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Research Article

Existence of Global Attractors for the Nonlinear Plate Equation with Memory Term

Lifang Niu and Jianwen Zhang

School of Mathematics, Taiyuan University of Technology, Yingze 79 West Street, Shanxi, Taiyuan 030024, China

Correspondence should be addressed to Jianwen Zhang, niulifang@tyut.edu.cn

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A two-dimensional nonlinear plate equation is revisited, which arises from the model of the viscoelastic thin rectangular plate with four edges supported. We establish that the system is exponentially decayed if the memory kernel satisfies the condition of the exponential decay. Furthermore, we show the existence of the global attractor by verifying the condition (C).

1. Introduction

In this paper, we investigate the nonlinear plate equation with memory type:

$$\begin{aligned}\rho u_{tt} + r_1 r_2 u_t + \phi(0)\triangle^2 u - \left(N_1 + \beta|u_x|^2\right)u_{xx} - \left(N_2 + \beta\left|u_y\right|^2\right)u_{yy} \\ + \int_0^\infty \phi'(s)\triangle^2 u(t-s)\,ds = r_1 f(x,y), \quad \text{in } \Omega \times R^+,\end{aligned} \tag{1.1}$$

verifying the initial conditions:

$$u(x,y,0) = u_0(x,y,t) \tag{1.2}$$

and the boundary conditions:

$$\begin{aligned}u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0, \\ u_{xx}(0,y,t) = u_{xx}(1,y,t) = u_{yy}(x,0,t) = u_{yy}(x,1,t),\end{aligned} \tag{1.3}$$

where $x, y \in [0,1]$, $t \in R, \eta, r_1, r_2, N_1, N_2, \beta$ are nonnegative constants, $\Omega = [0,1] \times [0,1]$ is a bounded domain with boundary $\partial\Omega$, $\phi(0), \phi(\infty) > 0$ and $\phi'(s) < 0$ for every $s \in R^+$.

The asymptotical behavior of solutions for the nonlinear plate equations had been studied by many authors [1–10]; of those, Santos and Junior [5] studied a kind of plate equation with memory type. Yang and Zhong [11] studies the plate equation:

$$u_{tt} + \alpha(x)g(u_t) + \Delta^2 u + \lambda u + f(u) = h(x), \quad x \in \Omega, \tag{1.4}$$

where $\Omega \subset R^n$ is a bounded domain and proves the existence of a global attractor in the space $H_0^1(\Omega) \times L^2(\Omega)$. After Yang and Zhong [11], Yue and Zhong [12] obtained the existence of a global attractor about some equations similar to (1.4). Xiao [13] discusses the long-time behavior of the plate equation:

$$\varepsilon u_{tt} + \Delta^2 u + \lambda u_t + \beta(x)u = f(x,u), \quad x \in \Omega = R^n, \ t \ge 0 \tag{1.5}$$

on the unbounded domain R^n and show that there exists a compact global attractor for the above equation satisfying certain initial-boundary data. Wang and Zhang [14] prove that the two-dimensional nonlinear equation

$$\rho u_{tt} + D\Delta^2 u + \varepsilon\mu u_t - \left(N_1 + \frac{T}{2}\int_\Omega u_x^2 dx\, dy\right)u_{xx} - \left(N_2 + \frac{T}{2}\int_\Omega u_y^2 dx\, dy\right)u_{yy} = 0 \tag{1.6}$$

has a global attractor in space $(H_0^1(\Omega) \bigcap L^2(\Omega))$.

2. Preliminaries

We denote by $H = L^2(\Omega), V = H_0^2(\Omega)$ endowed with the scalar product and the norm on H and $V (\cdot,\cdot), |\cdot|, ((\cdot,\cdot)), \|\cdot\|$, respectively, where $(u,v) = \int_\Omega u(x,y)v(x,y)dx\,dy$, $((u,v)) = \int_\Omega \Delta u(x,y)\Delta v(x,y)dx\,dy$. Define $D(A) = \{v \in V, Av \in H\}$, where $A = \Delta^2$. For the operator A, we assume that $A : D(A) \to H$ are isomorphism, and there exists $\alpha > 0$ such that $(Au,u) \ge \alpha\|u\|^2$, for all $u \in V$. We also define the power A^s of A for $s \in R$ which operates on the spaces $D(A^s)$, and we write $V_{2s} = D(A^s)$, $s \in R$. This is a Hilbert space with the inner product and norm defined

$$(u,v)_{2s} = (A^s u, A^s v), \quad \|u\|_{2s} = ((u,v)_{2s})^{1/2}, \quad \forall u,v \in D(A^s), \tag{2.1}$$

and A^r is an isomorphism from $D(A^s)$ onto $D(A^{s-r})$, for all $s, r \in R$. In particular, $D(A^0) = H$, $D(A^{1/2}) = V$, $D(A^{-1/2}) = V^*$, $D(A) \subset V \subset H = H^* \subset V^*$, where H^*, V^* are the dual space, respectively, and each space is dense in the following one and the injections are continuous. Using the Poincáre inequality we have

$$\|v\| \ge \lambda_1 |v|, \quad \forall v \in V, \tag{2.2}$$

where λ_1 denotes the first eigenvalue of $A^{1/2}$.

Let $\mu(s) = -\phi'(s)$, $\phi(\infty) = 1$ and assume that the memory kernel μ is required to satisfy the following assumptions:

(h_1) $\mu \in C^1(R^+) \bigcap L^1(R^+), \mu'(s) \leq 0$, for all $s \in R^+$;

(h_2) $\int_0^\infty \mu(s)ds = M > 0$;

(h_3) $\mu'(s) + \alpha\mu(s) \leq 0$, for all $s \in R^+$, $\alpha > 0$.

In view of h_1, let $L^2_\mu(R^+, H^2_0)$ be the Hilbert space of H^2_0-valued functions on R^+, endowed with the following inner product and the norm:

$$(\phi, \psi)_{\mu,V} = \int_0^\infty \mu(s)(\Delta\phi(s), \Delta\psi(s))ds \tag{2.3}$$

and $|\phi|^2_{\mu,V} = (\phi, \phi)_{\mu,V} = \int_0^\infty \mu(s)\|\phi\|^2 ds$. Finally, we introduce the following Hilbert spaces:

$$H_0 = V \times H \times L^2_\mu(R^+, V), \qquad H_1 = D(A) \times V \times L^2_\mu(R^+, D(A)). \tag{2.4}$$

We define

$$\eta^t(x, y, s) = u(x, y, t) - u(x, y, t - s) \widehat{=} \eta(x, y, s), \tag{2.5}$$

and (1.1) is transformed into the system

$$\begin{aligned} &\rho u_{tt} + r_1 r_2 u_t + \triangle^2 u - \left(N_1 + \beta|u_x|^2\right)u_{xx} - \left(N_2 + \beta\left|u_y\right|^2\right)u_{yy} \\ &\quad + \int_0^\infty \mu(s)\triangle^2\eta ds = r_1 f(x, y), \\ &\eta^t_t + \eta^t_s = u_t, \end{aligned} \tag{2.6}$$

where the second equation is obtained by differentiating (2.5). The corresponding initial-boundary value conditions are then given by

$$\begin{gathered} u(0, y, t) = u(1, y, t) = u(x, 0, t) = u(x, 1, t) = 0, \\ u_{xx}(0, y, t) = u_{xx}(1, y, t) = u_{yy}(x, 0, t) = u_{yy}(x, 1, t), \\ \eta^t(0, y, s) = \eta^t(1, y, s) = \eta^t(x, 0, s) = \eta^t(x, 1, s) = 0, \\ \eta^t_{xx}(0, y, s) = \eta^t_{xx}(1, y, s) = \eta^t_{yy}(x, 0, s) = \eta^t_{yy}(x, 1, s) = 0, \\ u(x, y, 0) = u_1(x, y), \qquad u_t(x, y, 0) = u_2(x, y), \\ \eta^0(x, y, s) = u_0(x, y, 0) - u_0(x, y, -s), \end{gathered} \tag{2.7}$$

where $u_1(x, y) = u_0(x, y, t)$, $u_2(x, y) = \partial_t u_0(x, y, t)|_{t=0}$, $x, y \in [0, 1]$, $t \geq 0$, $s \in R^+$.

According to the classical Faedo-Galerkin method it is easy to obtain the existence and uniqueness of solutions and the continuous dependence to the initial value, so we omit it and only give the following theorem.

Theorem 2.1 (see [15, 16]). *Let (h_1) hold and $f(x, y) \in L^2(\Omega)$. Then given any time interval $I = [0, 1]$, problems (2.6)-(2.7) have a unique solution (u, u_t, η) in I with initial data $(u_1, u_2, \eta_0) \in H_0$, and the mapping $(u_1, u_2, \eta_0) \rightarrow (u(t), u_t(t), \eta_s^t)$ is continuous in H_0.*

Thus, it admits to define a C^0 semigroup

$$S(t) : \{u_1, u_2, \eta_0\} \longrightarrow \{u(t), u_t(t), \eta_s^t\}, \quad t \in R^+, \tag{2.8}$$

and they map H_0 into themselves.

In addition, the following abstract results will be used in our consideration.

Theorem 2.2 (see [17]). *A C^0 semigroup $\{S(t)\}_{t\geq 0}$ in a Banach space X is said to satisfy condition (C) which arised by [17] if for any $\varepsilon > 0$ and for any bounded set B of X, there exists $t(B) > 0$ and a finite dimensional subspace X_1 of X, such that $\{\|PS(t)x\|_X, x \in B, t \geq t(B)\}$ is bounded and*

$$\|(I - P)S(t)x\|_X <_X \varepsilon, \tag{2.9}$$

where $P : X \rightarrow X_1$ is a bounded projector.

Lemma 2.3 (see [17]). *Let $\{S(t)\}_{t\geq 0}$ be a C^0 semigroup in a Hilbert space M. Then $\{S(t)\}_{t\geq 0}$ has a global attractor if and only if*

(1) *$\{S(t)\}_{t\geq 0}$ satisfies the condition C;*

(2) *there exists a bounded absorbing subset B of M.*

3. Global Attractor Λ in H_0

Theorem 3.1. *Assume (h_1)–(h_3) hold. Then the ball of $H_0, B_0 = B_{H_0}(0, R_1)$, centered at 0 with $R_1 = \sqrt{c/c_0\rho_0}$, is a bounded absorbing set in H_0 for the semigroup $\{S(t)\}_{t\geq 0}$.*

Proof. We fixed δ and take $\delta \in (0, \delta_0)$, where $\delta_0 = \min(r_1 r_2/4\rho, \lambda_1^2/2r_1 r_2)$.

First, taking the inner product of the first equation of (2.6) with $v = u_t + \delta u$, after computation we conclude

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left\{\rho|v|^2 + |\Delta u|^2 + N_1|u_x|^2 + N_2\left|u_y\right|^2 + \frac{\beta}{2}|u_x|^4 + \frac{\beta}{2}\left|u_y\right|^4\right\} \\ &\quad + I_1 + N_1\delta|u_x|^2 + N_2\delta\left|u_y\right|^2 + \beta\delta|u_x|^4 + \beta\delta\left|u_y\right|^4 + (\eta, v)_{\mu,V} = r_1(f, v), \end{aligned} \tag{3.1}$$

where $I_1 = -\delta\rho|v|^2 + r_1 r_2|v|^2 + \delta|\Delta u|^2 - (r_1 r_2\delta - \rho\delta^2)(u, v)$, and we can easily obtain

$$I_1 \geq \frac{\delta}{2}|\Delta u|^2 + \frac{1}{2}r_1 r_2|v|^2. \tag{3.2}$$

Combining with the second equation of (2.6) we have

$$(\eta, v)_{\mu,V} = \frac{1}{2}\frac{d}{dt}|\eta|^2_{\mu,V} + (\eta, \eta_s)_{\mu,V} + \delta(\eta, v)_{\mu,V}. \tag{3.3}$$

According to (h_2)–(h_3), we conclude

$$\begin{aligned}
(\eta, \eta_s)_{\mu,V} &= \frac{1}{2}\int_0^\infty \mu(s)\frac{d}{ds}\left|\Delta\eta^t(s)\right|^2 ds \\
&= -\frac{1}{2}\int_0^\infty \mu'(s)\frac{d}{ds}\left|\Delta\eta^t(s)\right|^2 ds \geq \frac{\alpha}{2}|\eta|^2_{\mu,V}, \\
\delta(\eta, v)_{\mu,V} &= \delta\int_0^\infty \mu(s)\left(\Delta\eta^t(s), \Delta u\right)ds \\
&\geq -\delta\left(\int_0^\infty \mu(s)\left|\Delta\eta^t(s)\right|^2 ds\right)^{1/2} \cdot \left(\int_0^\infty \mu(s)|\Delta u|^2 ds\right)^{1/2} \\
&\geq -\frac{\alpha}{4}\int_0^\infty \mu(s)ds - \frac{\delta^2}{\alpha}\int_0^\infty \mu(s)|\Delta u|^2 ds \\
&\geq -\frac{\alpha}{4}|\eta|^2_{\mu,v} - \frac{M\delta^2}{\alpha}|\Delta u|^2.
\end{aligned} \tag{3.4}$$

Integrating with (3.4), from (3.3), entails

$$(\eta, v)_{\mu,V} \geq \frac{1}{2}\frac{d}{dt}|\eta|^2_{\mu,v} + \frac{\alpha}{4}|\eta|^2_{\mu,v} - \frac{M\delta^2}{\alpha}|\Delta u|^2. \tag{3.5}$$

Write $E(t) = \{\rho|v|^2 + |\Delta u|^2 + N_1|u_x|^2 + N_2|u_y|^2 + (\beta/2)|u_x|^4 + (\beta/2)|u_y|^4 + |\eta|^2_{\mu,V}\}$, from (2.7), (3.2), and (3.5), we obtain

$$\begin{aligned}
&\frac{d}{dt}E(t) + \delta\left(1 - \frac{2M\delta}{\alpha}\right)|\Delta u|^2 + \frac{r_1 r_2}{2}|v|^2 + \frac{\alpha}{2}|\eta|^2_{\mu,V} \\
&\quad + 2\delta N_1|u_x|^2 + 2\delta N_2\left|u_y\right|^2 + \delta\beta|u_x|^4 + \delta\beta\left|u_y\right|^4 \leq \frac{2r_1}{r_2}|f|^2.
\end{aligned} \tag{3.6}$$

Take δ small enough, such that $1 - (2M\delta/\alpha) > 1/2$. Write $c = (2r_1/r_2)|f|^2$, $c_0 = \min\{\delta/2, r_1 r_2/2\rho, \alpha/2\}$, thus in line with (3.6), we have

$$\frac{d}{dt}E(t) + c_0 E(t) \leq C. \tag{3.7}$$

By the Gronwall Lemma, we conclude

$$E(t) \leq E(0)\exp(-c_0 t) + \frac{c}{c_0}\left(1 - \exp(c_0)\right), \qquad \forall t \geq 0. \tag{3.8}$$

Due to (3.8), write $\rho_0 = \min(\rho, 1)$, we have

$$|\Delta u|^2 \mid +|v|^2 \mid +|\eta|^2_{\mu,V} \leq \frac{1}{\rho_0} E(0)\exp(-c_0 t) + \frac{c}{c_0\rho_0}(1-\exp(c_0)), \quad \forall t \geq 0. \tag{3.9}$$

Write $R_1 = \sqrt{c/c_0\rho_0}$, we end up with

$$\lim\sup\left\{\|u\|^2 \mid +|v|^2 \mid +|\eta|^2_{\mu,V}\right\} \leq R_1^2, \quad \forall t \geq 0. \tag{3.10}$$

□

Theorem 3.2. *Suppose $f \in L^2(\Omega)$ and conditions (h_1)–(h_3) are hold. Then the solution semigroup $\{S(t)\}_{t\geq 0}$ associated with system (2.6) and (2.7) has a global attractor Λ in H_0, and it attracts all bounded subsets of H_0, in the norm of H_0.*

Proof. Applying Lemma 2.3, we only to prove that the condition (C) holds in H_0.

Let $\widetilde{w_i}$ be an orthonormal basis of $D(A)$ which consists of eigenvectors of A. It is also an orthonormal basis of H, V, respectively. The corresponding eigenvalues are denoted by

$$0 < \widetilde{\lambda_1} < \widetilde{\lambda_2} < \widetilde{\lambda_3} < \cdots, \quad \widetilde{\lambda_i} \longrightarrow \infty, \quad \text{as } i \longrightarrow \infty \tag{3.11}$$

with $A\widetilde{w_i} = \widetilde{\lambda_i}\widetilde{w_i}$, for all $i \in N$. We write $H_m = \operatorname{span}\{\widetilde{w_1}, \widetilde{w_2}, \ldots, \widetilde{w_m}\}$. For any $(u, u_t, \eta) \in H_0$, we decompose that $(u, u_t, \eta) = (u_1, u_{1t}, \eta_1) + (u_2, u_{2t}, \eta_2)$, where $(u_1, u_{1t}, \eta_1) = (P_m u, P_m u_t, P_m \eta)$, and $P_m : H \longrightarrow H_m$ is the orthogonal projector. Since $f \in H$, for any $\varepsilon > 0$, there exists some m, such that

$$|(I - P_m) f| < \frac{\varepsilon}{4}. \tag{3.12}$$

Taking the scalar product of the first equation of (2.6) in H with $v_2 = u_{2t} + Au_2$, combining with the second equation and using the same way with Theorem 3.1, we find

$$\frac{d}{dt}\widetilde{E(t)} + C_0\widetilde{E(t)} \leq \frac{2r_1}{r_2}|f|^2 \leq \frac{r_1\varepsilon^2}{8r_2}, \tag{3.13}$$

where $\widetilde{E(t)} = \{\rho|v_2|^2 + |\Delta u_2|^2 + N_1|u_{2x}|^2 + N_2|u_{2y}|^2 + (\beta/2)|u_{2x}|^4 + (\beta/2)|u_{2y}|^4 + |\eta_2|^2_{\mu,V}$. According to Theorem 3.1, some M_0 and t_0 exist, for any $t \geq t_0$, such that $E(t_0) \leq M_0$. By the Gronwall Lemma, we conclude

$$\begin{aligned}\widetilde{E(t)} &\leq E(t_0)\exp(-c_0(t - t_0)) + \frac{r_1\varepsilon^2}{8c_0 r_2} \\ &\leq M_0 \exp(-c_0(t - t_0)) + \frac{r_1\varepsilon^2}{8c_0 r_2}, \quad \forall t \geq t_0.\end{aligned} \tag{3.14}$$

Take t_1 large enough, such that $t_1 - t_0 \geq (1/c_0) \ln M_0/\varepsilon^2$, so we conclude

$$|v_2|^2 + |\Delta u_2|^2 + \left|\eta_2\right|^2_{\mu,V} \leq \frac{1}{\rho_0}\left(1 + \frac{r_1}{8c_0 r_2}\right)\varepsilon^2, \quad \text{for } t \geq t_0. \tag{3.15}$$

Thus $S\{t\}_{t\geq 0}$ satisfies the condition (C). □

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