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Research Article

An $\mathscr{H}_\infty$ Optimal Robust Pole Placement with Fixed Transparent Controller Structure on the Basis of Nonnegativity of Even Spectral Polynomials

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This paper presents the synthesis of an optimal robust controller with the use of pole placement technique. The presented method includes solving a polynomial equation on the basis of the chosen fixed characteristic polynomial and introduced parametric solutions with a known parametric structure of the controller. Robustness criteria in an unstructured uncertainty description with metrics of norm $\mathscr{H}_\infty$ are for a more reliable and effective formulation of objective functions for optimization presented in the form of a spectral polynomial with positivity conditions. The method enables robust low-order controller design by using plant simplification with partial-fraction decomposition, where the simplification remainder is added to the performance weight. The controller structure is assembled of well-known parts such as disturbance rejection, and reference tracking. The approach also allows the possibility of multiobjective optimization of robust criteria, application of mixed sensitivity problem, and other closed-loop limitation criteria, where the common criteria function can be composed from different unrelated criteria. Optimization and controller design are performed with iterative evolution algorithm.

1. Introduction

Synthesis of closed-loop control system is based on a mathematical model of the plant, which should enable reliable prediction of input/output response, so that it can be used in controller synthesis. Uncertainty of the plant and external disturbance are the central issues of robust closed-loop control. Modern robust control approaches are mostly presented as a set of optimization problems, where the solution and the optimization procedure depend on the mathematical property of the objective function. Many problems are naturally not convex [1], but few of them can be presented as quasiconvex or convex problems in a well-known LMI framework [2–5]. However, convex problems become difficult to solve if some constraints are imposed on the controller coefficient or on the feedback gain matrix in the state space approach. Linear convex problem with an imposed structure of the feedback loop or of the controller is called Structured Linear Control problem (SLC) [6] and includes a large class of various problems, such as fixed-order feedback, norm bounded gain controller, and multiobjective optimization problems. All these problems are classified as NP-hard, with many excellent solving procedures [7–10] and different separate optimization aspects such as optimal solution, stable controller, robust stability, robust performance, and others [11–13]. Many of these optimization conditions in previous citations are separate and unrelated to each method. Proper solutions of solving algorithms depend on an intersection of conditions, where only suboptimality is guaranteed, with no further information about the existence of the solution [6, 13]. Our approach proposes an optimal solution with combined weighted objective function, where each single weight is selected by the importance of optimization criteria. The proper solution of the problem can be proved before the optimization starts.

Robust control paradigms, such as $\mathscr{H}_\infty$, $\mathscr{H}_2$, and μ-synthesis, enable design of robust controllers with consideration of the uncertainty of nominal plants [14–16]. Plant uncertainty for LTI systems is usually formulated as unstructured or structured uncertainty, where different uncertainty models [16] and different descriptions of the uncertainty parameter space [17] require different approaches. In this paper, we will deal only with unstructured uncertainty for different models. Despite the excellence and quality of recent approaches, the developed methods also have certain limitations. The most notable limitation of the synthesis is the influence of the weighting function structures on the controller order and on hidden performances of the feedback system. Mostly, the design of robust controller is a compromise between the controller order and the precision of the plant uncertainty description [15], which requires many iterations and changes by weight selection and plant-order reduction. Also the performance of the closed-loop system is conditioned by the optimization procedure and the selected objective function, where the final closed-loop performance is fuzzy related to the designer's initial dynamic criteria. Possible solutions of the presented problem can be dealt with by choosing a fixed characteristic polynomial and introducing parametric solutions. Fixed characteristic polynomial sets the stability and dynamic performance of the controlled system. Parametric solution in a polynomial equation is closely related to *Y-K* parameterization and coprime factorization [18, 19], with a crucial difference: the parametric solution does not influence the characteristic polynomial. Closed-loop stability and performance do not depend on the optimization procedure. In other words, the design procedure of the robust controller has two steps. The first step of the method is selecting a fixed characteristic polynomial with a set of parametric solutions. The second part of the design procedure is ensuring robust stability and optimal closed-loop performance with no back impact on the first step. The chosen characteristic polynomial remains unchanged during the whole design procedure, which means that the position of closed-loop poles is fixed.

The main objective of this paper is to present a robust polynomial approach for unstructured uncertainty, where the order of weighting functions does not influence the order of the synthesised controller. The influence of parametric solutions on the value of the norm $\|\cdot\|_\infty$ will be presented. The objective function of the optimization procedure can be composed from different unrelated criteria, like robust stability, strong stabilization, reference tracking, disturbances rejection, and so forth, with prior defined region of suboptimal

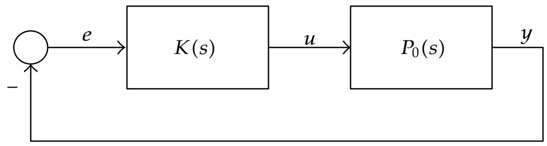

Figure 1: Negative feedback configuration.

solution on the basis of Šiljak's stability criterion [20, 21]. The condition of robustness is tested with metric $\mathscr{H}_\infty$, similarly as in papers [22–24], where conditions of robustness are formed in even spectral polynomials with a positivity condition. Such formulation of robust criteria allows for testing the robustness on the basis of polynomial coefficients and positivity conditions, with optimization algorithm (GA) and differential evolution (DE) [25].

This paper is organized as follows. The second section describes the solutions of the parametric polynomial equation in a matrix form. Based on the second section, the property of parametric solutions, introduced controller parameterizations and their influence on controller feasibility and polynomial equation solvability are proposed in Section 3. Sections 4 and 5 describe the influence of parametric solutions on the norm $\|\cdot\|_\infty$ and the assessment of robust criteria with a spectral polynomial nonnegativity test. Section 5 concludes with a formulation of cost functions for multiobjective optimization with DE. Section 6 describes synthesis examples to show the effectiveness of the proposed approach. This is followed by a conclusion.

2. Pole Placement and Parametric Solutions

Let us consider a given feedback system, with nominal plant $P_0(s)$ and controller $K(s)$ in direct branch (Figure 1).

Plant and controller configuration is shown in Figure 1: where $P_0(s)$ and $K(s)$ are as the following:

$$P_0(s) = \frac{B(s)}{A(s)}, \qquad K(s) = \frac{L(s)}{R(s)}. \tag{2.1}$$

Polynomials $A(s)$, $B(s)$, $R(s)$, and $L(s)$ can be written as the following:

$$\begin{aligned} A(s) &= a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0, \\ B(s) &= b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0, \\ R(s) &= r_j s^j + r_{j-1} s^{j-1} + \cdots + r_1 s + r_0, \\ L(s) &= l_k s^k + l_{k-1} s^{k-1} + \cdots + l_1 s + l_0. \end{aligned} \tag{2.2}$$

Complementary sensitivity and sensitivity functions are

$$\begin{aligned} T(s) &= B(s)L(s)(A(s)R(s)+B(s)L(s))^{-1} \\ &= B(s)L(s)C^{-1}(s), \\ S(s) &= A(s)R(s)(A(s)R(s)+B(s)L(s))^{-1} \\ &= A(s)R(s)C^{-1}(s), \end{aligned} \tag{2.3}$$

where $C(s)$ is a closed-loop characteristic polynomial,

$$C(s) = c_i s^i + c_{i-1}s^{i-1} + \cdots + c_1 s + c_0. \tag{2.4}$$

Controller coefficients $r_{j,j-1,\ldots 0}$ and $l_{k,k-1,\ldots 0}$ are determined by solving polynomial equation (2.5):

$$A(s)R(s) + B(s)L(s) = C(s). \tag{2.5}$$

Pole placement synthesis of the controller is based on the choice of the characteristic polynomial (2.4), [26]. The choice of $C(s)$ for high-order systems can be laborious, but with the use of different standard forms such as Manabe, Bessele, Kessler, Binomial function [27], Lipatov criterion [28], simple approximation of dominant dynamic/poles, and optimization procedure (ISE, IAE, etc.) in which a polynomial of any order can be chosen. Standard polynomial synthesis is presented as a 2DOF system structure. Such synthesis enables separate design of tracking the reference signal and elimination of external interferences. For simplification of control structure we can also assume that 2DOF configuration can be presented as 1DOF, where prefilter design is eliminated with an additional performance weight in the optimization procedure, similar to the $\mathscr{H}_\infty$ loop-shaping method [16].

The Diophantine equation (2.5) is solvable if and only if any greatest common divisor of $A(s)$ and $B(s)$ is a factor of $C(s)$ [26]. If polynomials $A(s)$ and $B(s)$ are coprime, then (2.5) can have no solutions, exactly one solution, or a family of parametric solutions. Only the last statement is used in our approach. Equation (2.5) has a family of parametric solutions if the following holds true:

$$\begin{aligned} \deg R(s) &\geq \deg A(s), \\ \deg C(s) &= \deg R(s) + \deg A(s). \end{aligned} \tag{2.6}$$

The number of parametric solutions is equal to

$$\widetilde{p} = \left\{ \begin{matrix} (j+k+2) \;-\; (i+1),\; \widetilde{j}+\widetilde{k}, \\ \widetilde{j} \;=\sum \widehat{z} \wedge \widetilde{k} \;=\sum \widehat{w} \\ \widetilde{r}_{\widetilde{j}} \in [r_j, r_0] \wedge \widetilde{l}_{\widehat{w}} \in [l_j, l_0] \end{matrix} \right\}, \tag{2.7}$$

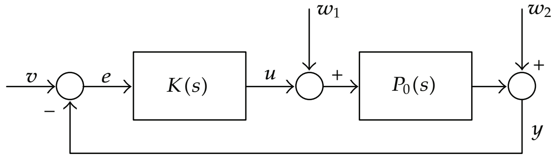

Figure 2: Closed-loop system with input, output disturbance, and reference signal.

where $\tilde{p}$ is the number of free parameters selected in controller polynomials $R(s)$ and $L(s)$. Indexes $\tilde{j}$ and $\tilde{k}$ denote the position of selected free parameters and refer to the power of the Laplace operator for each polynomial, $R(s)$ and $L(s)$.

Matrix form of the polynomial equation (2.5) with condition (2.6) is

$$C_y = R_y S_y + \tilde{P}\tilde{S}_y. \tag{2.8}$$

Solution of (2.8) is as follow:

$$R_y = \left(C_y - \tilde{P}\tilde{S}_y\right) \cdot S_y^{-1}, \tag{2.9}$$

where matrixes C_y and the Sylvester matrix S_y [23] contain a known coefficient of polynomials $C(s)$, $A(s)$, and $B(s)$. Matrix R_y contains the unknown controller parameters of polynomials $R(s)$ and $L(s)$, where $\tilde{S}_y$ is a modified S_y matrix, with the same structure and coefficients. Matrix $\tilde{P}$ contains a set of chosen free parameters $\tilde{p}$. Equation (2.9) is solvable only if matrix S_y is not singular. The arrangement of the coefficients of polynomials $A(s)$ and $B(s)$ in the matrix S_y depends on the position of the selected free parameters in the polynomials $R(s)$ and $L(s)$. The position of free parameters $\tilde{p}$ can be selected as any coefficient of the polynomial $R(s)$ or $L(s)$ under condition (2.6), with respect to the singularity of matrix S_y. Hence, it follows that the combination of free parameters in $R(s)$, $L(s)$ cannot be selected randomly. Each incorrect selection of free parameter combination in $R(s)$, $L(s)$ lowers matrix rank S_y at least by one, which means that the system of linear equation is linearly dependent and (2.9) has no solution.

As we have mentioned in the introduction, the polynomial approach is similar to the well-known *Y-K* parameterization, where Bezout's identity and Q stable transfer function must be solved and selected. Verification of closed-loop poles' position and closed-loop dynamics is thus somewhat blurred in the proposed approach. The next section will discuss transparent controller parameterization with introduced parametric solutions for ensuring good closed-loop performance in the sense of tracking and disturbance rejection.

3. Controller Parametrization with Parametric Solution

After stabilization of the closed-loop system with the selected closed-loop polynomial $C(s)$, it is also important to ensure proper behaviour of the controlled system in terms of reference tracking v, disturbance rejection w_1, w_2, robustness and time transient performances, and so forth (Figure 2). The issue of robustness will be discussed in the next section.

Good reference tracking of closed-loop system $e = 0$ can be ensured if the sensitivity function $|S(\omega)|$ is small at frequency ω_0 of the reference signal v:

$$\lim_{\omega \to \omega_0} |S(\omega)| \approx 0. \tag{3.1}$$

The system provides good tracking on the set of reference signals with the frequency span $B_\omega = [\omega_l, \omega_h]$, if $|S(B_\omega)| = 0$ [14] is true. From (2.3) we see that the tracking performance of the closed-loop system depends on the polynomial root location of $R(s)$ and $A(s)$. With this assumption, controller $K(s)$ can be parameterised with different polynomial structures, with a known effect on the sensitivity frequency response $|S(\omega)|$. The most often used structure in controller parameterization with a known effect on low frequency tracking performance is root $s = 0$ in polynomial $R(s)$. Also ramp tracking performance can be achieved with double roots at $s = 0$ in $R(s)$, and so forth. Guided systems often require tracking harmonic reference (HR) signals with frequency ω_r. Good performance can be achieved with Noch structure or with an added imaginary root $(s^2 + \omega_r^2)$ in $R(s)$, $|S(\omega_r)| = 0$.

The same effect as with tracking performance can be achieved also for input disturbance w_1, where the following holds true:

$$\frac{w_1}{y} = P_0(s)S(s). \tag{3.2}$$

And for output disturbance w_2, if

$$\frac{e}{v} = \frac{w_2}{y} = \frac{A(s)R(s)}{C(s)} = S(s). \tag{3.3}$$

From (3.2) and (3.3), we see that disturbance rejection (DR) is also related to sensitivity $S(s)$, (3.1), [16, 18], meaning that for input disturbance w_1 rejection is more effective with passive plants $P_0(s)$.

In some cases, for example, with higher operation safety or real time computation requirements, a strong stabilization system must be ensured. To be able to synthesize a strong stabilization controller the plant must fulfil the parity interlacing property (PIP) condition [16]. It often occurs that despite the PIP condition being fulfilled, synthesis leads to an unstable controller although all performance conditions are fulfilled [29]. Also many characteristics in the controller structure such as integral action, differential part (PID, PI structure), or complex zero can on one hand highly improve the closed-loop performance and on the other hand highly impair it (speed versus accuracy *D-I* part, overshot versus rise time, waterbed effect, etc.). In many cases, a compromise can be reached between performance requirements with optimal selection of controller structure and coefficients values. Let us take limited sensors accuracy for example. It is evident that a compromise can be achieved with a close approximation of the known structure. Integral action is approximated with stable zero $(s + \delta)$, $\mathbb{R} = \{\delta \mid 0 < \delta \ll 1\}$ in $R(s)$, where perfect accuracy $|S(0)| = 0$ is lowered for δ. With proper selection of δ: $|S(\delta)| < 0 \ll 1$ sufficient damping level of $|S(\delta)|$ can be guaranteed so that it does not exceed the limited sensor accuracy. A similar approach can be used for another property, double-integrator or complex zero $(s^2 + \varepsilon s + \omega_r^2)$, $\mathbb{R} = \{\varepsilon \mid 0 < \varepsilon \ll 1\}$. This approach offers a compromise and an optimal setting of parameters δ, ε, which slightly improves the overall closed-loop performance. With an added fixed structure in controller $K(s)$ a set of parametric solutions of the polynomial equation (2.5), (2.8) is provided.

Parameterized controller with an added known structure is

$$K(s) = K_a(s)K'(s),$$
$$K(s) = \frac{L'(s)}{R'(s)R_a(s)}, \tag{3.4}$$

where $K_a(s)$, $R_a(s)$ are known structures and $K'(s)$, $R'(s)$, $L'(s)$ are the solution of polynomial equation (2.8). The number of parametric solutions with parameterized controller (3.4) and condition (2.6) is

$$\widetilde{p} = \deg R - \deg A + 1,$$
$$\deg R = \widetilde{p} + \deg A - 1. \tag{3.5}$$

The degree of the characteristic polynomial $C(s)$ is

$$\deg C = \widetilde{p} + 2\deg A - 1. \tag{3.6}$$

Sensitivity function with controller (3.4) is

$$S(s) = \frac{A(s)R'(s)R_a(s)}{A(s)R'(s)R_a(s) + B(s)L'(s)}, \tag{3.7}$$

where

$$|S(\omega)| = \left|S'(\omega)\right| |R_a(\omega)|. \tag{3.8}$$

Minimizing the tracking error $e \approx 0$ or effective disturbance rejection of exogenous signals w_1, w_2 can be achieved with an added structure $R_a(s)$ if the following is true:

$$\lim_{\omega \to \omega_0} |R_a(\omega)| \approx 0. \tag{3.9}$$

Sensitivity function for step reference signals and disturbances with an added proxy structure of integral action $R_{ai}(s) = (s + \delta)$, $\mathbb{R} = \{\delta \mid 0 < \delta \ll 1\}$ is as the following:

$$\lim_{\omega \to 0} |S(\omega)| = \lim_{\omega \to 0} \left(\left|S'(\omega)\right| \sqrt{\omega^2 + \delta^2} \right),$$
$$\lim_{\omega \to 0} |S(\omega)| = \frac{a_0 r_0 \delta}{c_0}, \quad 0 < \delta \ll 1, \tag{3.10}$$
$$\lim_{\omega \to 0} |S(\omega)| \approx \delta \Longrightarrow |S(0, \delta)| \approx 0.$$

The value of parameter δ determines closed-loop tracking accuracy, disturbance rejection capability, and closed-loop dynamic with characteristic polynomial $C(s)$. Parameter δ can also be used as a limited arbitrary parametric solution in the further optimization technique, with an admissible interval $[0,\delta_{\max}] \in \{\delta \in \mathbb{R} \mid 0 < \delta \le \delta_{\max} \ll 1\}$. Considering (3.6), the degree of the characteristic polynomial with two parametric solutions $\{1,\delta\}$ is equal to $\deg C = 1 + 2\deg A$.

The same properties can be introduced for ramp reference signals and output disturbances, $R_{aii}(s) = (s^2 + 2\delta s + \delta^2)$, $\mathbb{R} = \{\delta \mid 0 < \delta \ll 1\}$, where $\deg C = 2 + 2\deg A$, as the following:

$$\begin{aligned}
&\lim_{\omega\to 0}|S(\omega)| = \lim_{\omega\to 0}\left(\left|S'(\omega)\right|\sqrt{\omega^4 + \omega^2\delta^2(2\delta^2 - 1) + \delta^4}\right),\\
&\lim_{\omega\to 0}|S(\omega)| = \frac{a_0 r_0 \delta^2}{c_0}, \quad 0 < \delta \ll 1,\\
&\lim_{\omega\to 0}|S(\omega)| \approx \delta^2 \Longrightarrow \left|S\left(0,\delta^2\right)\right| \approx 0.
\end{aligned} \tag{3.11}$$

For tracking harmonic reference signals and harmonic disturbance (HD) rejection, a structure with stable complex roots $R_{a\omega}(s) = (s^2 + \varepsilon s + \omega_r^2)\mathbb{R} = \{\varepsilon \mid 0 < \varepsilon \ll 1\}$ is proposed. ω_r is a well-dumped frequency of sensitivity function $|S(\omega)|$ with region $\varepsilon = \omega_r/\text{Q}$. Q is the "quality factor" of HR tracking, HD rejection and mean frequency gap width of $|S(\omega)|$, with symmetric centre ω_r in logarithmic scale. The quality factor is also defined with bandwidth (BW), where ω_l is a low frequency and ω_h a high frequency of the symmetric gap with a damping level −3 dB from the maximum,

$$Q = \frac{\omega_r}{\omega_l - \omega_h} = \frac{\omega_r}{BW}. \tag{3.12}$$

Selected parameter ε is equal to $\varepsilon = Bw$. Sensitivity function for harmonic property with ω_r is

$$\begin{aligned}
\lim_{\omega\to\omega_r}|S(\omega)| &= \lim_{\omega\to\omega_r}\left(\left|S'(\omega)\right|\sqrt{(-\omega + \omega_r)^2 + \omega^2\varepsilon^2}\right),\\
\lim_{\omega\to\omega_r}|S(\omega)| &= \left|S'(\omega_r)\right|\omega_r\varepsilon,\\
|S(0,\varepsilon)| &\approx 0, \quad \forall\varepsilon \ll 1 \ \wedge\ \forall\omega_r < 1.
\end{aligned} \tag{3.13}$$

Tracking and rejecting performance is slightly dependent on frequency ω_r and on the value of parameter ε in respect to polynomial $C(s)$. The tracking/rejecting performance can also be determined with a damping band between $[0,\omega_r]$, if zeros of $R_a(s)$ are closer to origin than zeros of $C(s)$. Damping band (DB) is equal to

$$\text{DB} = \left[\frac{a_0 r_0 \omega_r}{c_0}, \omega_r\right], \tag{3.14}$$

where the damping value of $|S(\mathrm{DB})|$ in dB is equal to

$$\zeta(\mathrm{dB}) = 20\log\left(\frac{a_0 r_0 \omega_r}{c_0}\right). \tag{3.15}$$

The same property can be used for step and low frequency signals. If inequality $a_0 r_0 \omega_r / c_0 \ll 1$ is true, then good tracking/rejection performance with damping (3.15) can be ensured for signals with frequency smaller than ω_r. The degree of the chosen characteristic polynomial $C(s)$ is equal to $\deg C = 2 + 2\deg A$.

Proper selection of coefficients δ, ω_r, ε according to optimal performance and robust criteria is the main issue in the optimization procedure. Objective functions of inequality of parameters $0 < \delta < \delta_{\max}$, $0 < \omega_r < \omega_{r\max}$, and $0 < \varepsilon < \varepsilon_{\max}$, will compose the main objective function with robust criteria, where the limiting values $\delta_{\max}$, $\omega_{r\max}$, $\varepsilon_{\max}$ determine the admissible property of the system in accordance with the tracking/rejecting closed-loop performance. Controller parameterization $K(s)$ can also be determined with fixed selection of δ, ω_r, ε, with an added auxiliary set of free parameters $\tilde{p}$ in controller structure $K'(s)$. The degree of controller $K(s)$ with fixed structure $R_{ai}(s)$ and a set of parameters $\tilde{p}$ is

$$\deg R = \tilde{p} + 1 + \deg A, \tag{3.16}$$

and with fixed structure $R_{ai}(s)$, $R_{a\omega}(s)$, it is

$$\deg R = \tilde{p} + 2 + \deg A. \tag{3.17}$$

Auxiliary parameters $\tilde{p}$ allow a wider range of optimization space, but on the other hand raise the degree of the controller $K(s)$. The possibility of simplification of the controller order for high-order plants with an emphasized low-order dominant dynamic will be discussed in the next section.

4. Robust Stability Assessment

After the first step of controller design, that is, selection of closed-loop poles and structure of the controller with a set of parametric solutions, we prepare leeway for further optimization of robustness criteria. Robustness criteria with consideration of $\mathcal{H}_\infty$ performance are proposed for models with nonstructured uncertainty. The closed-loop system with different types of non-structured uncertainty is robustly stable if the following holds true [15, 16]:

$$\|W_1 S\|_\infty < 1, \tag{4.1}$$

$$\|W_2 U\|_\infty < 1, \tag{4.2}$$

$$\|W_3 T\|_\infty < 1. \tag{4.3}$$

$S(s)$, $U(s)$, $T(s)$ are nominal sensitivity, controller output, and complementary sensitivity transfer functions, respectively. The selected stable and proper weights $W_1^{-1}(j\omega)$, $W_2^{-1}(j\omega)$, $W_3^{-1}(j\omega)$ represent the upper limit of the frequency band of closed-loop transfer functions $S(j\omega)$, $U(j\omega)$, $T(j\omega)$ [16]. The main goal of the proposed synthesis is optimized robust criteria (4.1)–(4.3) with a set of parametric solutions in controller coefficients

$$K(s) = f\left(\tilde{r}_{\hat{z}}, \tilde{l}_{\hat{w}}\right), \tag{4.4}$$

where

$$\begin{aligned}
\min_{K\left(\tilde{r}_{\hat{z}}, \tilde{l}_{\hat{w}}\right) \in R\mathcal{H}_\infty} \|W_1 S\|_\infty &= \gamma_{\min} < 1, \\
\min_{K\left(\tilde{r}_{\hat{z}}, \tilde{l}_{\hat{w}}\right) \in R\mathcal{H}_\infty} \|W_2 U\|_\infty &= \gamma_{\min} < 1, \\
\min_{K\left(\tilde{r}_{\hat{z}}, \tilde{l}_{\hat{w}}\right) \in R\mathcal{H}_\infty} \|W_3 T\|_\infty &= \gamma_{\min} < 1.
\end{aligned} \tag{4.5}$$

The $\mathcal{H}_\infty$ extended mixed sensitivity problem:

$$T_{zw} = \left\| \begin{matrix} W_1 S \\ W_2 U \\ W_3 T \end{matrix} \right\|_\infty . \tag{4.6}$$

with an optimal solution:

$$\min_{K \in R\mathcal{H}_\infty} \left\| T_{zw}\left(\tilde{r}_{\hat{z}}, \tilde{l}_{\hat{w}}\right) \right\|_\infty = \gamma_{\min}. \tag{4.7}$$

Pole placement with controller parameterization which involves free parameters $\tilde{p}$ or ω_r, ε, and metrics norm $\|\cdot\|_\infty$ can be used for loop shaping with known partial controller structure and its effect on the closed-loop performance discussed in the previous section.

4.1. Partial-Fractional Decomposition of $P_0(s)$ *and Robust Criteria*

As we have mentioned in Section 3, it is sometimes recommended in practice to use or implement a simpler controller structure. Low-order controller structure is more adequate for further closed-loop analysis and more appropriate for real-time computing on mid and low-range embedded systems or on efficient complex systems with high dynamic. Much research and many articles have been devoted to the question of model order simplification with different approaches such as balanced truncation, Hankle singular value decomposition, and others [30]. Most approaches assess the deviation between the simplified and the given model in frequency or time domain on the basis of optimal value of objective function, with no assessment of the difference in mathematical form such as the parametric model error in state space or transfer function. Advanced approach to robust optimal controller design for a simplified model is proposed under the condition that all closed-loop performance requirements for the simplified model are fulfilled also for the primary model. This can be ensured if we know the difference between both and can take it into account in controller design without back evaluation after each single calculation step. The proposed approach of model simplification with partial fraction decomposition is efficient for systems with emphasized low-order dynamics, where the remainder transfer function of order reduction is added to performance weight (4.5), with a minor loss in the robustness criterion in general. For example, partial fraction decomposition of the model can be used for many thermodynamic and electromechanical systems.

Partial fraction decomposition of the plant $P_0(s)$ is as follows:

$$\begin{gathered} P_0(s) = P_{\text{low}}(s) + Q_{\text{rem}}(s), \\ \frac{B(s)}{A(s)} = \frac{B_{\text{low}}(s)}{A_{\text{low}}(s)} + \frac{B_{\text{rem}}(s)}{A_{\text{rem}}(s)}, \end{gathered} \tag{4.8}$$

where

$$\begin{aligned} \frac{B(s)}{\Pi_{l=1}^{n-q}(s \pm p_l)\Pi_{c=1}^{q/2}(s^2 \pm d_{1c}s + d_0)} &= \frac{B_{\text{low}}(s)}{\Pi_{l=1}^{n_{\text{low}}-q_{\text{low}}}(s \pm p_l)\Pi_{c=1}^{q_{\text{low}}/2}(s^2 \pm d_{1c}s + d_0)} \\ &+ \frac{B_{\text{rem}}(s)}{\Pi_{l=1}^{n-n_{\text{low}}-q_{\text{low}}}(s + p_l)\Pi_{c=1}^{(q-q_{\text{low}})/2}(s^2 + d_{1c}s + d_0)}. \end{aligned} \tag{4.9}$$

$P_{\text{low}}(s)$ is the reduced model, $Q_{\text{rem}}(s)$ is the remainder function of reduction $P_0(s)$, n is the order of $A(s)$, q is the number of complex roots of $A(s)$, n_{low} is the order of $A_{\text{low}}(s)$, and q_{low} is the number of complex roots of $A_{\text{low}}(s)$. Function $P_{\text{low}}(s)$ contains dominant and unstable poles of $P_0(s)$. For good fitting at low frequency between $P_0(s)$ and $P_{\text{low}}(s)$ the following must be true:

$$B_{\text{low}}(0) \approx B(0). \tag{4.10}$$

Remainder $Q_{\text{rem}}(s)$ must be a stable and proper function. The stability property of $Q_{\text{rem}}(s)$ is used for further transformation of robust criterion (4.1)–(4.3) with reconstructed weights $W'(s)$. New weights $W'(s)$ are formed for each robust criterion separately. Before we introduce robust stability criteria for $P_{\text{low}}(s)$, it is necessary to consider the stability property for $P_0(s)$ with controller $P_{\text{low}}(s) \Rightarrow K_{\text{low}}(s)$.

The synthesized controller $K_{\text{low}}(s)$ for plant $P_{\text{low}}(s)$ stabilizes $P_0(s)$ if (4.8), (4.10), $Q_{\text{rem}}(s) \in \mathcal{RH}_\infty$, $\|Q_{\text{rem}}\|_\infty < 1$, and $\|Q_{\text{rem}}\|_\infty < \|P_{\text{low}}\|_\infty$ is true.

$$\begin{gathered} (1 + (P_{\text{low}} + Q_{\text{rem}})K_{\text{low}}) > 0, \\ (1 + P_{\text{low}}K_{\text{low}})\left(1 + (1 + P_{\text{low}}K_{\text{low}})^{-1}Q_{\text{rem}}K_{\text{low}}\right) > 0, \\ \|Q_{\text{rem}}K_{\text{low}}\|_\infty < \|1 + P_{\text{low}}K_{\text{low}}\|_\infty, \quad \forall Q_{\text{rem}} \land \forall P_{\text{low}}, \quad \|Q_{\text{rem}}\|_\infty \le \|P_{\text{low}}\|_\infty. \end{gathered} \tag{4.11}$$

All three properties are achieved with decomposition of $P_0(s)$ to dominant and unstable poles for $P_{\text{low}}(s)$ and remain stable poles for $Q_{\text{rem}}(s)$.

Robust stability property (4.1) considered by small gain theorem for reduced model $P_{\text{low}}(s)$ and inverse uncertainty model $P = P_0(1+W_1)^{-1}$ is

$$(1+KP_{\text{low}})\left(1+S_{\text{low}}\underbrace{(W_1+KQ_{\text{rem}})}_{W_1'(K)}\right)>0, \qquad \left\|S_{\text{low}}W_1'(K)\right\|_\infty<1. \tag{4.12}$$

$S_{\text{low}}(s)$ is the sensitivity closed-loop function of the reduced model $P_{\text{low}}(s)$ with synthesized controller $K(s)$ and reconstructed weight $W_1'(s)$.

Robust stability property (4.2) for reduced model $P_{\text{low}}(s)$ and additive uncertainty model $P = P_0 + W_2$ is

$$\|U_{\text{low}}(Q_{\text{rem}}+W_2)\|_\infty<1, \qquad \left\|U_{\text{low}}W_2'\right\|_\infty<1, \tag{4.13}$$

where $U_{\text{low}}(s)$ is the controller output transfer function with $K_{\text{low}}(s)$.

Robust stability property (4.3) for reduced model $P_{\text{low}}(s)$ and multiplicative uncertainty model $P = P_0(1+W_3)$ is

$$\|U_{\text{low}}(W_3P_0+Q_{\text{rem}})\|_\infty<1, \qquad \left\|U_{\text{low}}W_3'\right\|_\infty<1. \tag{4.14}$$

5. Optimization of Robust Criteria with Even Spectral Polynomials

Transformation of robust criteria (4.1)–(4.3) in even polynomial form is applied in many optimization techniques, particularly if objective functions ensure convex property [3]. Many efficient robust approaches use zero-order optimization method [31], where the selection of objective function is of key importance.

The norm $\|\cdot\|_\infty$ for $P_0(s)$ can be defined with function $\Phi(s)$:

$$\Phi(s)=\gamma^2I-P_0(s)P_0(-s). \tag{5.1}$$

$\Phi(s)$ is a continuous function for all $\omega\in\mathbb{R}\cup\{\infty\}$ and has no imaginary axis zero if $\|P_0\|_\infty<\gamma$. Proper condition of suboptimal robustness (4.1)–(4.3) is fulfilled, if $\|X\|_\infty<1$, where $\gamma=1$. From (5.1), inequality $\|P_0\|_\infty<1$ can be derived:

$$A(s)A(-s)-B(s)B(-s)>0, \qquad \left|A(j\omega)\right|^2-\left|B(j\omega)\right|^2=\Pi(\omega)>0, \qquad A\left(\omega^2\right)-B\left(\omega^2\right)=\Pi(\omega)>0. \tag{5.2}$$

Spectral polynomial $\Pi(\omega) = \sum_{w=0}^{v} q_{2w}\omega^{2w} > 0$ with real coefficients q_{2w} is an even function, where $\|P_0\|_\infty < 1$ is true, if polynomial $\Pi(\omega)$ has no real roots. Polynomial $\Pi(\omega)$ must be a strictly positive function: $q_{2w} > 0$ for all ω. Nonnegativity of even polynomials can be tested with algebraic Šiljak's stability test [21] if the condition $q_0 \wedge q_v > 0$ for all ω is fulfilled. Šiljak's test is used for testing the existence of proper solutions before the optimization method starts. A more detailed discussion of the whole algorithm and the use of Šiljak's test are presented in [32].

From condition (5.2), spectral polynomial $\Pi(\omega)$ can be derived for each robust criterion (4.1)–(4.3) separately.

Robust condition (4.1) with weight $W_1(s)$ or $W_1'(s)$ is

$$\left\| \frac{A(s)R(s)w_{b_1}(s)}{C(s)w_{a_1}(s)} \right\|_\infty < 1, \tag{5.3}$$

where $W_1(s) \wedge W_1'(s) = w_{b1}(s)w_{a1}^{-1}(s) = (w_{b1_g}s^g + \cdots + w_{b1_0})(w_{a1_g}s^g + \cdots + w_{a1_0})^{-1}$ and the derived spectral polynomial $\Pi_S(\omega)$ is

$$\begin{aligned} \Pi_s(s,\tilde{p}) &= C(s)C(-s)w_{a1}(s)w_{a1}(-s) - A(s)A(-s)R(s,\tilde{p})R(-s,\tilde{p})w_{b1}(s)w_{b1}(-s), \\ \Pi_s(\omega,\tilde{p}) &= Cw_{a1}(\omega) - ARw_{b1}(\omega,\tilde{p}) > 0. \end{aligned} \tag{5.4}$$

Robust condition (4.2) with weight $W_2(s)$, $W_2'(s) = w_{b2}(s)w_{a2}^{-1}(s)$ and spectral polynomial $\Pi_U(\omega)$ is

$$\left\| \frac{A(s)L(s)w_{b2}(s)}{C(s)w_{a_2}(s)} \right\|_\infty < 1, \tag{5.5}$$

$$\Pi_U(\omega,\tilde{p}) = Cw_{a2}(\omega) - ALw_{b2}(\omega,\tilde{p}) > 0. \tag{5.6}$$

Robust condition (4.3) with weight $W_3(s)$, $W_3'(s) = w_{b3}(s)w_{a3}^{-1}(s)$ and spectral polynomial $\Pi_T(\omega)$ is

$$\left\| \frac{B(s)L(s)w_{b3}(s)}{C(s)w_{a3}(s)} \right\|_\infty < 1, \tag{5.7}$$

$$\Pi_T(\omega,\tilde{p}) = Cw_{a3}(\omega) - BLw_{b3}(\omega,\tilde{p}) > 0. \tag{5.8}$$

With simple algebra we can determine the necessary edge conditions of controller parameters for ensuring nonnegativity of the spectral polynomials (5.4), (5.6), and (5.8). Edge conditions for criteria (5.3), (5.5), and (5.7) are as follows:

$$\begin{aligned} &0 < r_0 < |c_0 w_{a1_0}||a_0 w_{b1_0}|^{-1} \quad \forall \Pi_S(\omega,\tilde{p}), \\ &0 \le l_0 \le |c_0 w_{a2_0}||a_0 w_{b2_0}|^{-1} \quad \forall \Pi_U(\omega,\tilde{p}), \\ &0 \le l_0 \le |c_0 w_{a3_0}||b_0 w_{b3_0}|^{-1} \quad \forall \Pi_T(\omega,\tilde{p}). \end{aligned} \tag{5.9}$$

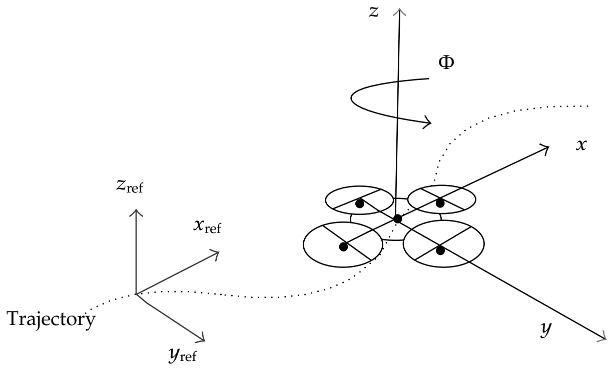

Figure 3: 4DOF quadcopter guidance problem.

Edge conditions (5.9) determine the solvability of the given problem, with the chosen controller structure. The central issue of optimization procedure is finding a global minimum of maximized nonnegative spectral polynomial $\Pi(\omega)$. The objective function f_{obj} of DE optimization procedure is composed of spectral polynomial and auxiliary condition f_{od}, like the strong stabilization property, and of additional closed-loop time performance criteria. The strong stabilization condition is ensured by testing the root location of $R(s)$ in LHP with Lipatov criterion [32]. The spectral polynomial is a quasiconvex function and can be considered as a convex function with the selection of $\omega \leq 0$ or $\omega \geq 0$. With bounded optimization space $\omega \leq 0$ or $\omega \geq 0$, the optimization procedure can be treated as a convex problem with no loss of generality. Objective function for single criterion is

$$f_{obj} = \left(\prod_{S}(\omega, \tilde{p}) \vee \prod_{U}(\omega, \tilde{p}) \vee \prod_{T}(\omega, \tilde{p}) \right) \wedge f_{ad}. \tag{5.10}$$

Objective function for the mixed sensitivity problem is

$$f_{obj} = \prod_{S}(\omega, \tilde{p}) \wedge \prod_{U}(\omega, \tilde{p}) \wedge \prod_{T}(\omega, \tilde{p}) \wedge f_{ad}. \tag{5.11}$$

6. Design Example

Example 6.1. In first example, the guidance of the autonomous flight systemquadcopter is taken into account. The system has already been internally stabilized around pitch and roll axis with gyro sensors (hovering mode ability). The problem of the presented controller design example is how to ensure proper guidance ability of the system to follow a desired path trajectory in three-dimensional spaces. The system has the following for degree of freedom 4DOF: altitude-z, longitudinal movement-x, y, and orientation (yaw)-ϕ (Figure 3).

The measured data of quadcopter's position and orientation were obtained from gyro, acceleration, and magnetic sensors with additional reconstruction measure filtering with Kalman recursive algorithm. Altitude measurement was performed with a barometric pressure and ultrasonic distance sensor. Nominal model of the plant in matrix form $P(s)$ with coupled dynamic $H(s)$ is

$$\begin{bmatrix} x(s) \\ y(s) \\ z(s) \\ \phi(s) \end{bmatrix} = \begin{bmatrix} P_x(s) & H_{xy}(s) & 0 & H_{\text{yaw}x}(s) \\ H_{yx}(s) & P_y(s) & 0 & H_{\text{yaw}y}(s) \\ 0 & 0 & P_{\text{alt}}(s) & H_{\text{yaw}a}(s) \\ H_{x\text{yaw}}(s) & H_{y\text{yaw}}(s) & 0 & P_{\text{yaw}}(s) \end{bmatrix} \times \begin{bmatrix} u_x(s) \\ u_y(s) \\ u_z(s) \\ u_\phi(s) \end{bmatrix}, \tag{6.1}$$

where individual transfer function is

$$\begin{aligned} P_x(s) = P_y(s) &= \frac{46345 \times (s + 0.837)}{s^4 + 7.049 \times s^3 + 13.73 \times s^2 + 4.484 \times s}, \\ P_{\text{alt}}(s) &= \frac{46345 \times s + 0.837}{s^3 + 21.05 \times s^2 + 0.965 \times s + 0.01}, \\ P_{\text{yaw}}(s) &= \frac{85.74 \times s + 26.46}{s^2 + 0.312 \times s}. \end{aligned} \tag{6.2}$$

The coupled dynamic transfer function is

$$\begin{aligned} H_{xy}(s) = H_{xy}(s) &= \frac{2.1 \times \left(s^2 + 0.039 \times s + 0.0000983\right)}{s^2 + 0.2 \times s + 0.0075}, \\ H_{\text{yaw}x}(s) = H_{\text{yaw}y}(s) &= \frac{4.5 \times \left(s^2 + 10.12 \times s + 0.0011\right)}{s^2 + 15.01 \times s + 0.135}, \\ H_{\text{yaw}a}(s) &= \frac{3 \times \left(s^4 + 190.1 \times s^3 + 4.355 \times s^2 + 10.11 \times s + 0.0001\right)}{s^4 + 64.01 \times s^3 + 724.6 \times s^2 + 1207 \times s + 12}, \\ H_{x\text{yaw}}(s) = H_{x\text{yaw}}(s) &= \frac{1.085 \cdot 10^{-4} \times \left(s^2 + 924.4 \times s + 279.1\right)}{s^2 + 0.421 \times s + 0.034}. \end{aligned} \tag{6.3}$$

For control design robust-mixed sensitivity problem (4.6) is used with weights as the following:

$$\begin{aligned} W_{Mx}(s) = W_{My}(s) &= \frac{0.015 \times s^4 + 0.1479 \times s^3 + 0.538 \times s^2 + 0.855 \times s + 0.5022}{s^4 + 4.713 \times s^3 + 7.099 \times s^2 + 3.651 \times s + 0.5602}, \\ W_{M\text{alt}}(s) &= \frac{0.112 \times s^3 + 5.615 \times s^2 + 0.7537 \times s + 7.045 \cdot 10^{-5}}{s^3 + 13.06 \times s^2 + 0.6501 \times s + 0.003406}, \end{aligned}$$

$$\begin{aligned}
W_{M\text{yaw}}(s) &= \frac{0.00126 \times s^2 + 0.0974 \times s + 0.02929}{s^2 + 0.4286 \times s + 0.03703}, \\
W_{ix}(s) = W_{iy}(s) &= \frac{s^2 + 0.2 \times s + 0.0075}{2.1s^2 + 0.0819 \times s + 0.000195}, \\
W_{i\text{alt}}(s) &= \frac{s^4 + 64.01 \times s^3 + 724.6 \times s^2 + 1207 \times s + 12}{3 \times s^4 + 570.3 \times s^3 + 13.01 \times s^2 + 30.3 \times s + 0.0003}, \\
W_{i\text{yaw}}(s) &= \frac{s^2 + 0.4286 \times s + 0.03703}{0.00126 \times s^2 + 0.0974 \times s + 0.02929},
\end{aligned} \tag{6.4}$$

where weights $W_{Mx}(s)$, $W_{My}(s)$, $W_{M\text{alt}}(s)$, and $W_{M\text{yaw}}(s)$ represent the multiplicative uncertainty of the nominal plant [15]. Uncertainty weights were determined on the basis of system parameter variation, like different load weight (battery, camera), different propeller characteristic, and so forth. Weight $W_M(s)$ describes the deviation of the plant at a lower frequency of nominal characteristic. Weight $W_i(s)$ is used to decuple a feedback MIMO plant on four single-axis SISO systems. The influences of other inputs for each axis are represented as output disturbances w_2 (Figure 2) with a frequency characteristic $W_i(s)$. Weight $W_i(s)$ is selected from coupled dynamic transfer functions $H(s)$ for each axis separately and represents the lowest upper boundary of the sensitivity function for single-axis SISO feedback system. Additional simplification of controller structure and design procedure is carried out with partial-fractional decomposition on stable higher dynamic transfer function $Q_{\text{rem}}(s)$ and the remaining lower dynamic, unstable transfer function $P_{\text{low}}(s)$. The new weight $W'_M(s)$ and robust criteria are transformed with expression (4.14). Partial-fractional decomposition is as follows:

$$\begin{aligned}
P_x(s) = P_y(s) = P_{\text{low}x/y}(s) + Q_{\text{rem}x/y}(s) &= \frac{2872 \times s + 3519}{s^2 + 0.484 \times s} - \frac{2872 \times s + 2.1 \cdot 10^4}{s^2 + 6.643 \times s + 11.03}, \\
P_{\text{alt}}(s) = P_{\text{low alt}}(s) + Q_{\text{rem alt}}(s) &= \frac{1297 \times s + 1.811}{s^2 + 0.0459 \times s + 4.7 \cdot 10^{-4}} - \frac{1297.23}{s + 21.03}, \\
P_{\text{yaw}}(s) = P_{\text{low yaw}}(s) + Q_{\text{rem yaw}}(s) &= \frac{84.81}{s} + \frac{0.93}{s + 0.312}.
\end{aligned} \tag{6.5}$$

Mixed sensitivity problem (4.6) with stability criterion for multiplicative uncertainty with modified weights $W'_M(s)$ and robust performance criterion with weights $W_i(s)$ is

$$T_x = T_y = \left\| \begin{matrix} W_{ix/y} S_{x/y} \\ W'_{Mx/y} U_{x/y} \end{matrix} \right\|_\infty, \qquad T_{\text{alt}} = \left\| \begin{matrix} W_{i\text{alt}} S_{\text{alt}} \\ W'_{M\text{alt}} U_{\text{alt}} \end{matrix} \right\|_\infty, \qquad T_{\text{yaw}} = \left\| \begin{matrix} W_{\text{yaw}} S_{\text{yaw}} \\ W'_{\text{yaw}} U_{\text{yaw}} \end{matrix} \right\|_\infty. \tag{6.6}$$

Time performance requirements of the controlled system are as the following:

(i) overshot $M_{\text{pr}}(\%)$: x/y-axes < 20%, z-axis < 30%, ϕ-axis < 30%;

(ii) settling time $t_s(\text{s})$: x/y-axis < 20 s, z-axis < 15 s, ϕ-axis < 6 s;

(iii) steady state error for all four axes: $e(\%) < 1\%$;

(iv) robust-stabilized system and good output low-frequency disturbance rejection over frequency band (0-0.1 rad/s) and good low-frequency reference signal tracking over (0–0.23 rad/s) for altitude control and (0–0.21 rad/s) for longitudinal movement and orientation;

(v) strong stabilization system.

According to control requirements, the controller structure for each axis can be determined. The selected controller structure for longitudinal movement $K_x(s)$ and $K_y(s)$ is

$$K_x\left(s,\tilde{r}_0,\tilde{r}_1,\tilde{l}_0\right) = K_y\left(s,\tilde{r}_0,\tilde{r}_1,\tilde{l}_0\right) = \frac{s+\tilde{l}_0}{s+\tilde{r}_1}\cdot\frac{l_2s^2+l_1s+l_0}{r_2s^2+r_1s+\tilde{r}_0}, \tag{6.7}$$

where free parameters $\tilde{r}_0$, $\tilde{r}_1$, $\tilde{l}_0$ enable good ability of optimization criteria with weights $W_i(s)$, $W_M(s)$. According to strong stabilization and overshot requirements, the acceptable interval limits for the free parameter are determined as follows: $0 < \tilde{r}_0 < 5$, $50 < \tilde{r}_1 < 750$, $10 < \tilde{l}_0 < 200$. Selected characteristic polynomial coefficients (Manabe polynomial form and settling time $t_s(\mathrm{s}) < 10\,\mathrm{s}$, [27]) and polynomial order requirements (2.6), (2.7) with controller structure $K_{x/y}(s)$ and plant $P_{\mathrm{low}x/y}(s)$ are as follows:

$$C_x(s) = C_y(s) = s^5 + 690\times s^4 + 2397\times s^3 + 2950\times s^2 + 1366\times s + 310. \tag{6.8}$$

Altitude controller $K_{\mathrm{alt}}(s)$ with added Noch characteristic at a damping frequency $\omega_{\mathrm{noch}} = 0.23\,\mathrm{rad/s}$ and three free parameters is

$$K_{\mathrm{alt}}\left(s,\tilde{r}_0,\tilde{r}_1,\tilde{l}_0\right) = \frac{\omega_{\mathrm{noch}}^2}{s^2+\tilde{r}_1s+\omega_{\mathrm{noch}}^2}\cdot\frac{\tilde{l}_0s^5+l_4s^4+l_3s^3+l_2s^2+l_1s+l_0}{r_5s^5+r_4s^4+r_3s^3+r_2s^2+r_1s+\tilde{r}_0}. \tag{6.9}$$

Noch characteristic is chosen from the property of weight $W_{i\mathrm{alt}}(s)$ with a frequency gap at $\approx 0.23\,\mathrm{rad/s}$.
The acceptable interval limit for free parameter $\tilde{r}_0$ is $0 < \tilde{r}_0 < 4$ for ensuring good low-frequency tracking property and disturbance rejection. Admissible interval for parameter $\tilde{r}_1$ is $0 < \tilde{r}_1 < 0.01$ for ensuring good damping at frequency ω_{noch}. The admissible interval for parameter $\tilde{l}_0$ is $0 < \tilde{l}_0 < 0.01$ for ensuring optimization property at higher frequency. Selected characteristic polynomial (Manabe form) according to controller order $K_{\mathrm{alt}}(s)$ and plant $P_{\mathrm{low\ alt}}(s)$ is:

$$\begin{aligned}C_{\mathrm{alt}}(s) = {} & s^9 + 70.4\times s^8 + 2478\times s^7 + 4.04\cdot 10^4\times s^6 + 9.44\cdot 10^4\times s^5 + 1.24\cdot 10^5\times s^4 \\ & + 6.5\cdot 10^4\times s^3 + 1.98\cdot 10^4\times s^2 + 53.59\times s + 0.037.\end{aligned} \tag{6.10}$$

Selected yaw controller structure $K_{\mathrm{yaw}}(s)$ with free parameters $\tilde{r}_0$, $\tilde{r}_1$ is

$$K_{\mathrm{yaw}}(s,\tilde{r}_0,\tilde{r}_1) = \frac{l_2s^2+l_1s+l_0}{r_2s^2+\tilde{r}_1s+\tilde{r}_0}. \tag{6.11}$$

Admissible interval for parameter $\tilde{r}_0$ is $0 < \tilde{r}_0 < 1$, and for $\tilde{r}_0$ it is $1 < \tilde{r}_1 < 10$ to ensure strong stabilization system and good low-frequency closed-loop system ability. The characteristic polynomial is

$$C_{\text{yaw}}(s) = s^3 + 2.75 \times s^2 + 3.92 \times s + 2.26. \tag{6.12}$$

The results of the optimization procedure with DE algorithm are described in [32]. Longitudinal movement controller $K_{x/y}(s)$ is as follows:

$$K_x(s) = K_y(s) = \frac{s + 93.8}{s + 686} \cdot \frac{2.2 \cdot 10^{-4} \times s^2 + 0.00127 \times s + 0.0009}{s^2 + 2.98 \times s + 2.464}. \tag{6.13}$$

The values of norms are $\|W_{ix/y}S_{ix/y}\|_\infty = 0.58$, $\|W'_{Mx/y}U_{ix/y}\|_\infty = 0.896$, $\|W_{Mx/y}T_{ix/y}\|_\infty = 0.896$.

Altitude controller $K_{\text{alt}}(s)$ is as follows:

$$\begin{aligned} K_{\text{alt}}(s) = {} & \frac{0.0531}{s^2 + 0.001 \times s + 0.0531} \\ & \cdot \frac{376.1 \times s^5 + 1308 \times s^4 + 1466 \times s^3 + 941.6 \times s^2 + 275.5 \times s + 0.384}{s^5 + 10.35 \times s^4 + 2475 \times s^3 + 1.37 \cdot 10^4 \times s^2 + 2610 \times s + 3.6}. \end{aligned} \tag{6.14}$$

The values of norms are $\|W_{i\text{alt}}S_{i\text{alt}}\|_\infty = 0.54$, $\|W'_{M\text{alt}}U_{i\text{alt}}\|_\infty = 0.95$, $\|W_{M\text{alt}}T_{i\text{alt}}\|_\infty = 0.959$.

Yaw controller $K_{\text{yaw}}(s)$ is as follows

$$K_{\text{yaw}}(s) = \frac{0.0145 \times s^2 + 0.0361 \times s + 0.0267}{s^2 + 1.524 \times s + 0.852}. \tag{6.15}$$

The values of norms are $\|W_{i\text{yaw}}S_{i\text{yaw}}\|_\infty = 0.51$, $\|W'_{M\text{yaw}}U_{i\text{yaw}}\|_\infty = 0.75$, $\|W_{M\text{yaw}}T_{i\text{yaw}}\|_\infty = 0.78$.

A closed-loop system responses to the step reference signal and output periodic disturbance.

Longitudinal movement control corresponds to $K_{x/y}(s)$ on step reference signal with low-frequency output disturbance $\omega = 0.08$ rad/s, Figure 4.

Altitude control with controller $K_{\text{alt}}(s)$ and output disturbance signal, $\omega = 0.22$ rad/s, is presented in Figure 5.

Orientation-yaw control with controller $K_{\text{yaw}}(s)$ and output disturbance signal, $\omega = 0.12$ rad/s, is presented in Figure 6.

A trajectory following with controllers $K_x(s)$, $K_y(s)$, $K_{\text{alt}}(s)$, is presented in Figure 7.

The quadcopter guidance controllers $K_x(s)$, $K_y(s)$, $K_{\text{alt}}(s)$, and $K_{\text{yaw}}(s)$ have satisfied the whole set of control requirements (see Figures 4 and 5). Controllers are robustly stable for multiplicative uncertainty models and allow the desired tracking performance for step and periodic reference signals at frequency (0–0.15 rad/s) for x, y, and ϕ axes and (0–0.27 rad/s) for altitude control z-axis. The feedback system also allows the desired output disturbance rejection for signal at frequency (0–0.18 rad/s) for x, y, ϕ-axes and (0–0.35 rad/s) for z-axis.

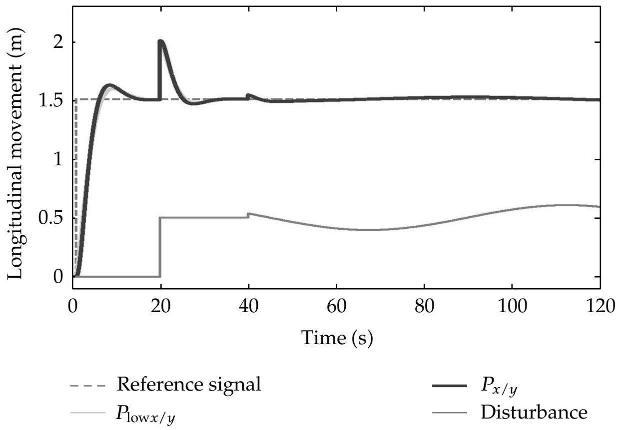

Figure 4: Longitudinal x/y movement control with controller $K_{x/y}(s)$ and output disturbance with frequency $\omega = 0.081$ rad/s.

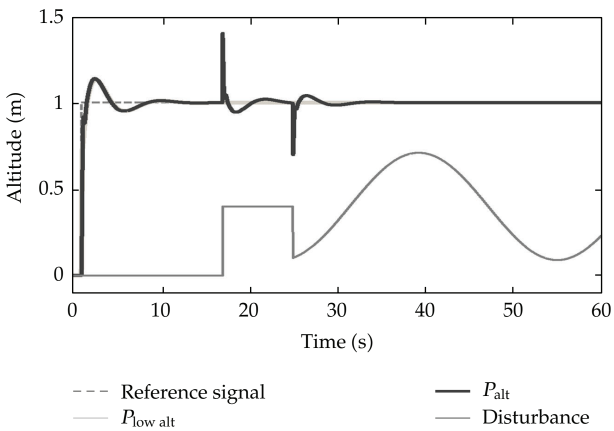

Figure 5: Altitude control with controller $K_{\text{alt}}(s)$ and output periodic disturbance.

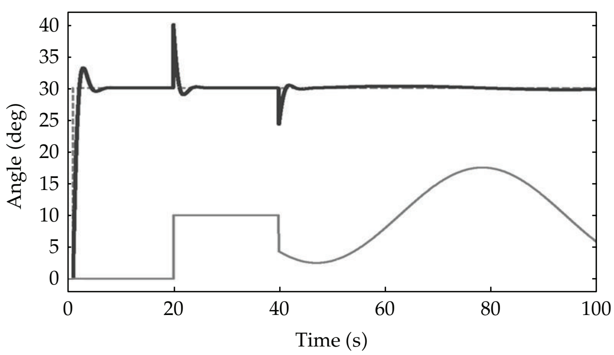

Figure 6: Orientation (yaw) control with controller $K_{\text{yaw}}(s)$ and output periodic disturbance.

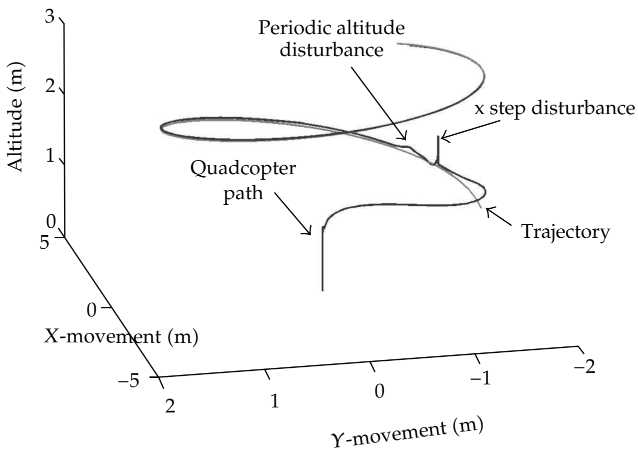

Figure 7: Quadcopter trajectory following.

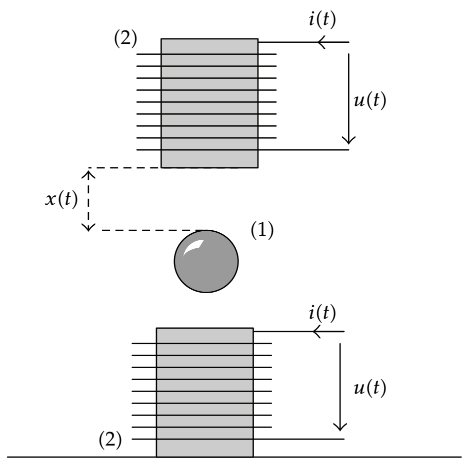

Figure 8: Single-axis magnetic system.

Example 6.2. The second design example is a stabilization of the single-axis magnetic ball suspension system.

The ball (1) in Figure 8 can move only vertically in the magnetic field of the electromagnet (2.3). The position of the ball is detected by the optical sensor. Nominal model of the plant [33] is

$$\frac{X(s)}{U(s)} = P_{\text{sus}}(s) = \frac{28.9}{s^4 + 147.4 \times s^3 + 454.7 \times s^2 - 85.49 \cdot 10^3 \times s - 656.6 \cdot 10^3}. \tag{6.16}$$

Used weights are

$$W_S(s) = \frac{2.88 \cdot 10^{-3} \times s^2 + 0.161 \times s + 0.8}{s^2 + 0.16 \cdot 10^{-3} \times s + 6.4 \cdot 10^{-9}},$$
$$W_A(s) = \frac{1.1 \cdot 10^{-5} \times s^2 + 0.011 \times s + 0.0011}{s^3 + 15 \times s^2 + 3 \times s + 2}, \tag{6.17}$$

where weights $W_S(s)$ and $W_A(s)$ represent desired sensitivity characteristic of closed-loop system and additive uncertainty of the nominal system, respectively. Uncertainty weights describe nonlinear characteristic of the magnetic field and the system parameter deviation, especially different ball weight.

Simplification of controller structure with partial-fractional decomposition on $P_{\text{low sus}}(s)$ and $Q_{\text{sus rem}}(s)$ (4.8) is as the following:

$$\begin{aligned} P_{\text{sus}}(s) &= P_{\text{sus low}}(s) + Q_{\text{sus rem}}(s) \\ &= \frac{0.06 \times s + 33.32}{s^3 + 7.371 \times s^2 - 577.2 \times s - 4690} + \frac{0.001274}{s + 139.34}. \end{aligned} \tag{6.18}$$

The new weights $W'_A(s)$ and $W'_S(s)$ are transformed with expressions (4.12), (4.13). Mixed sensitivity problem (4.6) with modified weights $W'_M(s), W'_S(s)$ are

$$T_{\text{mix}} = \left\| \begin{matrix} W'_S S_{\text{sus low}} \\ W'_M U_{\text{sus low}} \end{matrix} \right\|_\infty. \tag{6.19}$$

Performance requirements of the controlled system are as the following.

(i) overshoot: $M_{\text{pr}}(\%) < 10\%$.

(ii) settling time: $t_s(\text{s}) < 0.8\,\text{s}$.

(iii) the tracking error for step response and periodic reference signal over frequency band (0–4 rad/s): $e(\%) < 3\%$.

(iv) robust stabilized system and good output low frequency disturbance rejection (>−40 dB) over frequency band (0–5 rad/s).

(v) strong stabilization system.

Selected controller structure $K(s, \tilde{r}_0, \tilde{r}_1)$ with prefilter $F(s)$ designed according to [23],

$$K(s, \tilde{r}_0, \tilde{r}_1) = \frac{l_4 s^4 + l_3 s^3 + l_2 s^2 + l_1 s + l_0}{r_4 s^4 + r_3 s^3 + r_2 s^2 + \tilde{r}_1 s + \tilde{r}_0},$$
$$F(s) = \frac{\|TF\|_\infty}{l_4 s^4 + l_3 s^3 + l_2 s^2 + l_1 s + l_0}, \tag{6.20}$$

where free parameters $\tilde{r}_0$, $\tilde{r}_1$ improve the ability of optimization criteria with weights $W'_M(s)$, $W'_S(s)$. According to strong stabilization and overshoot requirements, the acceptable

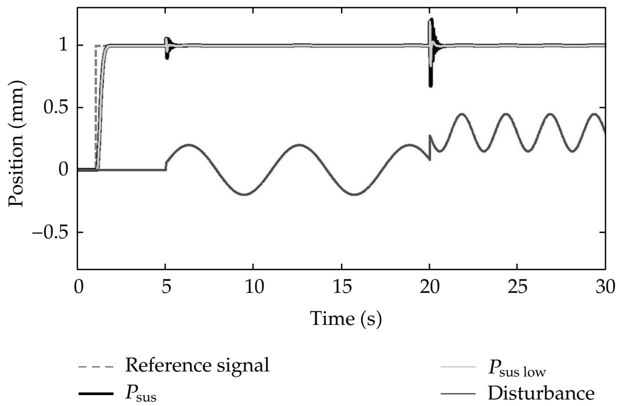

Figure 9: System response on step reference signal with two different output disturbances.

interval limits for the free parameter are determined as the following: $0 < \tilde{r}_0 < 1$, $150 < \tilde{r}_1 < 3000$. Selected characteristic polynomial coefficients according to (2.6), (2.7), and [27] are

$$C(s) = s^7 + 285.7 \times s^6 + 4.1 \cdot 10^4 \times s^5 + 2.9 \cdot 10^6 \times s^4 + 1.4 \cdot 10^8 \times s^3 + 1.8 \cdot 10^9 \times s^2 + 1.6 \cdot 10^{10} \times s + 5.9 \cdot 10^{10}. \tag{6.21}$$

The results of the optimization procedure are

$$K(s, \tilde{r}_0, \tilde{r}_1) = 9.6 \cdot 10^4 \frac{s^4 + 45.8 \times s^3 + 732.7 \times s^2 + 5.95 \cdot 10^3 \times s + 2.12 \cdot 10^4}{s^4 + 278 \times s^3 + 3.9 \cdot 10^4 \times s^2 + 783 \times s + 0.05},$$
$$F(s) = \frac{2.12 \cdot 10^4}{s^4 + 45.8 \times s^3 + 732.7 \times s^2 + 5.95 \cdot 10^3 \times s + 2.12 \cdot 10^4}. \tag{6.22}$$

The values of norms are $\|W'_S S_{\text{sus low}}\|_\infty = 0.508$, $\|W'_M U_{\text{sus low}}\|_\infty = 0.64$, $\|W_S S_{\text{sus}}\|_\infty = 0.501$, $\|W_M U_{\text{sus}}\|_\infty = 0.63$.

The closed-loop system responses to the step reference signal and output periodic disturbance.

System response on step reference signal with two different output disturbances, with frequency $\omega_1 = 1.5$ rad/s and $\omega_2 = 4$ rad/s and offset value 0.3 mm, Figure 9.

Step reference tracking with output disturbance Figure 10.

Periodic reference tracking with periodic output disturbance Figure 11.

The closed-loop systems with controller $K(s, \tilde{r}_0, \tilde{r}_1)$ and prefilter $F(s)$ have satisfied the whole set of control requirements (see Figures 10 and 11). Controllers are robustly stable for additive uncertainty model and allow the desired tracking performance for step and periodic reference signals at frequency (0–4 rad/s). The feedback system also allows the desired output disturbance rejection for signal at frequency (0–5 rad/s).

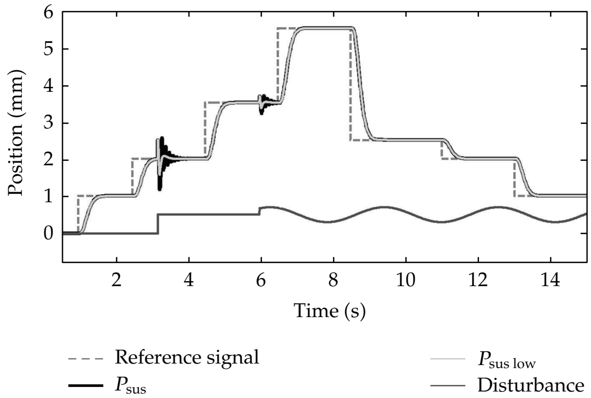

Figure 10: Step reference tracking with output disturbance.

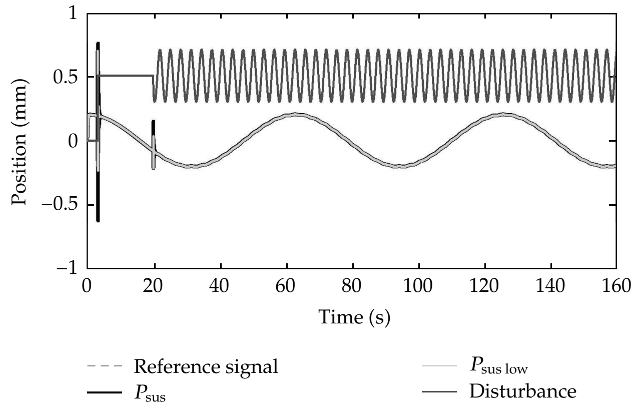

Figure 11: Periodic reference tracking with output disturbance.

7. Conclusion

The proposed method with transparent composed controller structure and optimization of robust criteria with free parameters in polynomial equation provides an efficient tool for feedback system design. Controller parameterization, with known characteristics, such as integral action, double-integrator, and Noch property, and their approximation with stable structure significantly improve feedback performance (overshot, settling time, overall feedback dynamic) and the ability of further implementation on a real system (operation safety, simplification of real-time algorithm). Fraction-partial decomposition is also a useful approach for a simplified controller structure in the polynomial approach, where exact feasibility of the controller depends on the plant degree. It is worth emphasizing that fractional decomposition is limited to plants with strongly expressed dominant and unstable poles which have the highest stored energy of the system (e.g., electromechanical system with a dominant mechanical part). The method can be easily used in combination with metric $\mathscr{H}_2$, where $\mathscr{H}_2$ means optimization of the feedback system energy. In general, the method is useful for all controller designs, where low-order transparent controllers are required and the structure of the controller is predefined.

References

[1] R. E. Skelton, T. Iwasaki, and K. M. Grigoriadis, *A Unified Algebraic Approach to Linear Control Design*, Taylor & Francis, 1994.

[2] J.-H. Kim, "New reduced-order H_∞ filter design method for discrete-time singular systems with lossy measurements by strict LMI approach," *International Journal of Control, Automation, and Systems*, vol. 9, no. 6, pp. 1095–1102, 2011.

[3] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, SIAM Studies in Applied Mathematics, 1994.

[4] H. Zhang, Y. Shi, and A. S. Mehr, "On H_∞ filtering for discrete-time Takagi-Sugeno fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 20, no. 2, pp. 396–401, 2012.

[5] H. Zhang, Y. Shi, and A. S. Mehr, "Robust H_∞ PID control for multivariable networked control systems with disturbance/noise attenuation," *International Journal of Robust and Nonlinear Control*, vol. 22, no. 2, pp. 183–204, 2012.

[6] J. Han and R. E. Skelton, "An LMI optimization approach for structured linear controllers," in *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 5143–5148, December 2003.

[7] V. L. Syrmos, C. T. Abdallah, P. Dorato, and K. Grigoriadis, "Static output feedback: a survey," *Automatica*, vol. 33, no. 2, pp. 125–137, 1997.

[8] H. Khatibi, A. Karimi, and R. Longchamp, "Fixed-order controller design for polytopic systems using LMIs," *IEEE Transactions on Automatic Control*, vol. 53, no. 1, pp. 428–434, 2008.

[9] D. Henrion, M. Šebek, and V. Kučera, "Positive polynomials and robust stabilization with fixed-order controllers," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1178–1186, 2003.

[10] T. Iwasaki, "The dual iteration for fixed-order control," *IEEE Transactions on Automatic Control*, vol. 44, no. 4, pp. 783–788, 1999.

[11] H. Zhang, Y. Shi, and A. S. Mehr, "Robust static output feedback control and remote PID design for networked motor systems," *IEEE Transactions on Industrial Electronics*, vol. 58, no. 12, pp. 5396–5405, 2011.

[12] X. Li and H. Gao, "Robust finite frequency H_∞ filtering for uncertain 2-D Roesser systems," *Automatica*, vol. 48, no. 6, pp. 1163–1170, 2012.

[13] Y. Zhao, L. Zhang, S. Shen, and H. Gao, "Robust stability criterion for discrete-time uncertain markovian jumping neural networks with defective statistics of modes transitions," *IEEE Transactions on Neural Networks*, vol. 22, no. 1, pp. 164–170, 2011.

[14] K. Zhou and J. Doyle, *Essentials of Robust Control*, Prentice-Hall, Upper Saddle River, NJ, USA, 1998.

[15] K. Zhou, *Glover, Robust and Optimal Control*, Prentice-Hall, Upper Saddle River, NJ, USA, 1997.

[16] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*, Macmillan Publishing Company, New York, NY, USA, 1990.

[17] I. Tuzcu, "A computational approach to stability of polytopic families of quasi-polynomials with single delay," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 17, pp. 1981–1992, 2010.

[18] V. Kučera, "The pole placement equation. A survey," *Kybernetika*, vol. 30, no. 6, pp. 578–584, 1994.

[19] D. Henrion, M. Šebek, and V. Kučera, "Positive polynomials and robust stabilization with fixed-order controllers," *IEEE Transactions on Automatic Control*, vol. 48, no. 7, pp. 1178–1186, 2003.

[20] J. S. Karmarkar, "On Šiljak's absolute stabitiy test," *Proceedings Letters*, vol. 58, no. 5, pp. 817–819, 1970.

[21] D. D. Šiljak, "New algebraic criteria for positive realness," *Journal of the Franklin Institute*, vol. 291, no. 2, pp. 109–120, 1971.

[22] C. C. Hsu, S. C. Chang, and C. Y. Yu, "Tolerance design of robust controllers for uncertain interval systems based on evolutionary algorithms," *IET Control Theory and Applications*, vol. 1, no. 1, pp. 244–252, 2007.

[23] A. Chowdhury, A. Sarjaš, P. Cafuta, and R. Svečko, "Robust controller synthesis with consideration of performance criteria," *Optimal Control Applications & Methods*, vol. 32, no. 6, pp. 700–719, 2011.

[24] F. Yang, M. Gani, and D. Henrion, "Fixed-order robust H_∞ controller design with regional pole assignment," *IEEE Transactions on Automatic Control*, vol. 52, no. 10, pp. 1959–1963, 2007.

[25] R. Storn and K. Price, "Differential evolution—a simple and efficient heuristic for global optimization over continuous spaces," *Journal of Global Optimization*, vol. 11, no. 4, pp. 341–359, 1997.

[26] V. Kučera, *Discrete Linear Control: The Polynomial Equation Approach*, John Wiley & Sons, Chichester, UK, 1979.

[27] S. Manabe, *Coefficient Diagram Method*, IFAC Automatic Control in Aerospace, Seoul, Korea, 1998.

[28] N. I. Sokolov and A. V. Lipatov, "On necessary conditions for stability of linear systems," *Moscow Aviation Institute Transactions*, vol. 240, pp. 26–30, 1972.

[29] Y. S. Chou, T. Z. Wu, and J. L. Leu, “On strong stabilization and H_∞ strong stabilization problems,” in *Proceedings of the 42nd IEEE Conference on Decision and Control*, pp. 5155–5160, December 2003.
[30] G. Obinata and B. D. O. Anderson, *Model Reduction For Control System Design*, Springer, 2001.
[31] J. Song, X. Chen, Z. Liu, and J. Zhang, “Robust optimal trajectory design by integrating genetic algorithm with min-max method,” *International Journal of Control, Automation, and Systems*, vol. 9, no. 5, pp. 1013–1019, 2011.
[32] A. Sarja, R. Sveko, and A. Chowdhury, “Strong stabilization servo controller with optimization of performance criteria,” *ISA Transactions*, vol. 50, no. 3, pp. 419–431, 2011.
[33] A. Chowdhury, M. Golob, P. Cafuta, and R. Svečko, “Robust controller design for position control of the metal ball in magnetic field,” in *Proceedings of the IEEE International Conference on Industrial Technology (ICIT ’03)*, vol. 2, pp. 1058–1063, December 2003.