] - E[h_{ij}]\rvert)/E[\tilde{h}_{ij}] \cdot 100$ (%)
I	0.8849	0.8201	0.8849	0.8201	0.6404	0.5756	4
	0.8894	0.8207	0.8894	0.8207	0.6215	0.5528	
II	0.9027	0.6815	0.9027	0.8607	0.685	0.4868	3.8
	0.9063	0.6856	0.9063	0.8579	0.6669	0.4684	
III	1.5908	1.4769	1.5908	1.4769	1.1498	1.0359	0.3
	1.5904	1.4729	1.5904	1.4729	1.1517	1.0342	
IV	1.4692	1.422	1.4692	1.422	1.2627	1.2154	1.1
	1.462	1.4166	1.462	1.4166	1.2745	1.2291	

TABLE IV Illustration of accuracy of $ms_{ij_{(k)}}$

Cases	$\widetilde{ms}_{12}$ ms_{12}	$\widetilde{ms}_{22}$ ms_{22}	$\widetilde{ms}_{01_1}$ ms_{01_1}	$\widetilde{ms}_{01_2}$ ms_{01_2}	$\widetilde{ms}_{02}$ ms_{02}	$\widetilde{ms}_{03}$ ms_{03}	$\max_{i,j,k}(\lvert\widetilde{ms}_{ij_{(k)}} ms_{ij_{(k)}}\rvert/\widetilde{ms}_{ij_{(k)}}) \cdot 100$ (%)
I	0.0921	0.0921	0.1439	0.1439	0.2877	0.3395	5.36
	0.0844	0.0844	0.1434	0.1434	0.3028	0.3577	
II	0.0584	0.0779	0.2866	0.1254	0.1953	0.4106	5.73
	0.0652	0.0749	0.2829	0.1279	0.2065	0.4253	
III	0.0404	0.0404	0.0629	0.0629	0.14	0.1733	1.5
	0.0406	0.0406	0.0634	0.0634	0.1421	0.1754	
IV	0.0611	0.0611	0.0776	0.0776	0.1113	0.1151	6.2
	0.0621	0.0621	0.0758	0.0758	0.1044	0.1089	

TABLE V Illustration of accuracy of mb_{ij}

Cases	$\widetilde{mb}_{11}$ mb_{11}	$\widetilde{mb}_{12}$ mb_{12}	$\widetilde{mb}_{21}$ mb_{21}	$\widetilde{mb}_{22}$ mb_{22}	$\widetilde{mb}_{01}$ mb_{01}	$\widetilde{mb}_{02}$ mb_{02}	$\max_{i,j}(\lvert\widetilde{mb}_{ij} - mb_{ij}\rvert/\widetilde{mb}_{ij}) \cdot 100$ (%)
I	0.3395	0.2877	0.3395	0.2877	0.1439	0.0921	5.36
	0.3577	0.3028	0.3577	0.3028	0.1434	0.0884	
II	0.4106	0.1753	0.4106	0.377	0.266	0.0604	6.39
	0.4253	0.1865	0.4253	0.3866	0.2629	0.0581	
III	0.1733	0.14	0.1733	0.14	0.0629	0.0404	1.5
	0.1754	0.1421	0.1755	0.1421	0.0634	0.0406	
IV	0.1151	0.1113	0.1151	0.1113	0.0776	0.0611	6.2
	0.1089	0.1044	0.109	0.1045	0.0759	0.0621	

7 CASE STUDY: PERFORMANCE ANALYSIS OF AN AUTOMOTIVE COMPONENT ASSEMBLY SYSTEM

In this section, an application of the performance analysis technique developed above is illustrated.

7.1 System Description and Problem Formulation

The topological structure of the production system under consideration is shown in Fig. 2. It consists of three circular conveyors, two of which produce component parts A_1 and A_2, respectively, and the third serves as a buffer. The mating of the two components occurs at Operation 150 (the assembly machine). All operations, from 10 to 200 (excluding 150) serve to either process component i, $i = 1, 2$, or to attach a purchased part to each of the components.

The operation of each component loop is as follows: Parts are transported by the conveyors on pallets. Raw material (parts A_1 and A_2) is loaded at the first operation of each loop (Operations 10 and 110) if a pallet is available; if not, Operations 10 and 110 are starved for pallets. After being loaded, the parts are transported by the conveyors to the subsequent operations $(20, \ldots, 80)$ in Component 1 loop and $(120, \ldots, 190)$ in Component 2 loop. Operation 90 transfers the

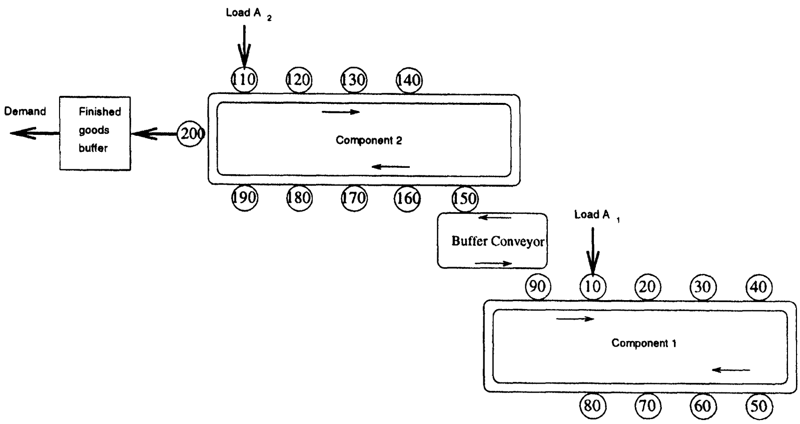

FIGURE 2 Assembly system layout.

finished Component 1 onto the buffer conveyor and releases the pallet for loading at Operation 10. The finished Component 1 is transported, by the buffer conveyor (again on pallets), to the assembly Operation 150 where Components 1 and 2 are mated (if the Component 2 is available at the time of the arrival of Component 1). The assembled part undergoes additional processing at Operations 160–190 and is released at Operation 200 in the finished goods buffer, making a pallet available for Operation 110.

The cycle time of each machine is 6 s, i.e., the nominal production rate of the system is 600 parts/h. The actual performance for six consecutive months is summarized in Table VI. As it follows from this table, the average production rate over the six months period is 362 parts/h, i.e., the system operates at 60% of its capacity. The goal of this case study was to identify and eliminate the causes of this significant loss of productivity. Below, we use this case study to illustrate the application of the production rate evaluation technique developed above. In Part II of this work the case study is described in its entirety.

7.2 Production Rate Evaluation

Neglecting the circular nature of the conveyors and the effect of the pallets, the system under consideration can be conceptualized as shown in Fig. 3.

Although it is quite important, we neglect here the effect of the circular nature of the conveyors. We do so because we do not have a theory for analyzing such structures. Nor the current literature offers

TABLE VI Measured production rate of the system

Month	*May*	*June*	*July*	*August*	*September*	*October*
$\widetilde{PR}$ (parts/h)	337	347	378	340	384	383

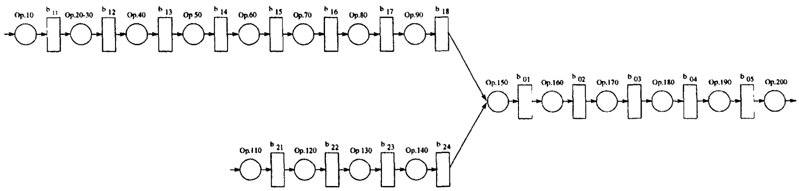

FIGURE 3 Structural model of the assembly system.

such a technique. The effect of starvation of Operations 10 and 110 for pallets will be, however, taken into account by estimating the probability of Operations 10 and 110 starvation from the real-time operation data and reducing the isolation production rate of these operations appropriately (see below). The blockage of Operation 200 due to the finished goods buffer will be treated analogously.

The model of the system with the machines and buffer parameters identified from the real-time data is shown in Fig. 4, using, as an example, the May data. In this figure, the number under each machine indicates its production rate in isolation. The numbers in the rectangles represent the buffers capacity. The effect of the closed nature of circular conveyors, i.e., the starvation of Operations 10 and 110 due to the lack of pallets and the blockage of Operation 200 due to full finished goods buffer, was taken into account as follows: The average fractions of time when Operations 10 and 110 were starved for pallets and Operation 200 was blocked were identified from the real-time data and the isolation production rates of Operations 10, 110 and 200 were multiplied by a factor (1 − average fraction of time when starvation or blockage take place) (see Fig. 4).

Thus, the model of the assembly system is identified. Using the iteration procedure (5.19) and (5.21), we calculate the production rate of the system for each of the consecutive six months. The results are shown in Table VII along with the actual production rate.

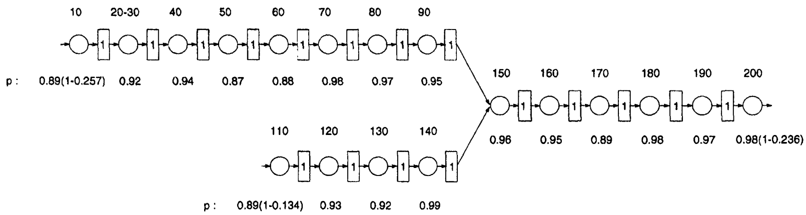

FIGURE 4 Assembly system with parameters identified (based on May data).

TABLE VII Comparison of measured and estimated production rates

Month	*May*	*June*	*July*	*August*	*September*	*October*
$\underset{\sim}{PR}$ (parts/h)	330	336	302	335	380	337
PR (parts/h)	337	347	378	340	384	383
Error (%)	2.1	3.2	20.1	1.6	1	11.9

As it follows from these data, the production rate estimates match closely the actual ones, with the exception of the months July and October. We rationalize the situation in July by the fact that during this month a two weeks shut-down period took place and, supposedly, some phenomena, not reflected in the data identified, were presented. We do not have an explanation for the large error in the month of October.

8 CONCLUSIONS

This paper formulates the problems of Improvability Theory for assembly systems. The development of this theory requires a method for evaluating system performance measures as functions of machine and buffer parameters. In this paper, such a method is developed, its convergence is proved, and an estimate of its accuracy is provided. In addition, the paper describes an application of this method to an assembly line at an automotive component plant. In Part II of this work, this method is used to derive improvability indicators for assembly systems.

APPENDIX A: PROOFS FOR SECTION 5

To prove Theorem 5.1, we need two auxiliary statements. The first one refers to the serial production lines defined and analyzed in [1]. The serial line is defined in [1] as $[p_1, \ldots, p_i, \ldots, p_M, N_1, \ldots, N_{M-1}]$, where p_i and N_i are the machines and buffer parameters (analogous to p_{ij}'s and N_{ij}'s of assumptions (iii) and (iv) of Section 2 above). For this auxiliary statement, we consider two serial lines with all but one identical machines and all identical buffers. We define these lines as

$$\begin{aligned} &\text{Line 1:} \quad [p_1, \ldots, p_{i-1}, p_i, p_{i+1}, \ldots, p_M, N_1, \ldots, N_{M-1}], \\ &\text{Line 2:} \quad [p_1, \ldots, p_{i-1}, \bar{p}_i, p_{i+1}, \ldots, p_M, N_1, \ldots, N_{M-1}]. \end{aligned} \tag{A.1}$$

LEMMA A.1 *Consider two serial lines defined by expressions* (A.1). *Assume that* $i \neq 1$ *or* M *and* $\bar{p}_i > p_i$. *Let* p_j^f, p_j^b *and* $\bar{p}_j^f$, $\bar{p}_j^b$, $j = 1, \ldots, M$, *be the steady states of the recursive procedure* (4) *of* [1] *applied to lines* 1 *and* 2, *respectively. Then*

$$\bar{p}_j^f > p_j^f, \quad \bar{p}_{j+1}^b < p_{j+1}^b, \qquad j = i, \ldots, M-2,$$
$$\bar{p}_{M-1}^f > p_{M-1}^f, \quad \bar{p}_M^b = p_M^b,$$

and

$$\bar{p}_1^f = p_1^f, \quad \bar{p}_2^b > p_2^b,$$
$$\bar{p}_{j-1}^f < p_{j-1}^f, \quad \bar{p}_j^b > p_j^b, \qquad j = 3, \ldots, i.$$

Proof For $j = M-1$, the statement of the lemma follows from the monotonicity of PR with respect to p_i^fs. For $j < M-1$, the proof is obtained by induction (see [10] for details).

The steady states of the recursive procedure (5.19) are defined by the following equations:

$$\begin{aligned}
\mu_{M_1+1} &= p_{01}[1 - Q(\nu_{M_2}^f, \nu_{M_2+1}^b, \Lambda_{M_2})],\\
\mu_i^b &= \mu_i[1 - Q(\mu_{i+1}^b, \mu_i^f, \Gamma_i)], \quad 1 \le i \le M_1 + M_0 - 1,\\
\mu_i^f &= \mu_i[1 - Q(\mu_{i-1}^f, \mu_i^b, \Gamma_{i-1})], \quad 2 \le i \le M_1 + M_0,\\
\nu_{M_2+1} &= p_{01}[1 - Q(\mu_{M_1}^f, \mu_{M_1+1}^b, \Gamma_{M_1})],\\
\nu_i^b &= \mu_i[1 - Q(\nu_{i+1}^b, \nu_i^f, \Lambda_i)], \quad 1 \le i \le M_2 + M_0 - 1,\\
\nu_i^f &= \nu_i[1 - Q(\nu_{i-1}^f, \nu_i^b, \Lambda_{i-1})], \quad 2 \le i \le M_2 + M_0,\\
\mu_1^f &= \mu_1, \quad \mu_{M_1+M_0}^b = \mu_{M_1+M_0},\\
\nu_1^f &= \nu_1, \quad \nu_{M_2+M_0}^b = \nu_{M_2+M_0}.
\end{aligned} \tag{A.2}$$

Introduce $(M_1 + M_2 + 2M_0 - 2)$ two machine–one buffer serial production lines, L_i', $i = 1, \ldots, M_1 + M_0 - 1$, and L_i'', $i = 1, \ldots, M_2 + M_0 - 1$, where the first machine has the isolation production rate μ_i^f (respectively, ν_i^f), the second μ_{i+1}^b (respectively, ν_{i+1}^b), and the buffer capacity is Γ_i (respectively, Λ_i). The following properties hold:

LEMMA A.2 *Let PR_i' be the production rate of line L_i', $i = 1, \ldots, M_1 + M_0 - 1$, and PR_i'' be the production rate of line L_i'', $i = 1, \ldots, M_2 +$*

$M_0 - 1$. *Denote* $\mu^f_{M_1+M_0}$ *as* $PR'_{M_1+M_0}$ *and* $\nu^f_{M_2+M_0}$ *as* $PR''_{M_2+M_0}$. *Then,*

(a) $PR'_i = (\mu^f_i \mu^b_i)/\mu_i$, $i = 1, \ldots, M_1 + M_0$, *and* $PR''_i = (\nu^f_i \nu^b_i)/\nu_i$, $i = 1, \ldots, M_2 + M_0$;
(b) $PR'_i = PR''_j, \forall i = 1, \ldots, M_1 + M_0, \forall j = 1, \ldots, M_2 + M_0$.

Proof The proof of statement (a) is similar to that of Lemma A.8 of [1].

Using Statement (a) of this lemma and Lemma A.8 of [1], we have

$$\begin{aligned} PR'_i &= PR'_{i-1}, \quad i = 2, \ldots, M_1 + M_0, \\ PR''_i &= PR''_{i-1}, \quad i = 2, \ldots, M_2 + M_0. \end{aligned} \tag{A.3}$$

Now we will show, by contradiction, that

$$PR'_i = PR''_j, \quad \forall i = 1, \ldots, M_1 + M_0, \ \forall j = 1, \ldots, M_2 + M_0.$$

This will complete the proof of statement (b). Suppose that $PR'_{M_1+1} > PR''_{M_2+1}$. Then, by induction, we prove that $\mu^b_{M_1+j} > \nu^b_{M_2+j}$ and $\mu^f_{M_1+j} < \nu^f_{M_2+j}$, $\forall 2 \le j \le M_0$. In particular, $\mu^f_{M_1+M_0} < \nu^f_{M_2+M_0}$, so by statement (a) and (A.3), $PR'_{M_1+M_0} < PR''_{M_2+M_0}$, and $PR'_{M_1+1} < PR''_{M_2+1}$, which contradicts the assumption. Therefore, we conclude that $PR'_{M_1+1} \le PR''_{M_2+1}$.

Assuming that $PR'_{M_1+1} < PR''_{M_2+1}$ and proceeding analogously, we obtain $PR'_{M_1+1} \ge PR''_{M_2+1}$. Therefore, $PR'_{M_1+1} = PR''_{M_2+1}$. Using (A.3), we finally conclude that

$$PR'_i = PR''_j, \quad \forall i = 1, \ldots, M_1 + M_0, \ \forall j = 1, \ldots, M_2 + M_0.$$

Proof of Theorem 5.1 Using Lemma A.1 and the monotonicity properties of function $Q(x, y, N)$ (see [1]), it is possible to show that $\mu_{M_1+1}(s)$, and $\nu_{M_2+1}(s)$, $s = 1, 2, \ldots$, are bounded on [0, 1] and monotonically decreasing and increasing, respectively. Then, the convergence of sequences $\mu^f_i(s)$, $\mu^b_i(s)$, $i = 1, \ldots, M_1 + M_0$ and $\nu^f_i(s)$, $\nu^b_i(s)$, $i = 1, \ldots, M_2 + M_0$, $s = 1, 2, 3, \ldots$, follows immediately from [1] based on that fact that $\mu_{M_1+1}(s)$ and $\nu_{M_2+1}(s)$, $s = 1, 2, 3, \ldots$, are convergent (see [10] for details).

In addition, since, from (A.2),

$$\begin{aligned}\mu^f_{M_1+1} &= p_{01}[1 - Q(\nu^f_{M_2}, \nu^b_{M_2+1}, \Lambda_{M_2})][1 - Q(\mu^f_{M_1}, \mu^b_{M_1+1}, \Gamma_{M_1})] \\ &= p_{01}[1 - Q(\mu^f_{M_1}, \mu^b_{M_1+1}, \Gamma_{M_1})][1 - Q(\nu^f_{M_2}, \nu^b_{M_2+1}, \Lambda_{M_2})] \\ &= \nu^f_{M_2+1}, \end{aligned} \tag{A.4}$$

by Lemma A.2 and Eqs. (5.3)–(5.6), we have

$$\begin{aligned} PR'_{M_1+1} &= \mu^f_{M_1+1}[1 - Q(\mu^b_{M_1+2}, \mu^f_{M_1+1}, \Gamma_{M_1+1})] \\ &= PR''_{M_2+1} \\ &= \nu^f_{M_2+1}[1 - Q(\nu^b_{M_2+2}, \nu^f_{M_2+1}, \Lambda_{M_2+1})]. \end{aligned} \tag{A.5}$$

It follows then that

$$\mu^b_{M_2+2} = \nu^b_{M_2+2}. \tag{A.6}$$

From Lemma A.2, we have

$$\frac{\mu^f_{M_1+2}\mu^b_{M_1+2}}{\mu_{M_1+2}} = \frac{\nu^f_{M_2+2}\nu^b_{M_2+2}}{\nu_{M_2+2}}. \tag{A.7}$$

Since $\mu^b_{M_2+2} = \nu^b_{M_2+2}$ and $\mu_{M_1+2} = \nu_{M_2+2}$, we obtain

$$\mu^f_{M_1+2} = \nu^f_{M_1+2}. \tag{A.8}$$

Similar arguments can be used to prove

$$\begin{aligned} \mu^f_{M_1+i} &= \nu^f_{M_2+i}, \quad i = 3, \ldots, M_0, \\ \nu^b_{M_1+i} &= \nu^b_{M_2+i}, \quad i = 3, \ldots, M_0. \end{aligned} \tag{A.9}$$

APPENDIX B: PROOFS FOR SECTION 6

Consider the following conditional probabilities:

$$\begin{aligned}
\tilde{\mu}_{M_1+1} &= \text{Prob}\{m_{01} \text{ produces} \mid m_{01} \text{ is not blocked and not starved} \\
&\qquad \text{due to } b'_{M_1} \text{ being empty}\}, \\
\tilde{\nu}_{M_2+1} &= \text{Prob}\{m_{01} \text{ produces} \mid m_{01} \text{ is not blocked and not starved} \\
&\qquad \text{due to } b''_{M_2} \text{ being empty}\}, \\
\tilde{\mu}_i^f (\text{or } \tilde{\nu}_i^f) &= \text{Prob}\{m'_i \text{ (or } m''_i) \text{ produces} \mid m'_i \text{ (or } m''_i) \text{ is not blocked}\}, \\
&\quad i = 1, \ldots, M_1 + M_0 \ \ (\text{or } i = 1, \ldots, M_2 + M_0), \\
\tilde{\mu}_i^b (\text{or } \tilde{\nu}_i^b) &= \text{Prob}\{m'_i \text{ (or } m''_i) \text{ produces} \mid m'_i \text{ (or } m''_i) \text{ is not starved}\}, \\
&\quad i = 1, \ldots, M_1 + M_0 \ \ (\text{or } i = 1, \ldots, M_2 + M_0).
\end{aligned} \tag{B.1}$$

These probabilities play important roles in the proof of Theorem 6.1. Specifically, we show below (Lemma B.3) that if $\tilde{\mu}_{M_1+1}$ (or $\tilde{\nu}_{M_2+1}$), $\tilde{\mu}_i^f$ (or $\tilde{\nu}_i^f$) and $\tilde{\mu}_{i+1}^b$ (or $\tilde{\nu}_{i+1}^b$) are known, the stationary probability distribution of buffer occupancy, $\tilde{X}_{i'}(\cdot)$ (or $\tilde{X}_{i''}(\cdot)$), can be calculated with the error $\mathcal{O}(\delta)$. Further, Lemma B.4 shows that $\tilde{\mu}_{M_1+1}$ (or $\tilde{\nu}_{M_2+1}$), $\tilde{\mu}_i^f$ (or $\tilde{\nu}_i^f$) and $\tilde{\mu}_{i+1}^b$ (or $\tilde{\nu}_{i+1}^b$) can be evaluated from the steady state of recursive procedure (5.19), with the error $\mathcal{O}(\delta)$. Therefore, since the production rate can be calculated as $\widetilde{PR} = [1 - \tilde{X}_{(M_1+M_0-1)'}(0)]\mu_{M_1+M_0}$ (or $\widetilde{PR} = [1 - \tilde{X}_{(M_2+M_0-1)''}(0)]\nu_{M_2+M_0}$), the first claim of Theorem 6.1 will follow. The other three claims are proved analogously.

LEMMA B.1 *Under Numerical Fact* 6.1, *the conditional probabilities* $\tilde{\mu}_{M_1+1}$ *and* $\tilde{\nu}_{M_2+1}$ *have the following form:*

(a) $\tilde{\mu}_{M_1+1} = p_{01}[1 - \tilde{X}_{M_2''}(0)] + \mathcal{O}(\delta)$,
(b) $\tilde{\nu}_{M_2+1} = p_{01}[1 - \tilde{X}_{M_1'}(0)] + \mathcal{O}(\delta)$.

Proof First we show that Prob$\{m_{01}$ is blocked or starved due to b'_{M_1} being empty$\}$ is $\mathcal{O}(\delta)$-close to Prob$\{m_{01}$ is blocked or starved due to b'_{M_1} being empty $|$ m_{01} is not starved due to b''_{M_2} being empty$\}$. Then, taking into account that $\tilde{\mu}_{M_1+1} =$ Prob$\{m_{01}$ produces $|$ m_{01} is not blocked and not starved due to b'_{M_1} being empty}, we use this relationship and the conditional probability formula to prove claim (a). Claim (b) is proved analogously (see [10] for details).

LEMMA B.2 *Under Numerical Fact* 6.1, *the conditional probabilities* $\tilde{\mu}_i^f, \tilde{\mu}_i^b, \tilde{\nu}_i^f, \tilde{\nu}_i^b$ *can be evaluated as follows:*

(a)

$$\tilde{\mu}_i^f = \begin{cases} \mu_i[1 - \tilde{X}_{(i-1)'}(0)] + \mathcal{O}(\delta), & i = 2, \dots, M_1, M_1 + 2, \dots, M_1 + M_0, \\ \tilde{\mu}_i[1 - \tilde{X}_{(i-1)'}(0)] + \mathcal{O}(\delta), & i = M_1 + 1; \end{cases}$$

(b)

$$\tilde{\mu}_i^b = \begin{cases} \mu_i\Big\{1 - \Big[\sum_{j=i+1}^{M_1}\Big(\prod_{r=i+1}^{j-1}\mu_r\Big)(1-\mu_j)\tilde{X}_{i',\dots,(j-1)'}(\Gamma_i,\dots,\Gamma_{j-1}) \\ \quad + \Big(\prod_{r=i+1}^{M_1}\mu_r\Big)(1-\tilde{\mu}_{M_1+1})\tilde{X}_{i',\dots,M_1'}(\Gamma_i,\dots,\Gamma_{M_1}) \\ \quad + \sum_{j=M_1+2}^{M_1+M_0}\Big(\prod_{r=i+1}^{M_1}\mu_r\Big)\tilde{\mu}_{M_1+1}\Big(\prod_{r=M_1+2}^{j-1}\mu_r\Big)(1-\mu_j) \\ \quad \times \tilde{X}_{i',\dots,(j-1)'}(\Gamma_i,\dots,\Gamma_{j-1})\Big]\Big\} + \mathcal{O}(\delta), \quad i = 1,\dots,M_1, \\ \tilde{\mu}_i\Big[1 - \sum_{j=i+1}^{M_1+M_0}\Big(\prod_{r=i+1}^{j-1}\mu_r\Big) \\ \quad \times (1-\mu_j)\tilde{X}_{i',\dots,(j-1)'}(\Gamma_i,\dots,\Gamma_{j-1})\Big] + \mathcal{O}(\delta), \quad i = M_1+1, \\ \mu_i\Big[1 - \sum_{j=i+1}^{M_1+M_0}\Big(\prod_{r=i+1}^{j-1}\mu_r\Big) \\ \quad \times (1-\mu_j)\tilde{X}_{i',\dots,(j-1)'}(\Gamma_i,\dots,\Gamma_{j-1})\Big] + \mathcal{O}(\delta), \\ \qquad i = M_1+2,\dots,M_1+M_0; \end{cases}$$

(c)

$$\tilde{\nu}_i^f = \begin{cases} \nu_i[1 - \tilde{X}_{(i-1)'}(0)] + \mathcal{O}(\delta), & i = 2, \dots, M_2, M_2 + 2, \dots, M_2 + M_0, \\ \tilde{\nu}_i[1 - \tilde{X}_{(i-1)'}(0)] + \mathcal{O}(\delta), & i = M_2 + 1; \end{cases}$$

(d)

$$\tilde{\nu}_i^b = \begin{cases} \nu_i\Big\{1 - \Big[\sum_{j=i+1}^{M_2}\Big(\prod_{r=i+1}^{j-1}\nu_r\Big)(1-\nu_j)\tilde{X}_{i'',\dots,(j-1)''}(\Lambda_i,\dots,\Lambda_{j-1}) \\ \quad + \Big(\prod_{r=i+1}^{M_2}\nu_r\Big)(1-\tilde{\nu}_{M_2+1})\tilde{X}_{i'',\dots,M_2''}(\Lambda_i,\dots,\Lambda_{M_2}) \\ \quad + \sum_{j=M_2+2}^{M_2+M_0}\Big(\prod_{r=i+1}^{M_2}\nu_r\Big)\tilde{\nu}_{M_2+1}\Big(\prod_{r=M_2+2}^{j-1}\nu_r\Big)(1-\nu_j) \\ \quad \times \tilde{X}_{i'',\dots,(j-1)''}(\Lambda_i,\dots,\Lambda_{j-1})\Big]\Big\} + \mathcal{O}(\delta), \quad i = 1,\dots,M_2, \\ \tilde{\nu}_i\Big[1 - \sum_{j=i+1}^{M_2+M_0}\Big(\prod_{r=i+1}^{j-1}\nu_r\Big) \\ \quad \times (1-\nu_j)\tilde{X}_{i'',\dots,(j-1)''}(\Lambda_i,\dots,\Lambda_{j-1})\Big] + \mathcal{O}(\delta), \quad i = M_2+1, \\ \nu_i\Big[1 - \sum_{j=i+1}^{M_2+M_0}\Big(\prod_{r=i+1}^{j-1}\nu_r\Big) \\ \quad \times (1-\nu_j)\tilde{X}_{i'',\dots,(j-1)''}(\Lambda_i,\dots,\Lambda_{j-1})\Big] + \mathcal{O}(\delta), \\ \quad i = M_2+2,\dots,M_2+M_0, \end{cases}$$

where $\tilde{X}_{i',\dots,j'}(h_i',\dots,h_j')$ (*or* $\tilde{X}_{i'',\dots,j''}(h_i'',\dots,h_j'')$) *is the steady state probability that consecutive buffers* $i',\dots,j'$ (*or* $i'',\dots,j''$) *in the upper line* (*or in the lower line*) *of the assembly system contain* $h_i',\dots,h_j'$ (*or* $h_i'',\dots,h_j''$) *parts, respectively, and* $\tilde{\mu}_{M_1+1}$ *and* $\tilde{\nu}_{M_2+1}$ *are given by Lemma* B.1

Proof Invoking Lemma B.1, it is possible to show that the proof of this lemma is similar to the proof of Lemma A.6 of [1]. Details can be found in [10].

Consider now $(M_1+M_2+M_0-1)$ two machine–one buffer lines L_i', $i=1,\dots,M_1+M_0-1$, where the first machine is defined by $\tilde{\mu}_i^f$, the second by $\tilde{\mu}_{i+1}^b$, and the buffer is of capacity Γ_i, and L_i'', $i=1,\dots,M_2$, where the first machine $\tilde{\nu}_i^f$, the second $\tilde{\nu}_{i+1}^b$, and the buffer Λ_i. Let $\check{X}_{i'}(\cdot)$ (or $\check{X}_{i''}(\cdot)$) be the equilibrium probability distribution of buffer occupancy of line L_i' (or L_i''). Along with these $M_1+M_2+M_0-1$ lines, consider the assembly line (i)–(vi) with $M_1+M_2+M_0$ machines. Let $\tilde{X}_{i'}(\cdot)$ (or $\tilde{X}_{i''}(\cdot)$), as before, be the equilibrium probability distribution of buffer occupancy of buffer i in the upper line (or in the lower line). Then, we have

LEMMA B.3 *Under Numerical Fact* 6.1, *the following is true:*

$$\tilde{X}_{i'(j)} = \check{X}_{i'}(j) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_1 + M_0 - 1, \; j = 0, \ldots, \Gamma_i, \quad \text{(B.2)}$$

$$\tilde{X}_{i''(j)} = \check{X}_{i''}(j) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_2, \; j = 0, \ldots, \Lambda_i. \quad \text{(B.3)}$$

Proof The proof of this lemma for $i = 1, \ldots, M_1 - 1, M_1 + 2, \ldots, M_1 + M_0 - 1$ in the upper line and $i = 1, \ldots, M_2 - 1$ in the lower line follows directly from Lemma A.7 of [1]. The proof for $i = M_1, M_1 + 1$ in the upper line and $i = M_2$ in the lower line is as follows: Consider assembly system (i)–(vi) with $M_1 + M_2 + M_0 - 1$ machines. Let $K_{i'} = [k_{1'}, \ldots, k_{(i-1)'}, k_{(i+1)'}, \ldots, k_{(M_1+M_0-1)'}, k_1, \ldots, k_{M_2''}]$, $1 \le i \le M_1 + M_0 - 1$, $0 \le k_{j'} \le \Gamma_j$, $j \neq i$ (or $K_{i''} = [k_{1'}, \ldots, k_{(M_1+M_0-1)'}, k_{1''}, \ldots, k_{(i-1)''}, k_{(i+1)''}, \ldots, k_{M_2''}]$, $1 \le i \le M_2$, $0 \le k_{j''} \le \Lambda_j$, $j \neq i$), be an $(M_1 + M_2 + M_0 - 2)$-dimensional vector.

Let $Y_{i'}(h_i', K_{i'})$, $1 \le i \le M_1 + M_0 - 1$, denote the probability that there are h_i' parts in buffer i, and k_j' parts in buffer j of the upper line, $\forall j \neq i$, and k_l parts in buffer l of the lower line, $l = 1, \ldots, M_2$. Similarly, let $Y_{i''}(h_i'', K_{i''})$, $1 \le i \le M_2$, denote the probability that there are h_i'' parts in buffer i, and k_j'' parts in buffer j of the lower line, $\forall j \neq i$, and k_l' parts in buffer l of the upper line, $I = 1, \ldots, M_1 + M_0 - 1$. Since assembly system (i)–(vi) can be described by an ergodic Markov chain with states $Y_{i'}(h_i', K_{i'})$ (or $Y_{i''}(h_i'', K_{i''})$) in the steady state we write:

$$\begin{aligned}
Y_{i'}(0, K_{i'}) = &\sum_{\bar{K}_{i'}} Y_{i'}(0, \bar{K}_{i'}) \text{Prob}\{m_i' \text{ does not produce} \,|\, 0, \bar{K}_{i'}\} \\
&\times \text{Prob}\{\bar{K}_{i'} \to K_{i'} \,|\, 0 \to 0\} + \sum_{\bar{K}_{i'}} Y_{i'}(1, \bar{K}_{i'}) \\
&\times \text{Prob}\{m_i' \text{ does not produce}, m_{i+1}' \text{ produces} \,|\, 1, \bar{K}_{i'}\} \\
&\times \text{Prob}\{\bar{K}_{i'} \to K_{i'} \,|\, 1 \to 0\} \Big(\text{or } Y_{i''}(0, K_{i''}) = \sum_{\bar{K}_{i''}} Y_{i''}(0, \bar{K}_{i''}) \\
&\times \text{Prob}\{m_i'' \text{ does not produce} \,|\, 0, \bar{K}_{i''}\} \\
&\times \text{Prob}\{\bar{K}_{i''} \to K_{i''} \,|\, 0 \to 0\} + \sum_{\bar{K}_{i''}} Y_{i''}(1, \bar{K}_{i''}) \\
&\times \text{Prob}\{m_i'' \text{ does not produce}, m_{i+1}'' \text{ produces} \,|\, 1, \bar{K}_{i'}\} \\
&\times \text{Prob}\{\bar{K}_{i''} \to K_{i''} \,|\, 1 \to 0\} \Big), \qquad \text{(B.4)}
\end{aligned}$$

where $\text{Prob}\{m_i' \text{ does not produce} \mid h_i', K_{i'}\}$ denotes the conditional probability that machine i in the upper line does not produce a part during a cycle, given that buffer b_i' contains h_i' parts and buffer b_j' contains k_j' parts, $\forall j \neq i$, and $\text{Prob}\{\bar{K}_{i'} \to K_{i'} \mid \bar{h}_i' \to h_i'\}$ denotes the conditional probability of the transition from the state where buffer b_j', $j \neq i$, contains $\bar{k}_j'$ parts to the state where buffer b_j' contains k_j' parts, given that the number of parts in buffer b_i' changes from $\bar{h}_i'$ to h_i'. First, for the buffers in the upper line, summation over all $K_{i'} \in \Re^{M_1+M_2+M_0-2}$ yields

$$
\begin{aligned}
\tilde{X}_{i'}(0) = &\sum_{\bar{K}_{i'}} Y_{i'}(0, \bar{K}_{i'}) \text{Prob}\{m_i' \text{ does not produce} \mid 0, \bar{K}_{i'}\} \\
&\times \sum_{K_{i'}} \text{Prob}\{\bar{K}_{i'} \to K_{i'} \mid 0 \to 0\} + \sum_{\bar{K}_{i'}} \bar{Y}_{i'}(1, \bar{K}_{i'}) \\
&\times \text{Prob}\{m_i' \text{ does not produce}, m_{i+1}' \text{ produces} \mid 1, \bar{K}_{i'}\} \\
&\times \sum_{K_{i'}} \text{Prob}\{\bar{K}_{i'} \to K_{i'} \mid 1 \to 0\}.
\end{aligned}
$$

Since $\sum_{K_{i'}} \text{Prob}\{\bar{K}_{i'} \to K_{i'} \mid 0 \to 0\} = 1$,

$$
\begin{aligned}
\tilde{X}_{i'}(0) = &\sum_{\bar{K}_{i'}} Y_{i'}(0, \bar{K}_{i'}) \text{Prob}\{m_i' \text{ does not produce} \mid 0, \bar{K}_{i'}\} \\
&+ \sum_{\bar{K}_{i'}} \bar{Y}_{i'}(1, \bar{K}_{i'}) \text{Prob}\{m_i' \text{ does not produce}, \\
&m_{i+1}' \text{ produces} \mid 1, \bar{K}_{i'}\}. \qquad \text{(B.5)}
\end{aligned}
$$

This expression is similar to (A.11) of [1], which makes the rest of the proof analogous to that of Lemma A.7 of [1].

Lemma B.3 shows that if the conditional probabilities $\tilde{\mu}_{M_1+1}, \tilde{\nu}_{M_2+1}$, $\tilde{\mu}_i^f$, $i = 1, \ldots, M_1 + M_0$, and $\tilde{\nu}_i^f, \tilde{\nu}_i^b$, $i = 1, \ldots, M_2 + M_0$, are known, then it is possible to determine, approximately, the steady state buffer occupancy probability distributions of $\tilde{X}_{i'}(\cdot)$, $i = 1, \ldots, M_1 + M_0 - 1$, and $\tilde{X}_{i''}(\cdot)$, $i = 1, \ldots, M_2$. The task of determining the values of these conditional probabilities, however, remains. Lemma B.4 shows that they are given, approximately, by recursive procedure (5.19).

LEMMA B.4 *Under Numerical Fact* 6.1,

$$\tilde{\mu}_{M_1+1} = \mu_{M_1+1} + \mathcal{O}(\delta),$$
$$\tilde{\nu}_{M_2+1} = \nu_{M_2+1} + \mathcal{O}(\delta),$$
$$\tilde{\mu}_i^f = \mu_i^f + \mathcal{O}(\delta), \quad \tilde{\mu}_i^b = \mu_i^b + \mathcal{O}(\delta), \qquad i = 1, \ldots, M_1 + M_0,$$
$$\tilde{\nu}_i^f = \nu_i^f + \mathcal{O}(\delta), \quad \tilde{\nu}_i^b = \nu_i^b + \mathcal{O}(\delta), \qquad i = 1, \ldots, M_2 + M_0,$$

where $\mu_{M_1+1}, \nu_{M_2+1}, \mu_i^f, \mu_i^b$, $i=1,\ldots,M_1+M_0$, *and* ν_i^f, ν_i^b, $i=1,\ldots,$ M_2+M_0, *are given by Eq.* (5.20).

Proof With the help of Lemmas B.1–B.3, it is possible to show that the proof of this lemma is similar to Lemma A.10 of [1] (see [10] for details).

LEMMA B.5 *Under Numerical Fact* 6.1, *the following is true:*

$$\check{X}_{i'}(0) = X_{i'}(0) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_1 + M_0 - 1,$$
$$\check{X}_{i''}(0) = X_{i''}(0) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_2,$$

where $\check{X}_{i'}(0)$ *and* $\check{X}_{i''}(0)$ *are, respectively, stationary probabilities of buffer* b_i' *being empty in line* L_i' *and buffer* b_i'' *being empty in line* L_i'', *and*

$$\begin{aligned} X_{i'}(0) &= Q(\mu_i^f, \mu_{i+1}, \Gamma_i), \quad i = 1, \ldots, M_1 + M_0 - 1, \\ X_{i''}(0) &= Q(\nu_i^f, \nu_{i+1}, \Lambda_i), \quad i = 1, \ldots, M_2. \end{aligned} \tag{B.6}$$

Proof Using Lemma A.5 in [1],

$$\check{X}_{i'}(0) = Q(\tilde{\mu}_i^f, \tilde{\mu}_{i+1}, \Gamma_i), \quad i = 1, \ldots, M_1 + M_0 - 1. \tag{B.7}$$

By Lemma B.4,

$$\check{X}_{i'}(0) = Q(\mu_i^f, \mu_{i+1}, \Gamma_i) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_1 + M_0 - 1. \tag{B.8}$$

Therefore, from (B.6),

$$\check{X}_{i'}(0) = X_{i'}(0) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_1 + M_0 - 1.$$

Analogously, we can conclude that

$$\check{X}_{i''}(0) = X_{i''}(0) + \mathcal{O}(\delta), \quad i = 1, \ldots, M_1 + M_0 - 1.$$

Proof of Theorem 6.1 *Proof of statement* (a) Using Lemma B.3 and Eq. (6.1), the production rate can be calculated as

$$\begin{aligned}\widetilde{PR} &= \mu_{M_1+M_0}[1 - \tilde{X}_{(M_1+M_0-1)'}(0)] \\ &= \mu_{M_1+M_0}[1 - \check{X}_{(M_1+M_0-1)'}(0)] + \mathcal{O}(\delta) \\ \Big(\text{or } \widetilde{PR} &= \nu_{M_1+M_0}[1 - \tilde{X}_{(M_2+M_0-1)''}(0)] \\ &= \nu_{M_2+M_0}[1 - \check{X}_{(M_2+M_0-1)''}(0)] + \mathcal{O}(\delta)\Big).\end{aligned}$$

From Lemma B.5, this may be expressed as

$$\begin{aligned}\widetilde{PR} &= \mu_{M_1+M_0}[1 - X_{(M_1+M_0-1)'}] + \mathcal{O}(\delta) \\ \big(\text{or } \widetilde{PR} &= \nu_{M_2+M_0}[1 - X_{(M_2+M_0-1)''}] + \mathcal{O}(\delta)\big).\end{aligned}$$

Using (B.6), we obtain

$$\begin{aligned}\widetilde{PR} &= \mu_{M_1+M_0}[1 - Q(\mu^f_{M_1+M_0-1}, \mu_{M_1+M_0}, \Gamma_{M_1+M_0-1})] + \mathcal{O}(\delta) \\ \big(\text{or } \widetilde{PR} &= \nu_{M_2+M_0}[1 - Q(\nu^f_{M_2+M_0-1}, \nu_{M_2+M_0}, \Lambda_{M_2+M_0-1})] + \mathcal{O}(\delta)\big).\end{aligned}$$

Therefore, using Eqs. (5.21) and (5.19), we conclude that

$$\widetilde{PR} = PR + \mathcal{O}(\delta).$$

Proof of statement (b) According to the definition of the average steady state in-process inventory, $E[\tilde{h}_{1j}]$ can be

$$E[\tilde{h}_{1j}] = \sum_{k=0}^{N_{1j}} k \cdot \check{X}_{1j}(k), \quad j = 1, \ldots, M_1. \tag{B.9}$$

From Lemma B.3, we have

$$E[\tilde{h}_{1j}] = \sum_{k=0}^{N_{1j}} k \cdot \check{X}_{1j}(k) + \mathcal{O}(\delta), \quad j = 1, \ldots, M_1. \tag{B.10}$$

By Lemma A.5 of [1],

$$\check{X}_{1j}(k) = \check{X}_{1j}(0) \left(\frac{1}{1 - \tilde{\mu}^b_{j+1}} \right) \alpha^k, \quad k = 1, \ldots, N_{1j}, \tag{B.11}$$

where $\alpha = \tilde{\mu}^f_j (1 - \tilde{\mu}^b_{j+1}) / \tilde{\mu}^b_{j+1} (1 - \tilde{\mu}^f_i)$. Then,

$$E[\tilde{h}_{1j}] = \sum_{k=0}^{N_{1j}} k \cdot \check{X}_{1j}(0) \left(\frac{1}{1 - \tilde{\mu}^b_{j+1}} \right) \alpha^k + \mathcal{O}(\delta), \quad j = 1, \ldots, M_1. \tag{B.12}$$

Using Lemmas B.4 and B.5 and Eqs. (5.3)–(5.6), we obtain

$$E[\tilde{h}_{1j}] = \sum_{k=0}^{N_{1j}} k \cdot X_{1j}(0) \left(\frac{1}{1 - \mu^b_{j+1}} \right) \left(\frac{\mu^f_j (1 - \mu^b_{j+1})}{\mu^b_{j+1} (1 - \mu^f_j)} \right)^k + \mathcal{O}(\delta),$$
$$j = 1, \ldots, M_1. \tag{B.13}$$

From Eq. (B.6),

$$E[\tilde{h}_{1j}] = \sum_{k=0}^{N_{1j}} k \cdot Q(\mu^f_j, \mu^b_{j+1}, N_{1j}) \left(\frac{1}{1 - \mu^b_{j+1}} \right) \left(\frac{\mu^f_j (1 - \mu^b_{j+1})}{\mu^b_{j+1} (1 - \mu^f_j)} \right)^k + \mathcal{O}(\delta),$$
$$j = 1, \ldots, M_1. \tag{B.14}$$

Therefore, we conclude

$$E[\tilde{h}_{1j}] = E[h_{1j}] + \mathcal{O}(\delta), \quad j = 1, \ldots, M_1. \tag{B.15}$$

Analogously,

$$E[\tilde{h}_{2j}] = E[h_{2j}] + \mathcal{O}(\delta), \quad j = 1, \ldots, M_2,$$
$$E[\tilde{h}_{0j}] = E[h_{0j}] + \mathcal{O}(\delta), \quad j = 1, \ldots, M_0 - 1.$$

Proof of statement (c) From the definitions of manufacturing starvations in (5.23) and Eqs. (5.3)–(5.6), $\widetilde{ms}_{1j}$ can be written as

$$\widetilde{ms}_{1j} = \mu_j \tilde{X}_{(j-1)'}(0), \quad j = 2, \ldots, M_1. \tag{B.16}$$

By Lemmas B.2 and B.4, this can be approximated as

$$\begin{aligned} \widetilde{ms}_{1j} &= \mu_j - \tilde{\mu}_j^f + \mathcal{O}(\delta) \\ &= \mu_j - \mu_j^f + \mathcal{O}(\delta), \quad j = 2, \ldots, M_1. \end{aligned} \tag{B.17}$$

Using Eq. (A.2), we have

$$\begin{aligned} \widetilde{ms}_{1j} &= \mu_j - \mu_j[1 - Q(\mu_{j-1}^f, \mu_j^b, \Gamma_{j-1})] + \mathcal{O}(\delta) \\ &= \mu_j Q(\mu_{j-1}^f, \mu_j^b, \Gamma_{j-1}) + \mathcal{O}(\delta), \quad j = 2, \ldots, M_1. \end{aligned} \tag{B.18}$$

Therefore, from (5.24),

$$\widetilde{ms}_{1j} = ms_{1j} + \mathcal{O}(\delta), \quad j = 2, \ldots, M_1.$$

Similar arguments can be used to prove that

$$\begin{aligned} \widetilde{ms}_{2j} &= ms_{2j} + \mathcal{O}(\delta), \quad j = 2, \ldots, M_2, \\ \widetilde{ms}_{0j} &= ms_{0j} + \mathcal{O}(\delta), \quad j = 2, \ldots, M_0, \\ \widetilde{mb}_{ij} &= mb_{ij} + \mathcal{O}(\delta), \quad \forall ij. \end{aligned}$$

Proof of Statement (d) For the probabilities of manufacturing starvations of the assembly machine, $\widetilde{ms}_{01_1}$ and $\widetilde{ms}_{01_2}$, from the definition in (5.23), they can be expressed as

$$\begin{aligned} \widetilde{ms}_{01_1} &= p_{01} \tilde{X}_{1M_1}(0), \\ \widetilde{ms}_{01_2} &= p_{01} \tilde{X}_{2M_2}(0). \end{aligned}$$

By Lemmas B.3 and B.5, using (5.3)–(5.6), we have

$$\begin{aligned} \widetilde{ms}_{01_1} &= p_{01} \check{X}_{1M_1}(0) + \mathcal{O}(\delta) = p_{01} X_{1M_1}(0) + \mathcal{O}(\delta), \\ \widetilde{ms}_{01_2} &= p_{01} \check{X}_{2M_2}(0) + \mathcal{O}(\delta) = p_{01} X_{2M_2}(0) + \mathcal{O}(\delta). \end{aligned}$$

Therefore, using Eqs. (B.6) and Eqs. (5.3)–(5.6) yields

$$\widetilde{ms}_{01_1} = p_{01} Q(\mu^f_{M_1}, \mu^b_{M_1+1}, \Gamma_{M_1}) + \mathcal{O}(\delta),$$
$$\widetilde{ms}_{01_2} = p_{01} Q(\nu^f_{M_2}, \nu^b_{M_2+1}, \Lambda_{M_2}) + \mathcal{O}(\delta).$$

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