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Research Article

Lyapunov Stability of Quasiperiodic Systems

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We present some observations on the stability and reducibility of quasiperiodic systems. In a quasiperiodic system, the periodicity of parametric excitation is incommensurate with the periodicity of certain terms multiplying the state vector. We present a Lyapunov-type approach and the Lyapunov-Floquet (L-F) transformation to derive the stability conditions. This approach can be utilized to investigate the robustness, stability margin, and design controller for the system.

1. Introduction

A large class of engineering systems, such as structures subjected to quasiperiodic excitations, is described by linear ordinary differential equations with time varying coefficients. These linear systems, in general, are described as

$$\dot{\mathbf{y}} = \mathbf{A}(t)\mathbf{y}, \tag{1.1}$$

where $\mathbf{A}(t)$ is an $n \times n$ quasiperiodic matrix and $\mathbf{y}$ is an n dimensional vector. In general, it is not a trivial problem to determine if (1.1) is asymptotically stable, simply stable, or unstable. The researchers have used perturbation-type techniques or numerical approaches to investigate the stability of this system [1–3].

In this work, we address the stability of a special class of quasiperiodic systems called as periodic quasiperiodic systems where (1.1) can be written as

$$\dot{\mathbf{y}} = [\mathbf{A}_0(t) + \mathbf{A}_1(t)]\mathbf{y}, \tag{1.2}$$

where $\mathbf{A}_0(t)$ has the principal period T and $\mathbf{A}_1(t)$ has the period T_1. It is noted that these periods are incommensurate. These types of equations arise in parametrically excited Micro

Electro Mechanical Systems (MEMS) [4]. It is noted that $\mathbf{A}(t)$ has a strong parametric excitation. In this paper, we present the methodology to investigate the stability of system given by (1.2) using the L-Ftransformation and Lyapunov's method.

This paper is organized as follows. In Section 2, a brief mathematical background on the L-F transformation is provided. Section 3 discusses the stability conditions followed by an example. We present the L-F-transformation-type approach for quasiperiodic systems in Section 4. The discussion and conclusions are presented in Section 5.

2. Mathematical Background

2.1. Floquet Theory and L-F Transformation

Consider (1.1), if $\mathbf{A}_1(t) = 0$, that is, the system is purely time periodic, then the State Transition Matrix (STM) $\mathbf{\Phi}(t)$ of (1.2) can be factored as [5]

$$\mathbf{\Phi}(t) = \mathbf{Q}(t)e^{\mathbf{R}t}, \qquad \mathbf{Q}(t) = \mathbf{Q}(t+2T), \qquad \mathbf{Q}(0) = \mathbf{I}, \tag{2.1}$$

where the matrix $\mathbf{Q}(t)$ is real and periodic with period $2T$, $\mathbf{R}$ is an $n \times n$ real time invariant matrix, and $\mathbf{I}$ is the identity matrix. Matrix $\mathbf{Q}(t)$ is known as the L-F transformation matrix [5].

The transformation $\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ produces a real-time invariant representation of purely time periodic system ((1.2) with $\mathbf{A}_1(t) = 0$) given by

$$\dot{\mathbf{z}}(t) = \overline{\mathbf{A}}\mathbf{z}(t). \tag{2.2}$$

It is to be noted that matrix $\overline{\mathbf{A}}$ in (2.2) is time invariant.

2.2. Construction of Lyapunov Functions

Lyapunov's direct method is widely used in the stability analysis of general dynamical systems. It makes use of a Lyapunov function $V(\mathbf{x}, t)$. This scalar function of the state and time may be considered as some form of time-dependent generalized energy. The basic idea of the method is to utilize the time rate of energy change in $V(\mathbf{x}, t)$ for a given system to judge whether the system is stable or not. The details about Lyapunov's method and stability theorems can be found in reference [6].

For a linear system with constant coefficients, it is rather simple to find a Lyapunov function. Consider the linear system

$$\dot{\underset{\sim}{\mathbf{x}}}(t) = \widetilde{\mathbf{A}}\underset{\sim}{\mathbf{x}}(t), \tag{2.3}$$

where $\widetilde{\mathbf{A}}$ is a constant matrix. A quadratic form of $V(\mathbf{x})$ may be assumed as

$$V\left(\underset{\sim}{\mathbf{x}}\right) = \underset{\sim}{\mathbf{x}}^T \mathbf{P} \underset{\sim}{\mathbf{x}}, \tag{2.4}$$

where $\mathbf{P}$ is a real, symmetric, and positive definite matrix. Then

$$\dot{V}\left(\underset{\sim}{\mathbf{x}}\right) = \dot{\underset{\sim}{\mathbf{x}}}^T \mathbf{P}\underset{\sim}{\mathbf{x}} + \underset{\sim}{\mathbf{x}}^T \mathbf{P}\dot{\underset{\sim}{\mathbf{x}}} = \left(\tilde{\mathbf{A}}\underset{\sim}{\mathbf{x}}\right)^T \mathbf{P}\underset{\sim}{\mathbf{x}} + \underset{\sim}{\mathbf{x}}^T \mathbf{P}\tilde{\mathbf{A}}\underset{\sim}{\mathbf{x}} \tag{2.5}$$

or

$$\dot{V}\left(\underset{\sim}{\mathbf{x}}\right) = \underset{\sim}{\mathbf{x}}^T\left(\tilde{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}}\right)\underset{\sim}{\mathbf{x}}. \tag{2.6}$$

According to the Lyapunov theorem for autonomous systems, if $\dot{V}(\mathbf{x})$ is negative definite, then the system is asymptotically stable [6]. Therefore, one can write

$$\tilde{\mathbf{A}}^T\mathbf{P} + \mathbf{P}\tilde{\mathbf{A}} = -\mathbf{C}, \tag{2.7}$$

where $\mathbf{C}$ is a positive definite matrix. Equation (2.7) is called the Lyapunov equation. It has been shown by Kalman and Bertram [7] that if there were eigenvalues with negative real parts (asymptotically stable), then for every given positive definite matrix $\mathbf{C}$, there exists a unique Lyapunov matrix $\mathbf{P}$. In this study, matrix $\mathbf{C}$ is always taken as the identity matrix.

3. Stability of Quasiperiodic Systems

Consider the quasiperiodic linear differential equation given by (1.2). In order to determine the stability bounds on $\mathbf{A}_1(t)$, we first use the L-F transformation $\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t)$ to (1.2). After the L-F transformation, (1.2) can be written as

$$\dot{\mathbf{z}} = \left[\overline{\mathbf{A}} + \mathbf{G}(t)\right]\mathbf{z}, \tag{3.1}$$

where $\mathbf{G}(t) = \mathbf{Q}^{-1}(t)\mathbf{A}_1(t)\mathbf{Q}(t)$. It is to be noted that $\overline{\mathbf{A}}$ is a constant matrix whose eigenvalues have negative real parts. We follow the approach presented by Infante [8] to obtain stability bounds.

Theorem 3.1 (see [8]). *If, for some positive definite matrix* $\mathbf{B}$ *and some* $\varepsilon > 0$*,*

$$E\left\{\lambda_{\max}\left[\overline{\mathbf{A}}^T + \mathbf{G}(t)^T + \mathbf{B}\left[\overline{\mathbf{A}} + \mathbf{G}(t)\right]\mathbf{B}^{-1}\right]\right\} \leq -\varepsilon, \tag{3.2}$$

then (3.1) *is almost surely asymptotically stable in the large, where* $E\{\cdot\}$ *is the expectation operator and* $\lambda_{\max}$ *is maximum real eigenvalues of a pencil [9].*

Proof. Consider the quadratic (Lyapunov) function $V(\mathbf{z}) = \mathbf{z}^T\mathbf{B}\mathbf{z}$. Then along the trajectories of (3.1), define

$$\lambda(t) = \frac{\dot{V}(\mathbf{z})}{V(\mathbf{z})} = \frac{\mathbf{z}^T\left[\left(\overline{\mathbf{A}} + \mathbf{G}(t)\right)^T\mathbf{B} + \mathbf{B}\left(\overline{\mathbf{A}} + \mathbf{G}(t)\right)\right]\mathbf{z}}{\mathbf{z}^T\mathbf{B}\ \mathbf{z}}. \tag{3.3}$$

It is noted that the numerator and denominator in (3.3) are quadratic forms. The pencil of quadratic forms $\widehat{\mathbf{A}}(\mathbf{x},\mathbf{x}) = \sum_{i,k=1}^{n} a_{ik}x_i x_k$ and $\widehat{\mathbf{B}}(\mathbf{x},\mathbf{x}) = \sum_{i,k=1}^{n} b_{ik}x_i x_k$ is a matrix-valued function defined over complex numbers λ given by $\widehat{\mathbf{A}}(\mathbf{x},\mathbf{x}) + \lambda\widehat{\mathbf{B}}(\mathbf{x},\mathbf{x})$ [9]. From the properties of pencils of quadratic forms [9], we can obtain the following inequality:

$$\lambda_{\min}\left[\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)^T + \mathbf{B}\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)\mathbf{B}^{-1}\right] \le \lambda(t) \le \lambda_{\max}\left[\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)^T + \mathbf{B}\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)\mathbf{B}^{-1}\right], \tag{3.4}$$

where $\lambda_{\max}$ is defined before and $\lambda_{\min}$ is the minimum real eigenvalues of a pencil. Consider $\|\mathbf{z}\|_p = (\mathbf{z}^T\mathbf{B}\mathbf{z})^{1/2}$; it can be shown [10] that $\|\mathbf{z}\|_p$ satisfies

$$\frac{d}{dt}\log\|\mathbf{z}\|_p = \frac{\mathbf{z}^T\left[\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)^T\mathbf{B} + \mathbf{B}\ \left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)\right]\mathbf{z}}{\mathbf{z}^T\mathbf{B}\ \mathbf{z}}. \tag{3.5}$$

Thus, integrating and dividing (3.5) by t,

$$\frac{1}{t}\left[\log\|\mathbf{z}(t)\|_p - \log\|\mathbf{z}(0)\|_p\right] = \frac{1}{t}\int_0^t \frac{\mathbf{z}^T\left[\left(\overline{\mathbf{A}}+\mathbf{G}(s)\right)^T\mathbf{B} + \mathbf{B}\ \left(\overline{\mathbf{A}}+\mathbf{G}(s)\right)\right]\mathbf{z}}{\mathbf{z}^T\mathbf{B}\ \mathbf{z}}ds. \tag{3.6}$$

For $(1/t)[\log\|\mathbf{z}(t)\|_p - \log\|\mathbf{z}(0)\|_p] < 0$ as $t \to \infty$ follows $\lim_{t\to\infty}\|\mathbf{z}\| = 0$. Thus, algebraic sign of $\lim_{t\to\infty}[(1/t)\int_0^t((\mathbf{z}^T[(\overline{\mathbf{A}}+\mathbf{G}(s))^T\mathbf{B}+\mathbf{B}\ (\overline{\mathbf{A}}+\mathbf{G}(s))]\ \mathbf{z})/\mathbf{z}^T\mathbf{B}\ \mathbf{z})ds]$ provides the condition for stability [10]. The solution of (3.3) can be given as

$$V[\mathbf{z}(t)] = V[\mathbf{z}(t_0)]e^{\int_{t_0}^t \lambda(\tau)d\tau} \equiv V[\mathbf{z}(t_0)]e^{(t-t_0)[(1/(t-t_0))\int_{t_0}^t \lambda(\tau)d\tau]}. \tag{3.7}$$

It can be observed that if $E\{\lambda(t)\} \le -\varepsilon$ for some $\varepsilon > 0$, $V[\mathbf{z}(t)]$ is bounded and that $V[\mathbf{z}(t)] \to 0$ as $t \to \infty$. This is the condition imposed by inequality given by (3.4), which proves the results. Since $\mathbf{y}(t) = \mathbf{Q}(t)\mathbf{z}(t)$, the stability of (3.1) implies the stability of (1.2).

It is remarked that a necessary condition for inequality (3.4) to hold is that the eigenvalues of matrix $\overline{\mathbf{A}}$ have negative real parts. It is also possible to obtain a result that is easier to compute but not as sharp. □

Corollary 3.2. *If, for some positive definite matrix* $\mathbf{B}$ *and some* $\varepsilon > 0$*,*

$$E\left\{\lambda_{\max}\left[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}\right]\right\} \le -\lambda_{\max}\left[\overline{\mathbf{A}}^T + \mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}\right] - \varepsilon, \tag{3.8}$$

then (3.1) *is almost surely asymptotically stable in the large.*

Proof. The proof follows immediately from theorem by noting that

$$\lambda(t) \le \lambda_{\max}\left[\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)^T + \mathbf{B}\left(\overline{\mathbf{A}}+\mathbf{G}(t)\right)\mathbf{B}^{-1}\right] \le \lambda_{\max}\left[\overline{\mathbf{A}}^T + \mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}\right] + \lambda_{\max}\left[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}\right]. \tag{3.9}$$

The second inequality is obtained by performing two maximizations separately. Further, using an $E\{\cdot\}$ operator

$$E\{\lambda(t)\} \leq \lambda_{\max}\left[\overline{\mathbf{A}}^T + \mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}\right] + E\left\{\lambda_{\max}\left[\mathbf{G}^T(t) + \mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}\right]\right\} \leq -\varepsilon, \tag{3.10}$$

yields the desired result. It is obvious that, unless the second inequality in (3.10) is an equality, the stability results obtained will not be as good as those given by the theorem. It is noted that this theorem and corollary can be extended to study stability and robustness of a linear time-periodic system subjected to random perturbations in a straightforward fashion, and for the details, we refer the reader to reference [11]. □

Example 3.3. Consider the system

$$\dot{\mathbf{y}} = \left[\tilde{\mathbf{A}}(t) + \hat{\mathbf{A}}(t)\right]\mathbf{y}, \tag{3.11}$$

where

$$\tilde{\mathbf{A}}(t) = \omega\begin{bmatrix} -1 + \alpha\cos^2(\omega t) & 1 - \alpha\sin(\omega t)\cos(\omega t) \\ -1 - \alpha\sin(\omega t)\cos(\omega t) & -1 + \alpha\sin^2(\omega t) \end{bmatrix}, \qquad \hat{\mathbf{A}}(t) = \begin{bmatrix} 0 & 0 \\ f(t) & 0 \end{bmatrix} \tag{3.12}$$

α is a system parameter and $\omega = 2\pi$. The state transition matrix (STM), $\mathbf{\Phi}(t)$, when the quasiperiodic term $\hat{\mathbf{A}}(t) = 0$, is given as [12]

$$\mathbf{\Phi}(t) = \begin{bmatrix} e^{(\alpha-1)\omega t}\cos(\omega t) & e^{-\omega t}\sin(\omega t) \\ -e^{(\alpha-1)\omega t}\sin(\omega t) & e^{-\omega t}\cos(\omega t) \end{bmatrix} = \mathbf{Q}(t)e^{\mathbf{R}t}. \tag{3.13}$$

Factoring the state transition matrix as shown above, the L-F transformation matrix $\mathbf{Q}(t)$ is found as

$$\mathbf{Q}(t) = \begin{bmatrix} \cos(\omega t) & \sin(\omega t) \\ -\sin(\omega t) & \cos(\omega t) \end{bmatrix}, \qquad e^{\mathbf{R}t} = \begin{bmatrix} e^{(\alpha-1)\omega t} & 0 \\ 0 & e^{-\omega t} \end{bmatrix}. \tag{3.14}$$

It is noted that the system is unstable for all $\alpha > 1$. Using the L-F transformation $\mathbf{z}(t) = \mathbf{Q}(t)\mathbf{y}(t)$ (c.f. (3.14)) (3.11) to yield a time-invariant system given by

$$\dot{\mathbf{z}}(t) = \overline{\mathbf{A}}\mathbf{z}(t). \tag{3.15}$$

Let $\mathbf{V} = \mathbf{z}^T(t)\mathbf{B}\mathbf{z}(t)$, where $\mathbf{B}$ is a constant, symmetric, positive definite matrix.

Then

$$\dot{\mathbf{V}} = \dot{\mathbf{z}}^T\mathbf{B}\mathbf{z} + \mathbf{z}^T\mathbf{B}\dot{\mathbf{z}} = \mathbf{z}^T\left[\overline{\mathbf{A}}^T\mathbf{B} + \mathbf{B}\overline{\mathbf{A}}\right]\mathbf{z} \equiv -\mathbf{z}^T\mathbf{C}\mathbf{z}. \tag{3.16}$$

Setting

$$\mathbf{B}=\begin{bmatrix}B_{11} & B_{12}\\ B_{12} & B_{22}\end{bmatrix},\qquad \mathbf{C}=\mathbf{I}_2=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \tag{3.17}$$

substituting (3.17) into (3.16), yields $B_{11}=-1/(2\omega(\alpha-1))$ $(\alpha<1)$, $B_{12}=0$ and ...
Therefore,

$$\mathbf{B}=\begin{bmatrix}-\dfrac{1}{2\omega(\alpha-1)} & 0\\ 0 & \dfrac{1}{2\omega}\end{bmatrix},\qquad \mathbf{B}^{-1}=\begin{bmatrix}-2\omega(\alpha-1) & 0\\ 0 & 2\omega\end{bmatrix}. \tag{3.18}$$

Since $B_{11}>0$ for $\alpha<1$ and

$$\operatorname{Det}(\mathbf{B})=\begin{vmatrix}-\dfrac{1}{2\omega(\alpha-1)} & 0\\ 0 & \dfrac{1}{2\omega}\end{vmatrix}=-\frac{1}{2\omega^2(\alpha-1)}>0. \tag{3.19}$$

Therefore, $\mathbf{B}$ is a positive definite symmetric matrix and Lyapunov stability conditions are satisfied.

Once the $\mathbf{B}$ matrix is constructed, the stability theorem and the corollary can be used to determine the stability conditions for the system. Simple computations yield

$$\begin{aligned}\mathbf{D}&=\overline{\mathbf{A}}^T+\mathbf{G}^T(t)+\mathbf{B}\left[\overline{\mathbf{A}}+\mathbf{G}(t)\right]\mathbf{B}^{-1}\\ &=\begin{bmatrix}2\omega(\alpha-1)+2f(t)\sin(\omega t)\cos(\omega t) & \dfrac{-\left[(\alpha-2)\cos^2(\omega t)+1\right]f(t)}{\alpha-1}\\ \left[(\alpha-2)\cos^2(\omega t)+1\right]f(t) & -2\omega-2f(t)\sin(\omega t)\cos(\omega t)\end{bmatrix}.\end{aligned} \tag{3.20}$$

Setting $\operatorname{Det}[\mathbf{D}-\lambda\mathbf{I}]=0$, the eigenvalues λ of the $\mathbf{D}$ matrix are computed as

$$\begin{aligned}\lambda_{1,2}&=-\omega(2-\alpha)\\ &\pm\sqrt{\omega^2\alpha^2+\frac{1}{\alpha-1}\left[2\omega\alpha(\alpha-1)f(t)\sin(\omega t)+\left(2\alpha\cos^2(\omega t)-\alpha^2\cos^4(\omega t)-1\right)f^2(t)\right]}.\end{aligned} \tag{3.21}$$

Application of the theorem yields

$$\begin{aligned}&E\{\lambda_{\max}[\mathbf{D}]\}\\ &\quad=-\omega(2-\alpha)\\ &\qquad+E\left\{\sqrt{\omega^2\alpha^2+\frac{1}{\alpha-1}\left[2\omega\alpha(\alpha-1)f(t)\sin(\omega t)+\left(2\alpha\cos^2(\omega t)-\alpha^2\cos^4(\omega t)-1\right)f^2(t)\right]}\right\}\leq 0\end{aligned} \tag{3.22}$$

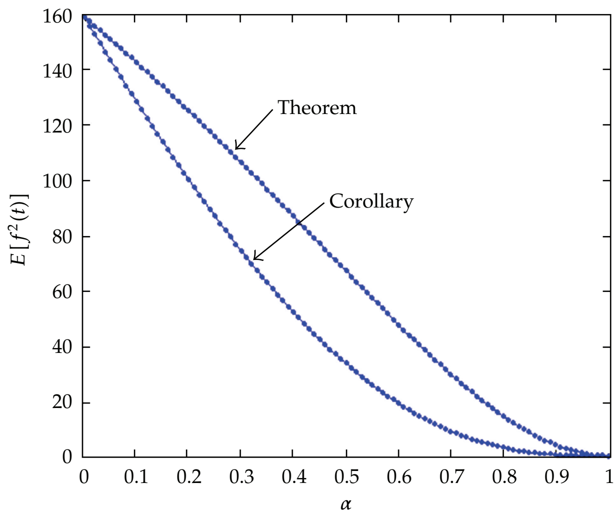

Figure 1: Stability results for example 1 obtained by the Theorem and Corollary.

or

$$E\left\{\sqrt{\omega^2\alpha^2+\frac{1}{\alpha-1}\left[2\omega\alpha(\alpha-1)f(t)\sin(\omega t)+\left(2\alpha\cos^2(\omega t)-\alpha^2\cos^4(\omega t)-1\right)f^2(t)\right]}\right\}\le\omega(2-\alpha). \tag{3.23}$$

Using Schwarz's Inequality [13], $(E\{f(t)\})^2 \le E\{f^2(t)\}$, and simplification yields

$$E\left\{f^2(t)\right\}\le\frac{32\omega^2(1-\alpha)^2}{8+3\alpha^2-8\alpha}. \tag{3.24}$$

The results obtained from condition (3.24) for α from 0 to 1 are shown in Figure 1.

In order to get the conditions for almost sure asymptotic stability from the corollary, matrices $[\mathbf{G}^T(t)+\mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}]$ and $[\overline{\mathbf{A}}+\mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}]$ are calculated as

$$\begin{aligned}\mathbf{G}^T(t)+\mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}&=f(t)\begin{bmatrix}2\sin(\omega t)\cos(\omega t) & -\dfrac{(\alpha-2)\cos^2(\omega t)+1}{\alpha-1}\\ (\alpha-2)\cos^2(\omega t)+1 & -2\sin(\omega t)\cos(\omega t)\end{bmatrix},\\ \overline{\mathbf{A}}+\mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}&=\begin{bmatrix}2(\alpha-1)\omega & 0\\ 0 & -2\omega\end{bmatrix}.\end{aligned} \tag{3.25}$$

The maximum eigenvalues of matrices given by (3.25) are computed as

$$\begin{aligned}\lambda_{\max}\left[\mathbf{G}^T(t)+\mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}\right]&=|f(t)|\sqrt{\frac{1-2\alpha\cos^2(\omega t)+\alpha^2\cos^4(\omega t)}{1-\alpha}},\\ \lambda_{\max}\left[\overline{\mathbf{A}}^{\mathbf{T}}+\mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}\right]&=-2(1-\alpha)\omega.\end{aligned} \tag{3.26}$$

Applying the corollary

$$E\left\{\lambda_{\max}\left[\mathbf{G}^{T}(t)+\mathbf{B}\mathbf{G}(t)\mathbf{B}^{-1}\right]\right\} \leq -\lambda_{\max}\left[\overline{\mathbf{A}}^{T}+\mathbf{B}\overline{\mathbf{A}}\mathbf{B}^{-1}\right]-\varepsilon \tag{3.27}$$

yields

$$E\left\{|f(t)|\sqrt{\frac{1-2\alpha\cos^{2}(\omega t)+\alpha^{2}\cos^{4}(\omega t)}{1-\alpha}}\right\} \leq 2(1-\alpha)\omega. \tag{3.28}$$

Then using Schwarz's Inequality in (3.28), one obtains

$$E\left\{f^{2}(t)\left[\frac{1-2\alpha\cos^{2}(\omega t)+\alpha^{2}\cos^{4}(\omega t)}{1-\alpha}\right]\right\} \leq 4(1-\alpha)^{2}\omega^{2} \tag{3.29}$$

or

$$E\left\{f^{2}(t)\right\}\left[1-2\alpha E\left\{\cos^{2}\omega t\right\}+\alpha^{2}E\left\{\cos^{4}\omega t\right\}\right] \leq 4(1-\alpha)^{3}\omega^{2}. \tag{3.30}$$

Since $E\{\cos^{2}(\omega t)\}=1/2$ and $E\{\cos^{4}(\omega t)\}=3/8$, inequality (3.30) provides the condition for almost sure asymptotic stability from corollary as

$$E\left\{f^{2}(t)\right\} \leq \frac{32(1-\alpha)^{3}\omega^{2}}{8-8\alpha+3\alpha^{2}}. \tag{3.31}$$

As expected, condition (3.31) is weaker than condition (3.24). Figure 1 displays the result obtained from (3.31) for α in the range of 0 to 1. A comparison of conditions yielding from the theorem and corollary is shown in Figure 1.

4. L-F Transformation Approach for Quasiperiodic System

In the previous section, we presented the theorem and corollary that provide the bounds on the quasiperiodic term so that the system described by (1.2) is stable. Alternatively, one can use L-F transformation type approach to ascertain the stability of quasiperiodic system. Unlike the theorem and corollary, this approach does not need $\overline{\mathbf{A}}$ to have negative real parts (c.f. (3.1)).

Consider a quasiperiodic system given in second order form [14] by

$$\ddot{x}+(\delta+\alpha(\cos t+\cos\omega t))x=0, \tag{4.1}$$

where δ, α are constants, quasiperiodicity $\omega \approx p/q$ where p, q are integers and $\ddot{x} = d^2x/dt^2$. Equation (4.1) can be written in the state space form as

$$\dot{\tilde{\mathbf{x}}} = \tilde{\mathbf{A}}(t)\tilde{\mathbf{x}}, \tag{4.2}$$

where $\tilde{\mathbf{x}} = \{x \ \dot{x}\}^T$, $\tilde{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -(\delta+\alpha(\cos t+\cos\omega t)) & 0 \end{bmatrix}$.

Now using the transformation $t = 2q\tau$, (4.1) can be transformed to

$$\left[\frac{1}{4q^2}\right]\frac{d^2x}{d\tau^2} + (\cos 2q\tau + \cos 2p\tau)x = 0. \tag{4.3}$$

It can be noted that (4.3) is a time periodic system with principle period π. The stability of (4.3) is governed by the Floquet theory. It is possible to find out the State Transtion Matrix (STM) $\mathbf{\Phi}(\tau)$ at the end of the principle period (also called as the Floquet Transition Matrix (FTM) $\mathbf{\Phi}(\tau = \pi)$) numerically [14, 15] or analytically using Picard iterations [16, 17]. For the details on the analytical computation of STM using Picard iteration approximation, we refer to reference [16]. It is noted that the Picard iteration approach yields an approximate closed form symbolic expression of the STM for time periodic system.

If eigenvalues of the FTM are inside the unit circle, then (4.3) is asymptotically stable. If the eigenvalues are on the unit circle then the system is simply stable, and if the eigenvalues are outside the unit circle, then the system is unstable. The stability (or instability) of the time periodic system given by (4.3) implies the stability (or instability) of (4.1). It can be noted that the Floquet theory states that the STM ($\mathbf{\Phi}(\tau)$) of (4.3) can be partitioned as

$$\mathbf{\Phi}(\tau) = \mathbf{Q}(\tau)\mathbf{e}^{\mathbf{B}\tau}, \tag{4.4}$$

where $\mathbf{Q}(\tau)$ is the time periodic L-F transformation matrix and $\mathbf{B}$ is the constant matrix of appropriate dimensions. The eigenvalues of $\mathbf{B}$ are called the Floquet exponent and govern the stability of the time periodic system given by (4.3). For computation of the L-F transformation matrix via Chebyshev polynomials, we refer the reader to reference [18].

5. Conclusion

In this paper, simple and efficient computational techniques to guarantee sufficient conditions for almost sure asymptotic stability of periodic quasiperiodic systems have been presented. First, the L-F transformation has been utilized to convert the periodic part of time-periodic system to a time-invariant form. For the linear periodic-quasiperiodic system, a theorem and related corollary have been suggested using the results previously obtained by Infante [8]. In order to apply the theorem and the corollary successfully, it is observed that the eigenvalues of matrix $\overline{\mathbf{A}}$, which governs the stability of the system, must have negative real parts and matrix $\mathbf{B}$ must be positive definite. One example is presented to show the application. Another apporach pressed here is based on the Floquet-type approach, where a quasiperiodic system is approximated as a periodic system and the Floquet theory can be applied to investigate the stability. Unlike the Infante type approch, the Floquet approach does not require eigenvalues of matrix $\overline{\mathbf{A}}$ to have negative real parts. In certain cases, Floquet type decomposition for quasiperiodic system can be used to reduce quasiperiodic system to LTI system. It is expected that these methodology would be useful in studying stability and designing controllers for a number of MEMS, where governing differential equations have time periodic quasiperiodic coefficients. The approaches presented in this paper can be extended to study stability and robustness of a linear time-periodic system subjected to random perturbations.

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