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A NEW TECHNIQUE IN CONSTRUCTING CLOSED-FORM SOLUTIONS FOR NONLINEAR PDEs APPEARING IN FLUID MECHANICS AND GAS DYNAMICS

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We develop a new unique technique in constructing closed-form solutions for several nonlinear partial differential systems appearing in fluid mechanics and gas dynamics. The obtained solutions include fewer arbitrary functions than needed for general solutions, fact that permits us to specify them according to the initial state, or the geometry, of each specific problem under consideration. In order to apply the before mentioned technique we construct closed-form solutions concerning the gas-dynamic equations with constant pressure, the dynamic equations of an ideal gas in isentropic flow, and the two-dimensional incompressible boundary layer flow.

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1. INTRODUCTION

The unsteady, one-dimensional isentropic flow of a perfect gas with variable pressure is one of the most discussed items of aerodynamics. The major difficulty in obtaining analytical solutions of the governing differential system is the nonlinearity ([2], [4]). Long ago the unsteady one-dimensional transonic flow of idealized compressible fluids has been studied by several ad hoc assumptions or other analytical methods and techniques. We mention here the classical works by Tamada [20] and Tomotika and Tamada [21], where exact solutions of the above problem are obtained by making use of some ad hoc assumptions or by means of separation technique, or other assumed forms of solution (see Ames [2]). Also, Pai ([15], [17]) dealt with the derivation of the equations of an ideal, one-dimensional, viscous, heat and electrically conducting gas. He constructed exact solutions concerning velocity and density for an isentropic flow making use of the ad hoc assumption that density is a function of the velocity alone.

On the other hand, Noh and Protter [14] have found that the technique of the utility of Lagrangian coordinates is a convenient vehicle for obtaining soft solutions ([2], [7]) of the equations of gas dynamics. Two cases of nonlinear partial differential equations (PDEs) concerning the gas-dynamic equations with or without constant pressure in Euler form were examined by Ames [2]. Another important and applicable technique in solving such types of nonlinear PDEs is that of similarity variables. Birkhoff [3] was probably the first to apply a general method of a one parameter group of transformation in order to develop similarity solutions in some area of fluid mechanics. This procedure was based upon a general theory due to Morgan [12], later elaborated by Schuh [18] and Manohar [10] for two-dimensional, unsteady, laminar, incompressible boundary layer flows. The method was also applied to the three-dimensional boundary layer equations as discussed by Geis [6] in the axisymmetric case, and Hansen [7] and Morgan [13] in the general case.

In this paper a successful attempt is made to develop a new unique technique in constructing closed form solutions for several kinds of nonlinear PDEs appearing in fluid mechanics and gas dynamics. The examined herein mathematical models of physical problems are classified into the mathematical category of well posed problems. The obtained solutions include fewer arbitrary functions than needed for general solutions, fact that permits us to specify them according to the initial state, or the geometry of each specific problem under consideration. The developed methodology is unique and can be applied to nonlinear PDEs exact solutions of which until now have been obtained by several different methods, as ad hoc methods, similarity concepts, Lagrange formulation and combined techniques. Thus, closed-form solutions of the gas-dynamic equations with constant pressure, the dynamic equations of an ideal gas in isentropic flow, and, the one-dimensional incompressible boundary-layer flow are extracted. Since nonlinear PDEs similar to the investigated herein arise in other domains of mechanics, the developed solution technique may be proved powerful for the research of other problems.

2. MATHEMATICAL FORMULATION

In this section we present several nonlinear partial differential systems of high interest in engineering practice, which appear in fluid mechanics and gas dynamics. A new technique in constructing closed-form solutions for these kinds of systems will be developed in the next sections. Note that the examined herein mathematical models of physical problems are classified into the mathematical category of well posed problems; namely, into the category of problems in which: a) The governing PDEs and sufficient auxiliary conditions are known to provide a unique solution, and b) The solutions continuously depend upon the auxiliary conditions [2].

The two-dimensional gas-dynamic equations with constant pressure in Euler form are [14]

$$
\begin{aligned}
&u_{,t} + uu_{,x} + vu_{,y} = 0;\\
&v_{,t} + uv_{,x} + vv_{,y} = 0;\\
&\rho_{,t} + (\rho u)_{,x} + (\rho v)_{,y} = 0; \qquad\qquad (1\text{a,b,c,d,e})\\
&\epsilon_{,t} + (\epsilon u)_{,x} + (\epsilon v)_{,y} = 0;\\
&\epsilon = e + p,
\end{aligned}
$$

with initial conditions

$$u(x, y, 0) = f(x, y); \; v(x, y, 0) = g(x, y); \tag{2}$$

$$\rho(x, y, 0) = h(x, y); \; \epsilon(x, y, 0) = k(x, y).$$

In the usual notation u; v are velocity components, ρ is density, ε denotes internal energy per unit volume, and p represents pressure. We note that all the above quantities are smooth and sufficiently differentiable functions of the distance-coordinates x, y and the time-variable t; also, sub index after comma denotes partial derivative with respect to this index.

Simple extention of equations (1) in the three-dimensional problem furnishes the system:

$$u_{,t} + uu_{,x} + vu_{,y} + wu_{,z} = 0;$$

$$v_{,t} + uv_{,x} + vv_{,y} + wv_{,z} = 0;$$

$$w_{,t} + uw_{,x} + vw_{,y} + ww_{,z} = 0; \tag{3a,b,c,d,e,f}$$

$$\rho_{,t} + (\rho u)_{,x} + (\rho v)_{,y} + (\rho w)_{,z} = 0;$$

$$\epsilon_{,t} + (\epsilon\, u)_{,x} + (\epsilon\, v)_{,y} + (\epsilon\, w)_{,z} = 0;$$

$$\epsilon = e + p,$$

with the initial conditions

$$u(x, y, z, 0) = f(x, y, z); \; v(x, y, z, 0) = g(x, y, z);$$

$$w(x, y, z, 0) = h(x, y, z); \; \rho(x, y, z, 0) = k(x, y, z); \tag{4}$$

$$\epsilon(x, y, z, 0) = m(x, y, z).$$

Both previous systems (1) and (3) derive from the more general form

$$u_{,t} + \varphi(u)u_{,x} + \psi(v)u_{,y} + \omega(w)u_{,z} = 0;$$

$$v_{,t} + \varphi(u)v_{,x} + \psi(v)v_{,y} + \omega(w)w_{,z} = 0; \tag{5a,b,c}$$

$$w_{,t} + \varphi(u)w_{,x} + \psi(v)w_{,y} + \omega(w)w_{,z} = 0$$

with the following initial conditions:

$$u(x, y, z, 0) = f(x, y, z);\ v(x, y, z, 0) = g(x, y, z); \tag{6}$$

$$w(x, y, z, 0) = h(x, y, z).$$

In equations (3) to (6) w denotes the third velocity component, while in (5) and (6) φ; ψ and ω are smooth arbitrary functions of the corresponding velocities.

A one-dimensional ideal gas which is inviscid, nonheat-conducting and has infinite electrical conductivity obeys in the following nonlinear PDEs ([16], [17]):

$$\rho_{,t} + (\rho u)_{,x} = 0;$$

$$\rho u_{,t} + \rho u u_{,x} + p_{,x} + \mu_e H H_{,x} = 0;$$

$$H_{,t} + (uH)_{,x} = 0; \tag{7a,b,c,d}$$

$$\rho h_{,t} + \rho u h_{,x} - p_{,t} + \mu_e u H H_{,x} = 0,$$

in which ρ, u, p, H and h are density, velocity, pressure, magnetic field (planar and perpendicular to u), and stagnation enthalpy respectively. The h-function is given by the equation

$$h = c_p T + \frac{1}{2} u^2, \tag{8}$$

where T is the temperature, while c_p denotes the specific heat at constant pressure. Finally, μ_e is the magnetic permeability. Furthermore, in case of isentropic flow the equation

$$p = \lambda \rho^{\gamma};\ \lambda = const. \tag{9}$$

must be added to relations (7). Here γ is the constant ratio of specific heats (usually taken equal to 1.40). In this case, the initial conditions are

$$\rho(x, 0) = g(x);\ u(x, 0) = f(x);\ H(x, 0) = k(x). \tag{10}$$

In the special case of the isentropic flow of a gas, when a sound wave of finite amplitude is being propagated through the governing equations of momentum, centinuity and gas laws are, [4]:

$$\rho u_{,t} + \rho u u_{,x} + p_{,x} = 0;$$

$$\rho_t + \rho u_{,x} + u \rho_{,x} = 0; \tag{11a,b,c,d}$$

$$p = \lambda \rho^{\gamma};\ c^2 = dp/d\rho$$

with initial conditions

$$u(x, 0) = f(x); \rho(x, 0) = g(x). \tag{12}$$

Here c is the velocity of sound in the gas, and γ the constant ratio of specific heats.

The last system which will be investigated is the one describing unsteady, incompressible, two-dimensional boundary layer flow. The consisting equations are the continuity and the momentum equations expressed in the form

$$u_{,x} + v_{,y} = 0; \tag{13a,b}$$

$$u_{,t} + uu_{,x} + vu_{,y} = -\frac{1}{\rho}p_{,x} + nu_{,yy}$$

with boundary conditions

$$u = v = 0 \; for \; y = 0; \tag{14a,b}$$

$$u = u_e(x, t) \; for \; very \; large \; values \; of \; y,$$

where u_e denotes the outer velocity, namely the velocity outside the boundary layer, and n is a positive constant. Note that pressure can be prescribed deriving from the potential flow solution [8], or, better, it can be discovered depending on the outer velocity u_e ([18], [2]).

Analytical solutions for systems of the form (13) were obtained by the well-known method of similarity via-one parameter groups [3], [16], [7], [12] and [13]. Also, the special case in which equations (13) an independent from time, (case of steady boundary layer flow), was successfully investigated in [16] and [11].

3. THE PROPOSED TECHNIQUE-CONSTRUCTION OF SOFT SOLUTIONS

Noh and Protter [14] have found the Lagrange formulation to be a convenient vehicle for obtaining "soft" solutions of the systems of equations in gas dynamics, namely of equations (1), (3) and (5). A function $u(x, t)$ is said to be a soft solution of the nonlinear PDE $u_{,t} + uu_{,x} = 0$, subjected to the initial conditions $u(x, 0) = f(x)$, if u satisfies the functional relation $u = f[x - tu(x, t)]$. Sufficient differentiability of f will insure that a soft solution is also a strict solution. The term "soft" was used by Ames [2] to distinguish these from the "weak" solutions of Lax [8]. Weak solutions of nonlinear equations automatically satisfy certain jump conditions across discontinuities, such as occur in choks. Soft solutions have no jump conditions. On the other hand, there are two well-known methods for developing similarity variables; that is, the use of transformation groups [3]; [6]; [7]; [12]; [13], and the separation of variables approach [1]; [3]; [8]. Both these techniques are convenient and applicable to problems concerning boundary layer flows [2], but they lead to solutions of special forms according to each specific problem under consideration.

Contrary to the above methodologies, we shall try to develop a unique technique in solving analytically nonlinear PDEs of the form (1), (3), (5), (7), (11) and (13). The obtained solutions are soft including arbitrary functions, determinable through the initial state, or the geometry of each specific problem under consideration.

Let us consider a fluid in a n-dimensional space. Then, each particle of the fluid has a velocity vector **U** given by

$$\mathbf{U} = [u_1(x_1, x_2,\ldots, x_n; t),\ldots, u_n(x_1, x_2,\ldots, x_n; t)],$$

where x_i ($i = 1,\ldots, n$) are the space-coordinates and t is the time-variable. If we assume constant pressure, the Navier-Stokes momentum nonlinear PDEs, including velocities u_i, are in Euler form the following

$$u_{i,t} + u_j u_{i,x_j} = 0; \quad i = 1, 2,\ldots, n; \quad j = sum.index\, 1, 2,\ldots, n, \tag{15}$$

in which sub index after comma indicates partial derivative with respect to this index.

The total differential of each velocity u_i is given through the relation

$$du_i = \frac{\partial u_i}{\partial t}\, dt + \frac{\partial u_i}{\partial x_j}\, dx_j,$$

and the total derivative (du_i/dt) by the relation

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + \frac{\partial u_i}{\partial x_j}\frac{dx_j}{dt}. \tag{16}$$

By definition we have

$$\frac{dx_j}{dt} = u_j,$$

and, consequently, equation (16) can be rewritten as

$$\frac{du_i}{dt} = u_{i,t} + u_j u_{i,x_j}. \tag{17}$$

In general, for any function $F(x_1,\ldots, x_n; t)$ one can write

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{\partial F}{\partial x_j}\frac{dx_j}{dt} = F_{,t} + u_j F_{,x_j}. \tag{18}$$

In the present case, combination of equations (15) and (17) results in

$$\frac{du_i}{dt} = 0; \quad u_{i,t} + u_j u_{i,x_j} = 0. \tag{19a,b}$$

Equations (19a) furnish

$$u_i = A_i; \; i = 1, \cdot \cdot \cdot, n, \tag{20}$$

where A_i are constants of integration. Since A_i are arbitrary, combination of Eqs (20) among them leads to the functional relations

$$u_1 = f_1(u_2); \; u_2 = f_2(u_3), \ldots, u_{n-1} = f_{n-1}(u_n) \tag{21}$$

in which f_i are arbitrary smooth and sufficient differentiable functions. Introducing (21) into equations (19b) and rewriting them explicitly we finally derive

$$\left(\prod_{k=1}^{n-1} f_k^{(u_{k+1})}\right) u_{n,t} + u_1 \left(\prod_{k=1}^{n-1} f_k^{(u_{k+1})}\right) u_{n,x_1} + u_2 \left(\prod_{k=1}^{n-1} f_k^{(u_{k+1})}\right) u_{n,x_2} + \ldots + u_n \left(\prod_{k=1}^{n-1} f_k^{(u_{k+1})}\right) u_{n,x_n} = 0; \tag{22}$$

$$\left(\prod_{k=2}^{n-1} f_k^{(u_{k+1})}\right) u_{n,t} + u_1 \left(\prod_{k=2}^{n-1} f_k^{(u_{k+1})}\right) u_{n,x_1} + u_2 \left(\prod_{k=2}^{n-1} f_k^{(u_{k+1})}\right) u_{n,x_2} + \ldots + u_n \left(\prod_{k=2}^{n-1} f_k^{(u_{k+1})}\right) u_{n,x_n} = 0;$$

$$u_{n,t} + u_1 u_{n,x_1} + u_2 u_{n,x_2} + \ldots + u_n u_{n,x_n} = 0,$$

where the upper-right index in partenthesis denotes derivative of the function f_k with respect to its argument u_{k+1}, while the symbol $\prod$ expresses finite product. For $f_k^{(u_{k+1})} \neq 0$ $(k = 1, \ldots, n-1)$ Eqns (22) furnish the unique equation

$$u_{n,t} + u_1 u_{n,x_1} + u_2 u_{n,x_2} + \ldots + u_n u_{n,x_n} = 0,$$

which, by way of the expressions (21), can be rewritten as

$$u_{n,t} + F_1 u_{n,x_1} + F_2 u_{n,x_2} + \ldots + F_{n-1} u_{n,x_{n-1}} + u_n u_{n,x_n} = 0 \tag{23}$$

where

$$F_1 = f_1(f_2(\ldots f_{n-1}(u_n))) = F_1(u_n); \; F_2 = f_2(f_3(\ldots f_{n-1}(u_n))) = F_2(u_n); \ldots; \; F_{n-1} = f_{n-1}(u_n) = F_{n-1}(u_n) \tag{24}$$

Eqn (23) is of quasi-linear form. Using the corresponding Lagrange subsidiary equations

$$\frac{dt}{1} = \frac{dx_1}{F_1} = \frac{dx_2}{F_2} = \cdots = \frac{dx_{n-1}}{F_{n-1}} = \frac{dx_n}{u_n} = \frac{du_n}{0}$$

and integrating, we derive the solution

$$u_n = H_n(x_1 - u_1 t, x_2 - u_2 t, \ldots, x_n - u_n t), \tag{25}$$

where H_n is an arbitrary function. This result permits us to obtain the soft solutions of the system (15), due to the initial conditions

$$u_i(x_1, \ldots, x_n; 0) = q_i(x_1, \ldots, x_n) \tag{26}$$

in the form

$$u_i = q_i(x_1 - u_1 t, x_2 - u_2 t, \ldots, x_n - u_n t). \tag{27}$$

The same solution technique is applicable to the more general systems of the form

$$u_{i,t} + \varphi_j(u_j) u_{i,x_j} = 0; i = 1, \ldots, n; j = sum.index\ 1, \ldots, n \tag{28}$$

which have the soft solutions

$$u_i = q_i[x_1 - \varphi_1(u_1)t, x_2 - \varphi_2(u_2)t, \ldots, x_n - \varphi_n(u_n)t] \tag{29}$$

satisfying the initial conditions

$$u_i(x_1, x_2, \ldots, x_n; 0) = q_i(x_1, x_2, \ldots, x_n). \tag{30}$$

4. APPLICATIONS

The already developed in section 3 technique will be applied for solving analytically the nonlinear partial differential systems presented in section 2.

4.1 Solution of the gas-dynamic equations with constant pressure

The two-dimensional gas-dynamic equations with constant pressure in Euler form are given by the expressions (1) with the initial conditions (2). Using the already defined solutions (27) by setting $u_1 = u$; $u_2 = v$; $u_3 = \ldots = u_n = 0$ in combination with the two first of the initial conditions (2), we deduce the following soft solutions for u and v

$$u = f(x - ut, y - vt);\ v = g(x - ut, y - vt). \tag{31a,b}$$

Furthermore setting

$$x - ut = \omega;\ y - vt = \sigma, \tag{32}$$

we find, by means of (31a,b) the partial derivatives $u_{,x}$ and $v_{,y}$ as follows

$$u_{,x} = \frac{f_{,\omega} + t(f_{,\omega}g_{,\sigma} - f_{,\sigma}g_{,\omega})}{(1 + tf_{,\omega})(1 + tg_{,\sigma}) - t^2 f_{,\sigma}g_{,\omega}};$$

$$v_{,y} = \frac{g_{,\sigma} + t(f_{,\omega}g_{,\sigma} - f_{,\sigma}g_{,\omega})}{(1 + tf_{,\omega})(1 + tg_{,\sigma}) - t^2 f_{,\sigma}g_{,\omega}}.$$

One observes that

$$u_{,x} + v_{,y} = \frac{f_{,\omega} + g_{,\sigma} + 2t(f_{,\omega}g_{,\sigma} - f_{,\sigma}g_{,\omega})}{J}, \tag{33}$$

where

$$J = (1 + tf_{,\omega})(1 + tg_{,\sigma}) - t^2 f_{,\sigma}g_{,\omega},$$

as well as that

$$\frac{dJ}{dt} = f_{,\omega} + g_{,\sigma} + 2t(f_{,\omega}g_{,\sigma} - f_{,\sigma}g_{,\omega}).$$

Therefore, equation (33) can be rewritten as

$$u_{,x} + v_{,y} = \frac{1}{J}\frac{dJ}{dt}. \tag{34}$$

By now, we are able to continue the solution of the system under consideration. In fact, observing that the third of Eqns (1) becomes,

$$\rho_{,t} + u\rho_{,x} + v\rho_{,y} = -\rho(u_{,x} + v_{,y}),$$

and using the relations (18) and (34), we deduce

$$\frac{d\rho}{dt} = -\rho\frac{1}{J}\frac{dJ}{dt}. \tag{35}$$

The integration of this equation furnishes

$$\rho J = \lambda = const. \tag{36}$$

Combining (36) with equations (20) for $u_I = u$, we obtain

$$\rho J = z(u), \tag{37}$$

where z is an arbitrary function. Because of (31a), (37) and the third of the initial conditions (2), we derive the following soft solution for the density

$$\rho = \frac{1}{J} h(x - ut, y - vt). \tag{38}$$

Finally, the Euler equation (1d) becomes

$$\epsilon_{,t} + u\epsilon_{,x} + v\epsilon_{,y} = -\epsilon(u_{,x} + v_{,y}),$$

fact that permits us to obtain a soft solution for the internal energy e as follows

$$e = \frac{1}{J} k(x - ut, y - vt) - p. \tag{39}$$

Through an exactly similar procedure one gets soft solutions for the three dimensional problem (Eqns (3)), as well as for the more complicated system (5).

4.2 Soft solutions for the dynamic equations of an ideal gas in isentropic flow

We begin our analysis with the more complicated nonlinear system of equations (7), (8) and (9) with the initial conditions (11). Since equations (7a) and (7c) are identical, we may write

$$H = A\rho; A = const. \tag{40}$$

On the other hand, Eqns (7a) and (7b) based on relations (18) and (40), become

$$\frac{d\rho}{dt} + \rho u_{,x} = 0; \qquad \frac{du}{dt} + \frac{1}{\rho} p_{,x} + \mu_e A^2 \rho_{,x} = 0. \tag{41a,b}$$

For isentropic flow introducing (9) into (41b) we derive

$$\frac{du}{dt} + \lambda\gamma\rho^{\gamma-2}\,\rho_{,x} + \mu_e A^2\,\rho_{,x} = 0.$$

The division by parts of this equation with equation (41a) leads to

$$\frac{d\rho}{du} = \pm\frac{1}{[\lambda\gamma\rho^{\gamma-2} + \mu_e A^2]^{1/2}}; \quad \lambda > 0,\, \mu_e > 0,\, \rho > 0$$

the integration of which furnishes

$$B \pm u = \int \sqrt{\lambda\gamma\rho^{\gamma-2} + \mu_e A^2}\, d\rho \tag{42}$$

where B is an integration constant.

This functional relation including velocity and density permits us to write

$$u = \omega(\rho)\ or\ \rho = \psi(u)\ (\psi \equiv \omega^{-1}), \tag{43}$$

where ω and ψ are known smooth functions. Therefore, equations (7a,b) in combination with (9) take the form

$$\rho_{,t} + \omega(\rho)\rho_{,x} + \omega'(\rho)\rho\rho_{,x} = 0; \tag{44a,b}$$

$$u_{,t} + uu_{,x} + \lambda\gamma\psi^{\gamma-2}(u)\overset{*}{\psi}(u)u_{,x} + \mu_e A\overset{*}{\psi}(u)u_{,x} = 0,$$

which, based on the already developed in section 3 solution technique, as well as on the initial conditions (10), have the soft solutions

$$\rho = g[x - (\omega + \omega'\rho)t]; \tag{45a,b}$$

$$u = f[x - (u + \lambda\gamma\psi^{\gamma-2}\,\overset{*}{\psi} + \mu_e A^2\,\overset{*}{\psi})t].$$

Here prime means differentiation with respect to ρ and asterisk with respect to u. Now, as far as the magnetic field H is concerned, combination of the first of equations (43) together with (40) and (7c) results in

$$H_{,t} + \omega\left(\frac{H}{A}\right)H_{,x} + H\dot{\omega}\left(\frac{H}{A}\right)H_{,x} = 0 \tag{46}$$

which, in accordance with the third of the initial conditions (10), has the soft solution

$$H = h\left[x - \left(\omega + \dot{\omega}\frac{H}{A}\right)t\right] \tag{47}$$

(dot means differentiation with respect to H/A).

The stagnation enthalpy h can be derived through the combination of equations (7d), (7b) and (9). Thus, solving (7b) for $\mu_e HH_{,x}$ and introducing the result into (7d) one yields

$$h_{,t} + uh_{,x} - \frac{1}{\rho}p_{,t} - (uu_{,t} + u^2u_{,x} + \frac{u}{\rho}p_{,x}) = 0.$$

Based on relation (18) the last equation becomes

$$\frac{dh}{dt} - u\frac{du}{dt} - \frac{1}{\rho}\frac{dp}{dt} = 0,$$

or, by means of (9),

$$\frac{dh}{dt} - \frac{1}{2}\frac{du^2}{dt} - \frac{\lambda\gamma}{\gamma - 1}\frac{d(\rho^{\gamma-1})}{dt} = 0.$$

Integration of this equation furnishes

$$h = \frac{1}{2}u^2 + \frac{\lambda\gamma}{\gamma - 1}\rho^{\gamma-1} + C,$$

where C is an integration constant. Finally, the temperature T can be directly evaluated through relation (8).

The simpler, but of great interest in engineering practice, system (11) refers to the one-dimensional isentropic flow of a perfect gas. One eliminates the pressure by using the gas laws (11c,d), which yield

$$p_{,x} = \gamma\lambda\rho^{\gamma-1}\,\rho_{,x};\; c^2 = dp/d\rho$$

namely

$$c^2 = \gamma\lambda\rho^{\gamma-1}.$$

Through this result one rewrites the above system in the form

$$u_{,t} + uu_{,x} + \rho^{\gamma-2}\,\rho_{,x} = 0; \tag{48a,b}$$

$$\rho_{,t} + \rho u_{,x} + u\rho_{,x} = 0.$$

According to (18) the simplified system (48a,b) furnishes

$$\frac{du}{dt} = -\rho^{\gamma-2}\,\rho_{,x};\ \frac{d\rho}{dt} = -\rho u_{,x} \tag{49a,b}$$

Dividing by parts we obtain

$$\frac{du}{d\rho} = \pm\rho^{\frac{\gamma-3}{2}}$$

and integrating, we deduce

$$u = A \pm \frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}};\ A = const. \tag{50}$$

This functional relation succeeds in giving, by way of (48a,b), the following nonlinear PDEs

$$\rho_{,t} + \left(A \pm \frac{\gamma+1}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\right)\rho_{,x} = 0;$$

$$u_{,t} + \left[u \pm \left(\frac{\gamma-1}{2}\right)^{\frac{2}{\gamma-1}}(u-A)^{\frac{2}{\gamma-1}}\right]u_{,x} = 0,$$

which, according to the initial conditions (12), have the soft solutions

$$\rho = g\left[x - \left(A \pm \frac{\gamma+1}{\gamma-1}\rho^{\frac{\gamma-1}{2}}\right)t\right]; \tag{51a,b}$$

$$u = f\left\{x - \left[u \pm \left(\frac{\gamma-1}{2}(u-A)\right)^{\frac{2}{\gamma-1}}\right]t\right\}.$$

4.3 Solutions for the Two-Dimensional Boundary Layer Flow

This problem in incompressible flow is described by the system (13a,b) with the boundary conditions (14a,b). In order to apply the solution technique of section 3, we consider the corresponding to (13a,b) homogeneous system

$$u_{,x} + v_{,y} = 0; \tag{52a,b}$$

$$u_{,t} + uu_{,x} + vu_{,y} = 0.$$

As it has been already proved, equation (52b) has the solutions

$$u = F(x - tu, y - tv); \quad v = G(x - tu, y - tv) \tag{53a,b}$$

where F and G are arbitrary functions. Setting

$$n = x - tu; \; \xi = y - tv, \tag{54a,b}$$

Eqn (52b) is identically satisfied, while Eqn (52a) furnishes

$$F_{,n} + G_{,\xi} + 2t(F_{,n}G_{,\xi} - F_{,\xi}G_{,n}) = 0. \tag{55}$$

Since Eqn (55) must hold true for any value of the time-variable $t \geq 0$, we conclude that

$$F_{,n} = -G_{,\xi}; \; F_{,n}G_{,\xi} - F_{,\xi}G_{,n} = 0. \tag{56a,b}$$

Introducing the first of these equations into the second we obtain

$$-F_{,n}^2 = F_{,\xi}G_{,n} \Rightarrow G_{,n} = -\frac{F_{,n}^2}{F_{,\xi}}; \quad F_{,\xi} \neq 0,$$

and then

$$G_{,n\xi} = -\frac{2F_{,\xi}F_{,n}F_{,n\xi} - F_{,n}^2F_{,\xi\xi}}{F_{,\xi}^2}.$$

Combining the last expression together with Eqn (56a) we lead to the second-order nonlinear PDE of the Plateau problem for the minimal surface area, in which the linear terms are vanished ([2], [19]), namely

$$F_{,\xi}^2F_{,nn} - 2F_{,n}F_{,\xi}F_{,n\xi} + F_{,n}^2F_{,\xi\xi} = 0. \tag{57}$$

This nonlinear PDE has the well-known general solution

$$F = H[\xi + n\varphi(F)], \tag{58}$$

where H and φ are arbitrary functions. Similarly, one obtains an analogous solution for the G-function, that is

$$G = Q[\xi + n\psi(G)]; \quad Q, \psi = \textit{arbitrary}. \tag{59}$$

Recalling equation (56a) we deduce that there exists a functional relation between H; Q; φ and ψ that is

$$\frac{\varphi \overset{\circ}{H}}{1 - n\varphi' \overset{\circ}{H}} = -\frac{\overset{*}{Q}}{1 - n\dot{\psi} \overset{*}{Q}}.$$

In this relation symbols "°" and "*" denote differentiations with respect to the groupings $[\xi + n\varphi(F)]$ and $[\xi + n\psi(G)]$ respectively, while prime and dot means differentiations with respect to F and G respectively. If we select

$$G = f(F); \; \varphi = -f'(F), \tag{60a,b}$$

the solutions (58) and (59) become

$$F = H[\xi - nf'(F)]; \quad G = f(F), \tag{61a,b}$$

and, therefore, the solutions (53a, b) take the form

$$u = H\{y - xf(u) - t[f(u) - f'(u)u]\}; \quad v = f(u) \tag{62a,b}$$

where H and f are arbitrary functions.

The next step is to specify the already introduced functions H and f such that the boundary conditions (14a,b) are satisfied. As long as this is achieved, the pressure p must be determined by the equation resulting from the vanishing of the right-hand side of equation (13b), that is

$$p_{,x} = n\,\rho u_{,yy}. \tag{63}$$

Evaluating the partial derivative $u_{,y}$ by way of (62a) we get

$$u_{,y} = \frac{\overset{\circ}{H}}{1 + (x - tu)f''\overset{\circ}{H}},$$

where here "°" denotes derivative with respect to the grouping $\{y - xf(u) - t[f(u) - uf'(u)]\}$. Based on this expression one sets

$$f'' = 2a = const.; \quad \overset{\circ}{H} = \lambda = const. \tag{64a,b}$$

Thus, the solutions (62a,b) become

$$u = \lambda(y - 2axu + atu^2); \quad v = au^2. \tag{65a,b}$$

Supposing now that

$$0 << \alpha\,\lambda \le N, \quad N \to \infty, \tag{66}$$

as well as that

$$0 << y \le M, \quad M \to \infty \tag{67}$$

and solving (64a) for u, we find

$$u = \frac{(2\lambda ax + 1) \pm \sqrt{(2\lambda\alpha x + 1)^2 - 4\lambda^2\alpha ty}}{2\lambda at}. \tag{68}$$

Noting by means of (66) that both α and λ must be of the same sign and considering only the sign minus in (68), we derive the solutions

$$u = \frac{(1 + 2|\lambda\alpha|x) - \sqrt{(1 + 2|\lambda\alpha|x)^2 - 4\lambda|\lambda\alpha|ty}}{2|\lambda\alpha|t}; \quad v = au^2. \tag{69}$$

Since for t; x, $y > 0$ the inequality

$$(1 + 2|\lambda\alpha|x)^2 - 4\lambda|\lambda\alpha|ty > 0,$$

or the inequality

$$4\lambda^2\alpha^2x^2 + 4|\lambda\alpha|x + 1 - 4\lambda|\lambda\alpha| + y > 0$$

must hold true, we deduce that

$$16\lambda^2\alpha^2|\lambda\alpha|\lambda ty < 0.$$

This means that $\lambda < 0$ and $a < 0$ and, therefore, the solutions (69) take the final form

$$u = \frac{(1 + 2|\lambda\alpha|x) - \sqrt{(1 + 2|\lambda\alpha|x)^2 + 4\lambda^2|a|ty}}{2|\lambda\alpha|t}; v = -|\alpha|u^2. \tag{70a,b}$$

One observes now that the boundary condition (14a) is identically satisfied. For the second boundary condition (14b) we rewrite the velocity u given in (70a) as

$$u = \frac{\left(\frac{1}{\sqrt{|\lambda\alpha|y}} + 2\sqrt{\frac{|\lambda\alpha|}{y}}x\right) - \sqrt{\left(\frac{1}{\sqrt{|\lambda\alpha y|}} + 2\sqrt{\frac{|\lambda\alpha|}{y}}x\right)^2 + 4|\lambda|t}}{2\sqrt{\frac{|\lambda\alpha|}{y}}t},$$

which, for very large values of y, according to the inequalities (66) and (67) furnishes the outer velocity u_e as a function of x and t only; namely,

$$u \to u_e(x, t) = \frac{x}{t} - \sqrt{\left(\frac{x}{t}\right)^2 + \frac{|\lambda|}{K^2 t}}, \tag{71}$$

where $K = M/N > 0$.

Evaluating also the second derivative $u_{,yy}$ and using (63) we calculate

$$p_{,x} = \frac{2\nu\, \rho\lambda^2 |\lambda\alpha| t}{[(1 + 2|\lambda\alpha| x)^2 + 4\lambda^2 |\alpha| ty]^{3/2}}. \tag{72}$$

For very large values of y, because of the inequalities (66) and (67), the last expression can be rewritten as

$$(p_{,x})_e = \frac{2\nu\, \rho t}{\alpha^2} \frac{K^{3/2}}{[(2Kx)^2 + 4|\lambda| t]^{3/2}}, \tag{73}$$

which means that the pressure outside the boundary layer is a function of x and t only, as exactly the outer velocity u_e given by Eqn (71). The results (71) and (72) are analogous to those discovered by Schuh [18] through the method of similarity via one-parameter groups [2]. Finally, the pressure p can be estimated by the integration of equation (63), through the relation (72), giving

$$p = \frac{\nu\, \rho}{4\,|\alpha|\, y} \frac{1 + 2\,|\lambda\alpha|\, x}{\sqrt{(1 + 2\,|\lambda\alpha|\, x)^2 + 4\lambda^2\,|\alpha|\, ty}} + G(y, t) \tag{74}$$

where G is an arbitrary function.

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