Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 239357, 32 pages
doi:10.1155/2012/239357

Research Article

Application of the Hori Method in the Theory of Nonlinear Oscillations

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Received 12 November 2011; Accepted 25 January 2012

Academic Editor: Antonio F. Bertachini A. Prado

Some remarks on the application of the Hori method in the theory of nonlinear oscillations are presented. Two simplified algorithms for determining the generating function and the new system of differential equations are derived from a general algorithm proposed by Sessin. The vector functions which define the generating function and the new system of differential equations are not uniquely determined, since the algorithms involve arbitrary functions of the constants of integration of the general solution of the new undisturbed system. Different choices of these arbitrary functions can be made in order to simplify the new system of differential equations and define appropriate near-identity transformations. These simplified algorithms are applied in determining second-order asymptotic solutions of two well-known equations in the theory of nonlinear oscillations: van der Pol equation and Duffing equation.

1. Introduction

In da Silva Fernandes [1], the general algorithm proposed by Sessin [2] for determining the generating function and the new system of differential equations of the Hori method for noncanonical systems has been revised considering a new approach for the integration theory which does not depend on the auxiliary parameter t^* introduced by Hori [3, 4].

In this paper, this new approach is applied to the theory of nonlinear oscillations for a second-order differential equation and two simplified versions of the general algorithm are derived. The first algorithm is applied to systems of two first-order differential equations corresponding to the second-order differential equation, and the second algorithm is applied to the equations of variation of parameters associated with the original equation. According to these simplified algorithms, the determination of the unknown functions $T_j^{(m)}$ and $Z_j^{*(m)}$, defined in the mth-order equation of the algorithm of the Hori method, is not unique, since these algorithms involve at each order arbitrary functions of the constants of integration of the general solution of the new undisturbed system. Different choices of the arbitrary functions can be made in order to simplify the new system of differential equations and define appropriate near-identity transformations. The problem of determining second-order asymptotic solutions of two well-known equations in the theory of nonlinear oscillations—van der Pol and Duffing equations—is taken as example of application of the simplified algorithms. For van der Pol equation, two generating functions are determined: one of these generating functions is the same function obtained by Hori [4], and, the other function provides the well-known averaged equations of variation of parameters in the theory of nonlinear oscillations. For Duffing equation, only one generating function is determined, and the second-order asymptotic solution is the same solution obtained through Krylov-Bogoliubov method [5], through the canonical version of Hori method [6] or through a different integration theory for the noncanonical version of Hori method [7]. For completeness, brief descriptions of the noncanonical version of the Hori method [4] and the general algorithm proposed by Sessin [2] are presented in the next two sections.

2. Hori Method for Noncanonical Systems

The noncanonical version of the Hori method [4] can be briefly described as follows.

Consider the differential equations:

$$\frac{dz_j}{dt} = Z_j(z,\varepsilon), \quad j = 1,\ldots,n, \tag{2.1}$$

where $Z_j(z,\varepsilon)$, $j = 1,\ldots,n$, are the elements of the vector function $Z(z,\varepsilon)$. It is assumed that $Z(z,\varepsilon)$ is expressed in power series of a small parameter ε:

$$Z(z,\varepsilon) = Z^{(0)}(z) + \sum_{m=1} \varepsilon^m Z^{(m)}(z). \tag{2.2}$$

The system of differential equations described by $Z^{(0)}(z)$ is supposed to be solvable.

Let the transformation of variables $(z_1,\ldots,z_n) \rightarrow (\zeta_1,\ldots,\zeta_n)$ be generated by the vector function $T(\zeta,\varepsilon)$. This transformation of variables is such that the new system:

$$\frac{d\zeta_j}{dt} = Z_j^*(\zeta,\varepsilon), \quad j = 1,\ldots,n, \tag{2.3}$$

is easier to solve or captures essential features of the system. $Z_j^*(\zeta,\varepsilon)$, $j = 1,\ldots,n$, are the elements of the vector function $Z^*(\zeta,\varepsilon)$, also expressed in power series of ε:

$$Z^*(\zeta,\varepsilon) = Z^{*(0)}(\zeta) + \sum_{m=1} \varepsilon^m Z^{*(m)}(\zeta). \tag{2.4}$$

It is assumed that the vector function $T(\zeta,\varepsilon)$, that defines a near-identity transformation, is also expressed in powers series of ε:

$$T(\zeta,\varepsilon) = \sum_{m=1} \varepsilon^m T^{(m)}(\zeta). \tag{2.5}$$

Following Hori [4], the transformation of variables $(z_1,\ldots,z_n) \rightarrow (\zeta_1,\ldots,\zeta_n)$ generated by $T(\zeta,\varepsilon)$ is given by

$$z_j = \zeta_j + \sum_{k=1} \frac{1}{k!} D_T^k \zeta_j = e^{D_T}\zeta_j, \quad j = 1,\ldots,n. \tag{2.6}$$

For an arbitrary function $f(z)$, the expansion formula is given by

$$f(z) = f(\zeta) + \sum_{k=1} \frac{1}{k!} D_T^k f(\zeta) = e^{D_T} f(\zeta). \tag{2.7}$$

The operator D_T is defined by

$$\begin{aligned} D_T f(\zeta) &= \sum_{j=1}^{n} T_j \frac{\partial f}{\partial \zeta_j}, \\ D_T^n f(\zeta) &= D_T^{n-1}\left(\sum_{j=1}^{n} T_j \frac{\partial f}{\partial \zeta_j}\right). \end{aligned} \tag{2.8}$$

According to the algorithm of the perturbation method proposed by Hori [4], the vector functions Z and T are obtained, at each order in the small parameter ε, from the following equations:

$$\text{order 0:} \quad Z_j^{(0)} = Z_j^{*(0)}, \tag{2.9}$$

$$\text{order 1:} \quad \left[Z^{(0)}, T^{(1)}\right]_j + Z_j^{(1)} = Z_j^{*(1)}, \tag{2.10}$$

$$\begin{aligned} &\text{order 2:} \quad \left[Z^{(0)}, T^{(2)}\right]_j + \frac{1}{2}\left[Z^{(1)} + Z^{*(1)}, T^{(1)}\right]_j + Z_j^{(2)} = Z_j^{*(2)}, \\ &\qquad\qquad \vdots \end{aligned} \tag{2.11}$$

$j = 1,\ldots,n$, where $[\,]_j$ stands for the generalized Poisson brackets

$$[Z,T]_j = \sum_{k=1}^{n} \left[T_k \frac{\partial Z_j}{\partial \zeta_k} - Z_k \frac{\partial T_j}{\partial \zeta_k}\right]. \tag{2.12}$$

$Z^{*(0)}$, $Z^{(m)}$, $Z^{*(m)}$, and $T^{(m)}$ are written in terms of the new variables $\zeta_1,\ldots,\zeta_n$.

The mth-order equation of the algorithm can be put in the general form:

$$\left[Z^{*(0)}, T^{(m)}\right]_j + \Psi_j^{(m)} = Z_j^{*(m)}, \quad j = 1, \ldots, n, \tag{2.13}$$

where the functions $\Psi_j^{(m)}$ are obtained from the preceding orders.

3. The General Algorithm

The determination of the functions $Z_j^{*(m)}$ and $T_j^{(m)}$ from (2.13) is based on the following proposition presented in da Silva Fernandes [1].

Proposition 3.1. *Let F be a $n \times 1$ vector function of the variables $\zeta_1, \ldots, \zeta_n$, which satisfy the system of differential equations:*

$$\frac{d\zeta_j}{dt} = Z_j^{*(0)}(\zeta) + R_j^{*}(\zeta; \varepsilon), \quad j = 1, \ldots, n, \tag{3.1}$$

where $Z^{*(0)}$ describes an integrable system of differential equations:

$$\frac{d\zeta_j}{dt} = Z_j^{*(0)}(\zeta), \quad j = 1, \ldots, n, \tag{3.2}$$

a general solution of which is given by

$$\zeta_j = \zeta_j(c_1, \ldots, c_n, t), \quad j = 1, \ldots, n, \tag{3.3}$$

being $c_1, \ldots, c_n$ arbitrary constants of integration; then

$$\left[F, Z^{*(0)}\right]_j = \frac{\partial F_j}{\partial t} - \sum_{k=1}^{n} \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} F_k, \quad j = 1, \ldots, n. \tag{3.4}$$

A corollary of this proposition can be stated.

Corollary 3.2. *Consider the same conditions of Proposition 3.1 with the general solution of* (3.2) *given by*

$$\zeta_j = \zeta_j(c_1, \ldots, c_{n-1}, M), \quad j = 1, \ldots, n, \tag{3.5}$$

being $c_1, \ldots, c_{n-1}$ arbitrary constants of integration and $M = t + \tau$, where τ is an additive constant; then

$$\left[F, Z^{*(0)}\right]_j = \frac{\partial F_j}{\partial M} - \sum_{k=1}^{n} \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} F_k, \quad j = 1, \ldots, n. \tag{3.6}$$

Now, consider (2.13). According to Proposition 3.1, this equation can be put in the form:

$$\frac{\partial T_j^{(m)}}{\partial t} - \sum_{k=1}^{n} \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} T_k^{(m)} = \Psi_j^{(m)} - Z_j^{*(m)}, \quad j = 1, \ldots, n, \tag{3.7}$$

with $\Psi_j^{(m)}$ written in terms of the general solution (3.3) of the undisturbed system (3.2), involving n arbitrary constants of integration—$c_1, \ldots, c_n$. $Z_j^{*(m)}$ and $T_j^{(m)}$ are unknown functions.

Equation (3.7) is very similar to the one presented by Hori [4], which is written in terms of an auxiliary parameter t^* through an ordinary differential equation, that is,

$$\frac{dT_j^{(m)}}{dt^*} - \sum_{k=1}^{n} \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} T_k^{(m)} = \Psi_j^{(m)} - Z_j^{*(m)}, \quad j = 1, \ldots, n. \tag{3.8}$$

To determine $Z_j^{*(m)}$ and $T_j^{(m)}$, $j = 1, \ldots, n$, Hori [4] extends the averaging principle applied in the canonical version: $Z_j^{*(m)}$ are determined so that the $T_j^{(m)}$ are free from secular or mixed secular terms. However, this procedure is not sufficient to determine Z^* such that the new system of differential equations (2.3) becomes more tractable, and, a tractability condition is imposed

$$\left[Z^{(0)}, Z^*\right]_j = 0, \quad j = 1, \ldots, n. \tag{3.9}$$

This condition is analogous to the condition $\{F^{(0)}, F^*\} = 0$ in the canonical case, which provides the first integral $F_0(\xi, \eta) = \text{const}$, where ξ, η denotes the new set of canonical variables, and, $F^{(0)}$ and F^* are the undisturbed Hamiltonian and the new Hamiltonian, respectively, and $\{\}$ stands for Poisson brackets [3, 4].

In the next paragraphs, the general algorithm for determining $Z_j^{*(m)}$ and $T_j^{(m)}$, $j = 1, \ldots, n$, proposed by Sessin [2] and revised in da Silva Fernandes [1] is briefly presented.

Introducing the $n \times 1$ matrices:

$$T^{(m)} = \left(T_j^{(m)}\right), \quad \Psi^{(m)} = \left(\Psi_j^{(m)}\right), \quad Z^{*(m)} = \left(Z_j^{*(m)}\right), \quad j = 1, \ldots, n, \tag{3.10}$$

and the $n \times n$ Jacobian matrix

$$J(t) = \left(\frac{\partial Z_j^{*(0)}}{\partial \zeta_k}\right), \quad j, k = 1, \ldots, n, \tag{3.11}$$

the system of partial differential equations (3.7) can be put in the following matrix form:

$$\frac{\partial T^{(m)}}{\partial t} - J(t) T^{(m)} = \Psi^{(m)} - Z^{*(m)}. \tag{3.12}$$

The vector functions $Z^{*(m)}$ and $T^{(m)}$ are determined from the following equations:

$$Z^{*(m)} = \Delta_\zeta \frac{\partial}{\partial t}\left\{ \Delta_\zeta^{-1}\left[\Delta_\zeta D^{(m)} + \Delta_\zeta \int \Delta_\zeta^{-1}\Psi^{(m)} dt\right]_s \right\}, \tag{3.13}$$

$$T^{(m)} = \left[\Delta_\zeta D^{(m)} + \Delta_\zeta \int \Delta_\zeta^{-1}\Psi^{(m)} dt\right]_p, \tag{3.14}$$

where $\Delta_\zeta = [\partial\zeta_j(c_1, \ldots, c_n, t)/\partial c_k]$ is the Jacobian matrix associated to the general solution (3.3) of the undisturbed system (3.2), s denotes the secular or mixed secular terms, and p denotes the remaining part. $D^{(m)}$ is the $n \times 1$ vector, $D^{(m)} = (D_j^{(m)})$, which depends only on the arbitrary constants of integration $c_1, \ldots, c_n$ of the general solution (3.3). The choice of $D^{(m)}$ is arbitrary. Recall that in the integration process, the arbitrary constants of integration of the general solution (3.3) are taken as parameters.

Equations (3.13) and (3.14) assure that $T^{(m)}$ is free from secular or mixed secular terms. Moreover, these equations provide the tractability condition (3.9) as it will be shown in the case of nonlinear oscillation problems presented in the next section.

Finally, it should be noted that $D^{(m)}$ can be chosen at each order to simplify the generating function T and the new system of differential equations (2.3). This aspect is discussed thoroughly in the examples of Section 5.

4. Simplified Algorithms in the Theory of Nonlinear Oscillations

In this section two simplified algorithms will be derived from the general algorithm in the case of nonlinear oscillations described by a second-order differential equation of the general form:

$$\ddot{x} + x = \varepsilon f(x, \dot{x}). \tag{4.1}$$

The first algorithm is applied to the system of first-order differential equations with x and $\dot{x}$ as elements of the vector z, and, the second algorithm is applied to the system of equations of variation of parameters associated to the differential equation with c' and θ' as elements of the vector z; c' and $\theta'$are defined in (4.26).

4.1. Simplified Algorithm I

For completeness, we present now the first simplified algorithm [1]. Additional remarks are included at the end of section.

Introducing the variables $z_1 = x$ and $z_2 = \dot{x}$, (4.1) can be put in the form:

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = -z_1 + \varepsilon f(z_1, z_2). \tag{4.2}$$

According to the notation introduced in (2.1) and (2.2):

$$Z^{(0)} = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix}, \qquad Z^{(1)} = \begin{bmatrix} 0 \\ f(z_1, z_2) \end{bmatrix}. \tag{4.3}$$

Following the algorithm of the Hori method for noncanonical systems, one finds from zero-th-order equation, (2.9), that

$$Z^{*(0)} = \begin{bmatrix} \zeta_2 \\ -\zeta_1 \end{bmatrix}. \tag{4.4}$$

Applying Proposition 3.1, it follows that the undisturbed system (3.2) is given by

$$\dot{\zeta}_1 = \zeta_2, \qquad \dot{\zeta}_2 = -\zeta_1, \tag{4.5}$$

general solution of which can be written in terms of the exponential matrix as [2]:

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \end{bmatrix} = e^{Et} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{4.6}$$

where

$$e^{Et} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}, \tag{4.7}$$

and E is the symplectic matrix:

$$E = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4.8}$$

and c_i, $i = 1, 2$, are constants of integration. The Jacobian matrix Δ_ζ associated with the solution (4.6) is then given by

$$\Delta_\zeta = e^{Et}, \tag{4.9}$$

with inverse $\Delta_\zeta^{-1} = e^{-Et} = \Delta_\zeta^T$, since Δ_ζ is an orthogonal matrix.

In view of (4.6), the functions $\Psi_j^{(m)}$ defined at each order of the algorithm (see (2.13) or (3.7)) are expressed by Fourier series with multiples of t as arguments such that the vector function $\Psi_I^{(m)}$ can be written as

$$\Psi_I^{(m)} = \sum_{k=0}^{\infty} \begin{bmatrix} a_{k,I}^{(m)} \cos kt + b_{k,I}^{(m)} \sin kt \\ c_{k,I}^{(m)} \cos kt + d_{k,I}^{(m)} \sin kt \end{bmatrix}, \tag{4.10}$$

where the coefficients $a_{k,I}^{(m)}$, $b_{k,I}^{(m)}$, $c_{k,I}^{(m)}$, and $d_{k,I}^{(m)}$ are functions of the constants c_i, $i = 1, 2$. The Fourier series can also be put in matrix form:

$$\Psi_I^{(m)} = \sum_{k=0}^{\infty} \left(e^{Ekt} A_{k,I}^{(m)} + e^{-Ekt} B_{k,I}^{(m)} \right), \tag{4.11}$$

with

$$A_{k,I}^{(m)} = \frac{1}{2} \begin{bmatrix} a_{k,I}^{(m)} - d_{k,I}^{(m)} \\ b_{k,I}^{(m)} + c_{k,I}^{(m)} \end{bmatrix}, \qquad B_{k,I}^{(m)} = \frac{1}{2} \begin{bmatrix} a_{k,I}^{(m)} + d_{k,I}^{(m)} \\ c_{k,I}^{(m)} - b_{k,I}^{(m)} \end{bmatrix}. \tag{4.12}$$

The subscript I is introduced to denote the first simplified algorithm.

Substituting (4.9) and (4.11) into (3.13), one finds

$$Z_I^{*(m)} = e^{Et} \frac{\partial}{\partial t} \left\{ e^{-Et} \left[t e^{Et} A_{1,I}^{(m)} + \text{periodic terms} \right]_s \right\} = e^{Et} A_{1,I}^{(m)}. \tag{4.13}$$

On the other hand,

$$\left\langle e^{-Et} \Psi_I^{(m)} \right\rangle = A_{1,I}^{(m)}, \tag{4.14}$$

where $\langle\,\rangle$ stands for the mean value of the function.

Therefore, from (4.9), (4.13), and (4.14), it follows that

$$Z_I^{*(m)} = e^{Et} \left\langle e^{-Et} \Psi_I^{(m)} \right\rangle = \Delta_\zeta \left\langle \Delta_\zeta^{-1} \Psi_I^{(m)} \right\rangle. \tag{4.15}$$

The second equation of the general algorithm, (3.14) can be simplified as described bellow.

From (3.14), (4.14), and (4.15), one finds

$$t e^{Et} A_{1,I}^{(m)} = \Delta_\zeta \int \left\langle \Delta_\zeta^{-1} \Psi_I^{(m)} \right\rangle dt = \left[\Delta_\zeta D_I^{(m)} + \Delta_\zeta \int \Delta_\zeta^{-1} \Psi_I^{(m)} dt \right]_s. \tag{4.16}$$

On the other hand,

$$\Delta_\zeta D_I^{(m)} + \Delta_\zeta \int \Delta_\zeta^{-1} \Psi_I^{(m)} dt = \left[\Delta_\zeta D_I^{(m)} + \Delta_\zeta \int \Delta_\zeta^{-1} \Psi_I^{(m)} dt \right]_p + t e^{Et} A_{1,I}^{(m)}. \tag{4.17}$$

Thus, introducing (4.16) and (4.17) into (3.14), one finds

$$T_I^{(m)} = \left[\Delta_\zeta D_I^{(m)} + \Delta_\zeta \int \Delta_\zeta^{-1} \Psi_I^{(m)} dt \right]_p = \Delta_\zeta D_I^{(m)} + \Delta_\zeta \int \left[\Delta_\zeta^{-1} \Psi_I^{(m)} - \left\langle \Delta_\zeta^{-1} \Psi_I^{(m)} \right\rangle \right] dt. \tag{4.18}$$

Equations (4.15) and (4.18) define the first simplified form of the general algorithm applicable to the nonlinear oscillations problems described by (4.1) with x and $\dot{x}$ as elements of vector z.

Finally, we note that (4.15) satisfies the tractability condition (3.9) up to order m. In order to show this equivalence, one proceeds as follows. Since, from (4.14), $\langle \Delta_{\zeta}^{-1}\Psi_I^{(m)} \rangle$ does not depend explicitly on the time t, it follows that

$$\frac{\partial Z^{*(m)}}{\partial t} = \frac{\partial \Delta_{\zeta}}{\partial t}\left\langle \Delta_{\zeta}^{-1}\Psi_I^{(m)} \right\rangle = J\Delta_{\zeta}\left\langle \Delta_{\zeta}^{-1}\Psi_I^{(m)} \right\rangle = JZ^{*(m)}. \tag{4.19}$$

Using Proposition 3.1 and taking into account that $J = \partial Z_j^{*(0)}/\partial \zeta_k$, this equation can be put in the following form:

$$\left[Z^{*(m)}, Z^{(0)}\right]_j = \frac{\partial Z_j^{*(m)}}{\partial t} - \sum_{k=1}^{n} \frac{\partial Z_j^{*(0)}}{\partial \zeta_k} Z_k^{*(m)} = 0, \quad j = 1, 2, \tag{4.20}$$

which is the tractability condition (3.9) up to order m.

Remark 4.1. It should be noted that (4.15) and (4.18) for determining the vector functions $Z_I^{*(m)}$ and $T_I^{(m)}$, respectively, are invariant with respect to the form of the general solution of the undisturbed system described by $Z^{*(0)}$. This means that if the general solution of the undisturbed system is written in terms of a second set of constants of integration, for instance, if this solution is given by

$$\begin{aligned} \zeta_1 &= c\cos(t+\theta), \\ \zeta_2 &= -c\sin(t+\theta), \end{aligned} \tag{4.21}$$

where c and θ denote new constants of integration, then $Z_I^{*(m)}$ and $T_I^{(m)}$ are determined through (4.15) and (4.18), with the Jacobian matrix Δ_{ζ} given by

$$\Delta_{\zeta} = \begin{bmatrix} \cos(t+\theta) & -c\sin(t+\theta) \\ -\sin(t+\theta) & -c\cos(t+\theta) \end{bmatrix}. \tag{4.22}$$

This result can be proved as follows. The two sets of constants of integration (c_1, c_2) and (c, θ) are related through the following transformation:

$$\begin{aligned} c^2 &= c_1^2 + c_2^2, \\ \tan\theta &= -\frac{c_2}{c_1}. \end{aligned} \tag{4.23}$$

In view of this transformation, the Jacobian matrix Δ_{ζ}^1 can be written in terms of the Jacobian matrix Δ_{ζ}^2 as

$$\Delta_{\zeta}^1 = \Delta_{\zeta}^2 \Delta_C, \tag{4.24}$$

where the superscripts 1 and 2 are introduced to denote the form of the general solution of the undisturbed system described by $Z^{*(0)}$ with respect to the set of constants of integration (c_1, c_2) and (c, θ), respectively. Δ_C is the Jacobian matrix of the transformation. Since Δ_C does not depend on the time t, it follows from (4.15) and (4.18) that

$$\begin{aligned}
Z_I^{*(m)} &= \Delta_\zeta^1 \left\langle \left(\Delta_\zeta^1\right)^{-1} \Psi_I^{1(m)} \right\rangle = \Delta_\zeta^2 \Delta_C \left\langle \Delta_C^{-1} \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} \right\rangle \\
&= \Delta_\zeta^2 \Delta_C \Delta_C^{-1} \left\langle \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} \right\rangle = \Delta_\zeta^2 \left\langle \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} \right\rangle, \\
T_I^{(m)} &= \Delta_\zeta^1 D_I^{1(m)} + \Delta_\zeta^1 \int \left[\left(\Delta_\zeta^1\right)^{-1} \Psi_I^{1(m)} - \left\langle \left(\Delta_\zeta^1\right)^{-1} \Psi_I^{1(m)} \right\rangle \right] dt \\
&= \Delta_\zeta^2 \Delta_C D_I^{1(m)} + \Delta_\zeta^2 \Delta_C \int \left[\Delta_C^{-1} \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} - \left\langle \Delta_C^{-1} \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} \right\rangle \right] dt \\
&= \Delta_\zeta^2 \Delta_C D_I^{1(m)} + \Delta_\zeta^2 \Delta_C \Delta_C^{-1} \int \left[\left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} - \left\langle \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} \right\rangle \right] dt \\
&= \Delta_\zeta^2 D_I^{2(m)} + \Delta_\zeta^2 \int \left[\left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} - \left\langle \left(\Delta_\zeta^2\right)^{-1} \Psi_I^{2(m)} \right\rangle \right] dt.
\end{aligned} \tag{4.25}$$

Finally, we note that the general solution given by (4.21) is more suitable in practical applications than the general solution given by (4.6), that, in turn, is more suitable for theoretical purposes.

4.2. Simplified Algorithm II

In this section, a second simplified algorithm is derived from the general one. Introducing the transformation of variables $(x, \dot{x}) \rightarrow (c', \theta')$ defined by the following equations

$$\begin{aligned}
x &= c' \cos(t + \theta'), \\
\dot{x} &= -c' \sin(t + \theta'),
\end{aligned} \tag{4.26}$$

equation (4.1) is transformed into

$$\begin{aligned}
\frac{dc'}{dt} &= -\varepsilon f\left(c' \cos(t + \theta'), -c' \sin(t + \theta')\right) \sin(t + \theta'), \\
\frac{d\theta'}{dt} &= -\varepsilon \frac{1}{c'} f\left(c' \cos(t + \theta'), -c' \sin(t + \theta')\right) \cos(t + \theta').
\end{aligned} \tag{4.27}$$

These differential equations are the well-known variation of parameters equations associated to the second-order differential equation (4.1). Equation (4.27) define a nonautonomous system of differential equations.

The sets (c, θ) and (c', θ'), defined, respectively in (4.21) and (4.26), have different meanings in the theory: in (4.21), c and θ are constants of integration of the general solution of the new undisturbed system described by $Z^{*(0)}(\zeta_1, \zeta_2)$; in (4.26), c' and θ' are new variables which represent the constants of integration of the general solution of the original undisturbed system described by $Z^{(0)}(z_1, z_2)$ in the variation of parameter method. These sets, (c, θ) and (c', θ'), are connected through a near identity transformation.

Remark 4.2. It should be noted that a second transformation of variables involving a fast phase, $(x, \dot{x}) \to (c', \phi')$, can also be defined. This second transformation is given by

$$x = c' \cos \phi', \qquad \dot{x} = -c' \sin \phi'. \tag{4.28}$$

In this case, (4.1) is transformed into

$$\frac{dc'}{dt} = -\varepsilon f\left(c' \cos \phi', -c' \sin \phi'\right) \sin \phi', \qquad \frac{d\phi'}{dt} = 1 - \varepsilon \frac{1}{c'} f\left(c' \cos \phi', -c' \sin \phi'\right) \cos \phi'. \tag{4.29}$$

These equations define an autonomous system of differential equations. In what follows, the first set of variation of parameters equations, (4.27), will be considered.

Now, introducing the variables $z_1 = c'$ and $z_2 = \theta'$, one gets from (4.27) that

$$Z^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \qquad Z^{(1)} = \begin{bmatrix} -f(z_1 \cos(t + z_2), -z_1 \sin(t + z_2)) \sin(t + z_2) \\ -\dfrac{1}{z_1} f(z_1 \cos(t + z_2), -z_1 \sin(t + z_2)) \cos(t + z_2) \end{bmatrix}. \tag{4.30}$$

Applying Proposition 3.1, it follows that the undisturbed system (3.2) is given by

$$\dot{\zeta}_1 = 0, \qquad \dot{\zeta}_2 = 0, \tag{4.31}$$

and its general solution is very simple,

$$\zeta_i = c_i, \quad i = 1, 2, \tag{4.32}$$

where c_i, $i = 1, 2$, are constants of integration. The Jacobian matrix Δ_ζ associated with this general solution is also very simple, and it is given by

$$\Delta_\zeta = I, \tag{4.33}$$

where I is the identity matrix.

Substituting (4.33) into (3.13) and (3.14), it follows that

$$Z_{II}^{*(m)} = \frac{\partial}{\partial t}\left\{ \left[D_{II}^{(m)} + \int \Psi_{II}^{(m)} dt \right]_s \right\}, \qquad T_{II}^{(m)} = \left[D_{II}^{(m)} + \int \Psi_{II}^{(m)} dt \right]_p. \tag{4.34}$$

The subscript II is introduced to denote the second simplified algorithm.

Equation (4.34) can be put in a more suitable form as follows. In view of (4.30), the functions $\Psi_j^{(m)}$ defined at each order of the algorithm (see (2.13) or (3.7)) are expressed by Fourier series with multiples of $t + z_2$ as arguments such that the vector function $\Psi_{II}^{(m)}$ can be written as

$$\Psi_{II}^{(m)} = \sum_{k=0}^{\infty} \begin{bmatrix} a_{k,II}^{(m)} \cos k(t + c_2) + b_{k,II}^{(m)} \sin k(t + c_2) \\ c_{k,II}^{(m)} \cos k(t + c_2) + d_{k,II}^{(m)} \sin k(t + c_2) \end{bmatrix}, \tag{4.35}$$

where the coefficients $a_{k,II}^{(m)}$, $b_{k,II}^{(m)}$, $c_{k,II}^{(m)}$, and $d_{k,II}^{(m)}$ are functions of the constant c_1. The vector function $\Psi_{II}^{(m)}$ can also be put in matrix form:

$$\Psi_{II}^{(m)} = \sum_{k=0}^{\infty} \left(e^{Ek(t+c_2)} A_{k,II}^{(m)} + e^{-Ek(t+c_2)} B_{k,II}^{(m)} \right), \tag{4.36}$$

with

$$A_{k,II}^{(m)} = \frac{1}{2} \begin{bmatrix} a_{k,II}^{(m)} - d_{k,II}^{(m)} \\ b_{k,II}^{(m)} + c_{k,II}^{(m)} \end{bmatrix}, \qquad B_{k,II}^{(m)} = \frac{1}{2} \begin{bmatrix} a_{k,II}^{(m)} + d_{k,II}^{(m)} \\ c_{k,II}^{(m)} - b_{k,II}^{(m)} \end{bmatrix}. \tag{4.37}$$

Note that $\Psi_{II}^{(m)}$ is very similar to $\Psi_{I}^{(m)}$, defined by (4.11). They represent different forms of Fourier series of $\Psi^{(m)}$, but they are not the same, since they involve different sets of arbitrary constants of integration.

Thus, it follows from (4.36) that

$$D_{II}^{(m)} + \int \Psi_{II}^{(m)} dt = D_{II}^{(m)} + \left(A_{0,II}^{(m)} + B_{0,II}^{(m)} \right) t + \text{periodic terms}, \tag{4.38}$$

with the periodic terms given by

$$\sum_{k=1}^{\infty} \left((Ek)^{-1} e^{Ek(t+c_2)} A_{k,II}^{(m)} + (-Ek)^{-1} e^{-Ek(t+c_2)} B_{k,II}^{(m)} \right). \tag{4.39}$$

Therefore,

$$\begin{gathered} \left[D_{II}^{(m)} + \int \Psi_{II}^{(m)} dt \right]_s = D_{II}^{(m)} + \left(A_{0,II}^{(m)} + B_{0,II}^{(m)} \right) t, \\ \left[D_{II}^{(m)} + \int \Psi_{II}^{(m)} dt \right]_p = \sum_{k=1}^{\infty} \left((Ek)^{-1} e^{Ek(t+c_2)} A_{k,II}^{(m)} + (-Ek)^{-1} e^{-Ek(t+c_2)} B_{k,II}^{(m)} \right). \end{gathered} \tag{4.40}$$

Substituting (4.40) into (4.34), one finds

$$Z_{II}^{*(m)} = A_{0,II}^{(m)} + B_{0,II}^{(m)} = \left\langle \Psi_{II}^{(m)} \right\rangle, \tag{4.41}$$

$$T_{II}^{(m)} = D_{II}^{(m)} + \int \left(\Psi_{II}^{(m)} - \left\langle \Psi_{II}^{(m)} \right\rangle \right) dt. \tag{4.42}$$

Note that $D_{II}^{(m)}$ depends only on $c_1 = \zeta_1$.

It should be noted that (4.41) and (4.42) can be straightforwardly obtained from (3.12) by applying the averaging principle if $D_{II}^{(m)}$ is assumed to be zero, since in this second approach:

$$J(t) = \left(\frac{\partial Z_j^{*(0)}}{\partial \zeta_k} \right) = O, \tag{4.43}$$

where O denotes the null matrix. Thus, the general algorithm defined by (3.13) and (3.14) is equivalent to the averaging principle usually applied in the theory of nonlinear oscillations [5, 7].

Remark 4.3. Equations (4.41) and (4.42) are also obtained, if the second set of variation of parameters equations is considered (see Remark 4.2). In this case, the undisturbed system (3.2) is given by

$$\dot{\zeta}_1 = 0, \qquad \dot{\zeta}_2 = 1, \tag{4.44}$$

with general solution defined by

$$\zeta_1 = c_1, \qquad \zeta_1 = t + c_2, \tag{4.45}$$

and Jacobian matrix $\Delta_\zeta = I$.

Finally, we note that (4.41) is the tractability condition (3.9) up to order m. Since $\langle \Psi_{II}^{(m)} \rangle$ does not depend explicitly on the time t, it follows that

$$\frac{\partial Z_j^{*(m)}}{\partial t} = \left[Z^{*(m)}, Z^{(0)} \right]_j = 0, \quad j = 1, 2, \tag{4.46}$$

which is the tractability condition (3.9) up to order m.

5. Application to Nonlinear Oscillations Problems

In order to illustrate the application of the simplified algorithms, two examples are presented. The noncanonical version of the Hori method will be applied in determining second-order asymptotic solutions for van der Pol and Duffing equations. For the van der Pol equation, two different choices of the vector $D^{(m)}$ will be made, and two generating functions $T^{(m)}$ will be determined, one of these generating functions is the same function obtained by Hori [4] through a different approach, and, the other function gives the well-known averaged variation of parameters equations in the theory of nonlinear oscillations obtained through Krylov-Bogoliubov method [5]. It should be noted that the solution presented by Hori defines a new system of differential equations with a different frequency for the phase in comparison with the solution obtained by Ahmed and Tapley [7] and by Nayfeh [5], using different perturbation methods. For the Duffing equation, only one generating function is determined, and the second simplified algorithm gives the same generating function obtained through Krylov-Bogoliubov method.

The section is organized in two subsections: in the first subsection, the asymptotic solutions are determined through the first simplified algorithm, and, in the second subsection, they are determined through the second simplified algorithm.

5.1. Determination of Asymptotic Solutions through Simplified Algorithm I

5.1.1. Van der Pol Equation

Consider the well-known van der Pol equation:

$$\ddot{x} + \varepsilon\left(x^2 - 1\right)\dot{x} + x = 0. \tag{5.1}$$

Introducing the variables $z_1 = x$ and $z_2 = \dot{x}$, this equation can be written in the form:

$$\frac{dz_1}{dt} = z_2, \qquad \frac{dz_1}{dt} = -z_1 - \varepsilon\left(z_1^2 - 1\right)z_2. \tag{5.2}$$

Thus

$$Z^{(0)} = \begin{bmatrix} z_2 \\ -z_1 \end{bmatrix}, \tag{5.3}$$

$$Z^{(1)} = \begin{bmatrix} 0 \\ -(z_1^2 - 1)z_2 \end{bmatrix}. \tag{5.4}$$

As described in preceding paragraphs, two different choices of $D^{(m)}$ will be made, and two generating functions $T^{(m)}$ will be determined. Firstly, we present the solution obtained by Hori [4].

(1) First Asymptotic Solution: Hori's [4] Solution

Following the simplified algorithm I defined by (4.15) and (4.18), the first-order terms $Z^{*(1)}$ and $T^{(1)}$ are calculated as follows.

Introducing the general solution given by (4.21) of the undisturbed system described by $Z^{*(0)}(\zeta_1,\zeta_2)$ into (5.4), with ζ replacing z, one gets

$$Z^{(1)} = \begin{bmatrix} 0 \\ \left(-c + \frac{1}{4}c^3\right)\sin(t+\theta) + \frac{1}{4}c^3\sin 3(t+\theta) \end{bmatrix}. \tag{5.5}$$

Computing $\Delta_\zeta^{-1}Z^{(1)}$,

$$\Delta_\zeta^{-1}Z^{(1)} = \begin{bmatrix} \frac{1}{2}c\left(1-\frac{1}{4}c^2\right) - \frac{1}{2}c\cos 2(t+\theta) + \frac{1}{8}c^3\cos 4(t+\theta) \\ \frac{1}{2}\left(1-\frac{1}{2}c^2\right)\sin 2(t+\theta) - \frac{1}{8}c^2\sin 4(t+\theta) \end{bmatrix}, \tag{5.6}$$

and taking its secular part, one finds

$$\left\langle \Delta_\zeta^{-1}Z^{(1)} \right\rangle = \begin{bmatrix} \frac{1}{2}c\left(1-\frac{1}{4}c^2\right) \\ 0 \end{bmatrix}. \tag{5.7}$$

From (4.15) and (4.22), it follows that $Z^{*(1)}$ is given by

$$Z^{*(1)} = \begin{bmatrix} \frac{1}{2}c\left(1-\frac{1}{4}c^2\right)\cos(t+\theta) \\ -\frac{1}{2}c\left(1-\frac{1}{4}c^2\right)\sin(t+\theta) \end{bmatrix}. \tag{5.8}$$

In view of (4.21), $Z^{*(1)}$ can be written explicitly in terms of the new variables ζ_1 and ζ_2 as follows:

$$Z^{*(1)} = \begin{bmatrix} \frac{1}{2}\zeta_1\left(1-\frac{1}{4}(\zeta_1^2+\zeta_2^2)\right) \\ \frac{1}{2}\zeta_2\left(1-\frac{1}{4}(\zeta_1^2+\zeta_2^2)\right) \end{bmatrix}. \tag{5.9}$$

To determine $T^{(1)}$, the indefinite integral $\int[\Delta_\zeta^{-1}Z^{(1)} - \langle\Delta_\zeta^{-1}Z^{(1)}\rangle]dt$ is calculated:

$$\int\left[\Delta_\zeta^{-1}Z^{(1)} - \left\langle\Delta_\zeta^{-1}Z^{(1)}\right\rangle\right]dt = \begin{bmatrix} -\frac{1}{4}c\sin 2(t+\theta) + \frac{1}{32}c^3\sin 4(t+\theta) \\ -\left(\frac{1}{4}-\frac{1}{8}c^2\right)\cos 2(t+\theta) + \frac{1}{32}c^2\cos 4(t+\theta) \end{bmatrix}. \tag{5.10}$$

Thus, from (4.18), it follows that $T^{(1)}$ is given by

$$T^{(1)} = \begin{bmatrix} -\frac{1}{4}c\left(1-\frac{1}{4}c^2\right)\sin(t+\theta) - \frac{1}{32}c^3\sin 3(t+\theta) \\ \frac{1}{4}c\left(1-\frac{1}{4}c^2\right)\cos(t+\theta) - \frac{3}{32}c^3\cos 3(t+\theta) \end{bmatrix} + \Delta_\zeta D^{(1)}. \tag{5.11}$$

In view of (4.21), $T^{(1)}$ can be written explicitly in terms of the new variables ζ_1 and ζ_2 as follows:

$$T^{(1)} = \begin{bmatrix} \frac{1}{4}\zeta_2\left(1+\frac{1}{8}(\zeta_1^2+\zeta_2^2)\right) - \frac{1}{8}\zeta_2^3 \\ \frac{1}{4}\zeta_1\left(1+\frac{7}{8}(\zeta_1^2+\zeta_2^2)\right) - \frac{3}{8}\zeta_1^3 \end{bmatrix} + \Delta_\zeta D^{(1)}, \tag{5.12}$$

with $\Delta_\zeta D^{(1)}$ put in the form:

$$\Delta_\zeta D^{(1)} = \begin{bmatrix} d_1^{(1)}\zeta_1 + d_2^{(1)}\zeta_2 \\ d_1^{(1)}\zeta_2 - d_2^{(1)}\zeta_1 \end{bmatrix}, \tag{5.13}$$

being $d_i^{(1)} = d_i^{(1)}(c), i = 1,2, D_1^{(1)} = cd_1^{(1)}$, and $D_2^{(1)} = d_2^{(1)}$. The auxiliary vector $d^{(1)}$ is introduced in order to simplify the calculations, and, it is calculated in the second-order approximation as described below.

Following the algorithm of the Hori method described in Section 2, the second-order equation, (2.11), involves the term $\Psi^{(2)}$ given by

$$\Psi^{(2)} = \frac{1}{2}\left[Z^{(1)} + Z^{*(1)}, T^{(1)}\right]. \tag{5.14}$$

The determination of $\Psi^{(2)}$ is very arduous. The generalized Poisson brackets must be calculated in terms of ζ_1 and ζ_2 through (2.12), and, the general solution of the undisturbed system, defined by (3.1), must be introduced. It should be noted that $d_i^{(1)}$, $i = 1,2$, in (5.12) are functions of the new variables ζ_1 and ζ_2 through $c^2 = \zeta_1^2 + \zeta_2^2$. So, their partial derivatives must be considered in the calculation of the generalized Poisson brackets. After lengthy calculations performed using MAPLE software, one finds

$$\begin{aligned}
\left(\Delta_{\zeta}^{-1}\Psi^{(2)}\right)_1 &= \left(\frac{7}{64}c^3 - \frac{3}{128}c^5\right)\sin 2(t+\theta) \\
&\quad - \frac{1}{32}c^3 \sin 4(t+\theta) - \frac{1}{128}c^5 \sin 6(t+\theta) \\
&\quad + d_1^{(1)}\left(-\frac{1}{4}c^3 + \frac{1}{8}c^3\cos 4(t+\theta)\right) \\
&\quad + d_2^{(1)}\left(\frac{1}{2}c\sin 2(t+\theta) - \frac{1}{4}c^3 \sin 4(t+\theta)\right) \\
&\quad + \frac{dd_1^{(1)}}{d\zeta_1}\left(-\frac{1}{4}\left(c^2 - \frac{1}{4}c^4\right)\cos(t+\theta)\right) \\
&\quad + \frac{dd_1^{(1)}}{d\zeta_2}\left(\frac{3}{4}\left(c^2 - \frac{1}{4}c^4\right)\sin(t+\theta) - \frac{1}{8}c^4 \sin 3(t+\theta)\right), \\
\left(\Delta_{\zeta}^{-1}\Psi^{(2)}\right)_2 &= -\frac{1}{8} + \frac{3}{16}c^2 - \frac{11}{256}c^4 - \left(\frac{1}{32}c^2 + \frac{1}{64}c^4\right)\cos 2(t+\theta) \\
&\quad + \left(-\frac{1}{32}c^2 + \frac{1}{128}c^4\right)\cos 4(t+\theta) - \frac{1}{128}c^4 \cos 6(t+\theta) \\
&\quad + d_1^{(1)}\left(-\frac{1}{4}c^2 \sin 2(t+\theta) - \frac{1}{8}c^2 \sin 4(t+\theta)\right) \\
&\quad + d_2^{(1)}\left(\left(\frac{1}{2} - \frac{1}{4}c^2\right)\cos 2(t+\theta) - \frac{1}{4}c^2\cos 4(t+\theta)\right) \\
&\quad + \frac{dd_2^{(1)}}{d\zeta_1}\left(-\frac{1}{4}\left(c - \frac{1}{4}c^3\right)\cos(t+\theta)\right) \\
&\quad + \frac{dd_2^{(1)}}{d\zeta_2}\left(\frac{3}{4}\left(c - \frac{1}{4}c^3\right)\sin(t+\theta) - \frac{1}{8}c^3 \sin 3(t+\theta)\right).
\end{aligned} \tag{5.15}$$

In order to obtain the same result presented by Hori [4] for the new system of differential equations and the near-identity transformation, the following choice is made for the auxiliary vector $d^{(1)}$. Taking

$$d^{(1)} = \begin{bmatrix} 0 \\ -\frac{1}{4} + \frac{1}{16}c^2 \end{bmatrix}, \tag{5.16}$$

it follows from (5.15) that

$$\left\langle \Delta_{\zeta}^{-1}\Psi^{(2)} \right\rangle = \begin{bmatrix} 0 \\ -\frac{1}{8} + \frac{1}{8}c^2 - \frac{7}{256}c^4 \end{bmatrix}. \tag{5.17}$$

From (4.15), (4.21), and (5.17), one finds

$$Z^{*(2)} = \begin{bmatrix} -\frac{1}{8}\zeta_2\left(1-(\zeta_1^2+\zeta_2^2)+\frac{7}{32}(\zeta_1^2+\zeta_2^2)^2\right) \\ \frac{1}{8}\zeta_1\left(1-(\zeta_1^2+\zeta_2^2)+\frac{7}{32}(\zeta_1^2+\zeta_2^2)^2\right) \end{bmatrix}. \tag{5.18}$$

In view of the choice the auxiliary vector $d^{(1)}$, (5.12) can be simplified, and $T^{(1)}$ is then given by

$$T^{(1)} = \begin{bmatrix} \frac{1}{32}\zeta_2(3\zeta_1^2-\zeta_2^2) \\ \frac{1}{32}\zeta_1(16-7\zeta_1^2+5\zeta_2^2) \end{bmatrix}. \tag{5.19}$$

Computing the indefinite integral $\int[\Delta_\zeta^{-1}\Psi^{(2)} - \langle\Delta_\zeta^{-1}\Psi^{(2)}\rangle]dt$ and substituting the general solution of the new undisturbed system, it follows that $T^{(2)}$ is given by

$$T^{(2)} = \begin{bmatrix} \frac{1}{16}\zeta_1 - \frac{5}{64}\zeta_1^3 + \frac{13}{768}\zeta_1^5 + \frac{1}{96}\zeta_1^3\zeta_2^2 + \frac{11}{768}\zeta_1\zeta_2^4 \\ -\frac{1}{16}\zeta_2 + \frac{3}{64}\zeta_1^2\zeta_2 + \frac{1}{16}\zeta_2^3 - \frac{29}{768}\zeta_2\zeta_1^4 - \frac{5}{192}\zeta_1^2\zeta_2^3 - \frac{7}{768}\zeta_2^5 \end{bmatrix} + \Delta_\zeta D^{(2)}, \tag{5.20}$$

with $\Delta_\zeta D^{(2)}$ put in the form:

$$\Delta_\zeta D^{(2)} = \begin{bmatrix} d_1^{(2)}\zeta_1 + d_2^{(2)}\zeta_2 \\ d_1^{(2)}\zeta_2 - d_2^{(2)}\zeta_1 \end{bmatrix}, \tag{5.21}$$

being $d_i^{(2)} = d_i^{(2)}(c)$, $i = 1,2$, $D_1^{(2)} = cd_1^{(2)}$, and $D_2^{(2)} = d_2^{(2)}$. $D^{(2)}$ is obtained from the third-order approximation.

In order to get the same result presented by Hori [4], one finds, repeating the procedure described in the preceding paragraphs, that the auxiliary vector $d^{(2)}$ must be taken as follows:

$$d^{(2)} = \begin{bmatrix} -\frac{1}{16} + \frac{15}{256}c^2 - \frac{7}{512}c^4 \\ 0 \end{bmatrix}. \tag{5.22}$$

Accordingly, $T^{(2)}$ is given by

$$T^{(2)} = \begin{bmatrix} \frac{5}{1536}\zeta_1^5 - \frac{13}{768}\zeta_1^3\zeta_2^2 + \frac{1}{1536}\zeta_1\zeta_2^4 - \frac{5}{256}\zeta_1^3 + \frac{15}{256}\zeta_1\zeta_2^2 \\ -\frac{35}{1536}\zeta_2^5 - \frac{41}{768}\zeta_1^2\zeta_2^3 - \frac{79}{1536}\zeta_2\zeta_1^4 + \frac{31}{256}\zeta_2^3 + \frac{27}{256}\zeta_1^2\zeta_2 - \frac{1}{8}\zeta_2 \end{bmatrix}. \tag{5.23}$$

The new system of differential equations and the generating function are given, up to the second-order of the small parameter, by

$$\frac{d\zeta_1}{dt} = \zeta_2 + \varepsilon\frac{1}{2}\zeta_1\left(1 - \frac{1}{4}\left(\zeta_1^2 + \zeta_2^2\right)\right) - \varepsilon^2\frac{1}{8}\zeta_2\left(1 - \left(\zeta_1^2 + \zeta_2^2\right) + \frac{7}{32}\left(\zeta_1^2 + \zeta_2^2\right)^2\right),$$
$$\frac{d\zeta_2}{dt} = -\zeta_1 + \varepsilon\frac{1}{2}\zeta_2\left(1 - \frac{1}{4}\left(\zeta_1^2 + \zeta_2^2\right)\right) + \varepsilon^2\frac{1}{8}\zeta_1\left(1 - \left(\zeta_1^2 + \zeta_2^2\right) + \frac{7}{32}\left(\zeta_1^2 + \zeta_2^2\right)^2\right), \tag{5.24}$$

$$T_1 = \varepsilon\frac{1}{32}\zeta_2\left(3\zeta_1^2 - \zeta_2^2\right) + \varepsilon^2\left(\frac{5}{1536}\zeta_1^5 - \frac{13}{768}\zeta_1^3\zeta_2^2 + \frac{1}{1536}\zeta_1\zeta_2^4 - \frac{5}{256}\zeta_1^3 + \frac{15}{256}\zeta_1\zeta_2^2\right),$$
$$T_2 = \varepsilon\frac{1}{32}\zeta_1\left(16 - 7\zeta_1^2 + 5\zeta_2^2\right) + \varepsilon^2\left(-\frac{35}{1536}\zeta_2^5 - \frac{41}{768}\zeta_1^2\zeta_2^3 - \frac{79}{1536}\zeta_2\zeta_1^4 + \frac{31}{256}\zeta_2^3 + \frac{27}{256}\zeta_1^2\zeta_2 - \frac{1}{8}\zeta_2\right). \tag{5.25}$$

These results are in agreement with the ones obtained by Hori [4] using a different approach.

Following da Silva Fernandes [1], the Lagrange variational equations—equations of variation of parameters—for the noncanonical version of the Hori method are given by

$$\frac{dC}{dt} = \Delta_\zeta^{-1}R^*, \tag{5.26}$$

where $R^* = \sum_{m=1}\varepsilon^m Z^{*(m)}$, and C is the $n \times 1$ vector of constants of integration of the general solution of the new undisturbed system (3.3). In view of (4.15), Lagrange variational equations can be put in the form:

$$\frac{dC}{dt} = \sum_{m=1}\varepsilon^m\left\langle\Delta_\zeta^{-1}\Psi_I^{(m)}\right\rangle. \tag{5.27}$$

Accordingly, the Lagrange variational equations for the new system of differential equations, (5.24), are given by

$$\frac{dc}{dt} = \frac{\varepsilon c}{2}\left(1 - \frac{1}{4}c^2\right), \tag{5.28a}$$

$$\frac{d\theta}{dt} = \varepsilon^2\left(-\frac{1}{8} + \frac{1}{8}c^2 - \frac{7}{256}c^4\right). \tag{5.28b}$$

The solution of the new system of differential equations can be obtained by introducing the solution of the above variational equations into (4.21).

The originalvariables x and $\dot{x}$ are calculated through (2.6), and the second-order asymptotic solution is

$$x = \zeta_1 + \varepsilon\frac{1}{32}\zeta_2\left(3\zeta_1^2 - \zeta_2^2\right) + \varepsilon^2\left(-\frac{43}{6144}\zeta_1^5 + \frac{29}{3072}\zeta_1^3\zeta_2^2 - \frac{59}{6144}\zeta_1\zeta_2^4 + \frac{1}{256}\zeta_1^3 + \frac{9}{256}\zeta_1\zeta_2^2\right),$$
$$\dot{x} = \zeta_2 + \varepsilon\frac{1}{32}\zeta_1\left(16 - 7\zeta_1^2 + 5\zeta_2^2\right) + \varepsilon^2\left(-\frac{155}{6144}\zeta_2^5 - \frac{35}{3072}\zeta_2^3\zeta_1^2 - \frac{715}{6144}\zeta_2\zeta_1^4 + \frac{29}{256}\zeta_2^3 + \frac{53}{256}\zeta_2\zeta_1^2 - \frac{1}{8}\zeta_2\right). \tag{5.29}$$

Equations (5.29) define exactly the same solution presented by Hori.

(2) Second Asymptotic Solution

Now, let us to consider a different choice of the auxiliary vector $d^{(1)}$. Taking $d^{(1)}$ as a null vector, it follows straightforwardly from (5.15) that

$$\left\langle \Delta_{\zeta}^{-1}\Psi^{(2)} \right\rangle = \begin{bmatrix} 0 \\ -\dfrac{1}{8} + \dfrac{3}{16}c^2 - \dfrac{11}{256}c^4 \end{bmatrix}. \tag{5.30}$$

Thus, from (4.15), (4.21), (5.28a), and (5.28b), one finds

$$Z^{*(2)} = \begin{bmatrix} -\dfrac{1}{8}\zeta_2\left(1 - \dfrac{3}{2}\left(\zeta_1^2 + \zeta_2^2\right) + \dfrac{11}{32}\left(\zeta_1^2 + \zeta_2^2\right)^2\right) \\ \dfrac{1}{8}\zeta_1\left(1 - \dfrac{3}{2}\left(\zeta_1^2 + \zeta_2^2\right) + \dfrac{11}{32}\left(\zeta_1^2 + \zeta_2^2\right)^2\right) \end{bmatrix}. \tag{5.31}$$

Since $d^{(1)}$ is a null vector, (5.12) simplifies, and $T^{(1)}$ is given by

$$T^{(1)} = \begin{bmatrix} \dfrac{1}{4}\zeta_2\left(1 + \dfrac{1}{8}\left(\zeta_1^2 + \zeta_2^2\right)\right) - \dfrac{1}{8}\zeta_2^3 \\ \dfrac{1}{4}\zeta_1\left(1 + \dfrac{7}{8}\left(\zeta_1^2 + \zeta_2^2\right)\right) - \dfrac{3}{8}\zeta_1^3 \end{bmatrix}. \tag{5.32}$$

Now, repeating the procedure described in the previous section, that is, computing the indefinite integral $\int\lfloor \Delta_{\zeta}^{-1}\Psi^{(2)} - \langle \Delta_{\zeta}^{-1}\Psi^{(2)} \rangle \rfloor dt$, substituting the general solution of the new undisturbed system defined by (4.21), and taking $d^{(2)}$ is a null vector, it follows that $T^{(2)}$ is given by

$$T^{(2)} = \begin{bmatrix} -\dfrac{3}{64}\zeta_1^3 + \dfrac{1}{16}\zeta_1\zeta_2^2 + \dfrac{5}{384}\zeta_1^5 - \dfrac{1}{192}\zeta_1^3\zeta_2^2 + \dfrac{7}{384}\zeta_1\zeta_2^4 \\ -\dfrac{7}{64}\zeta_1^2\zeta_2 + \dfrac{1}{16}\zeta_2^3 - \dfrac{1}{384}\zeta_2\zeta_1^4 - \dfrac{1}{96}\zeta_1^2\zeta_2^3 - \dfrac{5}{384}\zeta_2^5 \end{bmatrix}. \tag{5.33}$$

So, the new system of differential equations and the generating function are given, up to the second-order of the small parameter ε, by

$$\begin{aligned}
\frac{d\zeta_1}{dt} &= \zeta_2 + \varepsilon\frac{1}{2}\zeta_1\left(1 - \frac{1}{4}\left(\zeta_1^2 + \zeta_2^2\right)\right) - \varepsilon^2\frac{1}{8}\zeta_2\left(1 - \frac{3}{2}\left(\zeta_1^2 + \zeta_2^2\right) + \frac{11}{32}\left(\zeta_1^2 + \zeta_2^2\right)^2\right),\\
\frac{d\zeta_2}{dt} &= -\zeta_1 + \varepsilon\frac{1}{2}\zeta_2\left(1 - \frac{1}{4}\left(\zeta_1^2 + \zeta_2^2\right)\right) + \varepsilon^2\frac{1}{8}\zeta_1\left(1 - \frac{3}{2}\left(\zeta_1^2 + \zeta_2^2\right) + \frac{11}{32}\left(\zeta_1^2 + \zeta_2^2\right)^2\right),
\end{aligned} \tag{5.34}$$

$$\begin{aligned}
T_1 &= \varepsilon\left(\frac{1}{4}\zeta_2\left(1 + \frac{1}{8}\left(\zeta_1^2 + \zeta_2^2\right)\right) - \frac{1}{8}\zeta_2^3\right)\\
&\quad + \varepsilon^2\left(-\frac{3}{64}\zeta_1^3 + \frac{1}{16}\zeta_1\zeta_2^2 + \frac{5}{384}\zeta_1^5 - \frac{1}{192}\zeta_1^3\zeta_2^2 + \frac{7}{384}\zeta_1\zeta_2^4\right),\\
T_2 &= \varepsilon\left(\frac{1}{4}\zeta_1\left(1 + \frac{7}{8}\left(\zeta_1^2 + \zeta_2^2\right)\right) - \frac{3}{8}\zeta_1^3\right)\\
&\quad + \varepsilon^2\left(-\frac{7}{64}\zeta_1^2\zeta_2 + \frac{1}{16}\zeta_2^3 - \frac{1}{384}\zeta_2\zeta_1^4 - \frac{1}{96}\zeta_1^2\zeta_2^3 - \frac{5}{384}\zeta_2^5\right).
\end{aligned} \tag{5.35}$$

The Lagrange variational equations for the new system of differential equations, defined by (5.34), are given by

$$\frac{dc}{dt} = \frac{\varepsilon c}{2}\left(1 - \frac{1}{4}c^2\right), \tag{5.36a}$$

$$\frac{d\theta}{dt} = \varepsilon^2\left(-\frac{1}{8} + \frac{3}{16}c^2 - \frac{11}{256}c^4\right). \tag{5.36b}$$

These differential equations are the well-known averaged equations obtained through Krylov-Bogoliubov method [5]. Note that (5.28b) and (5.36b) define the phase θ with slightly different frequencies.

As described in the preceding subsection, the solution of the new system of differential equations, defined by (5.34), can be obtained by introducing the solution of the above variational equations into (4.21).

The original variables x and $\dot{x}$ are calculated through (2.6), which provides the following second-order asymptotic solution,

$$\begin{aligned}
x &= \zeta_1 + \varepsilon\frac{1}{4}\zeta_2\left(1 + \frac{1}{8}\left(\zeta_1^2 - 3\zeta_2^2\right)\right)\\
&\quad + \varepsilon^2\left(\frac{65}{6144}\zeta_1^5 + \frac{65}{3072}\zeta_1^3\zeta_2^2 - \frac{95}{6144}\zeta_1\zeta_2^4 - \frac{1}{16}\zeta_1^3 + \frac{1}{16}\zeta_1\zeta_2^2 + \frac{1}{32}\zeta_1\right),\\
\dot{x} &= \zeta_2 + \varepsilon\frac{1}{4}\zeta_1\left(1 - \frac{1}{8}\left(5\zeta_1^2 - 7\zeta_2^2\right)\right)\\
&\quad + \varepsilon^2\left(-\frac{143}{6144}\zeta_2^5 + \frac{193}{3072}\zeta_2^3\zeta_1^2 - \frac{271}{6144}\zeta_2\zeta_1^4 + \frac{5}{64}\zeta_2^3 - \frac{7}{64}\zeta_2\zeta_1^2 + \frac{1}{32}\zeta_2\right).
\end{aligned} \tag{5.37}$$

Finally, note that (5.29) and (5.37) give different second-order asymptotic solution for van der Pol equation.

5.1.2. Duffing Equation

Consider the well-known Duffing equation:

$$\ddot{x} + \varepsilon x^3 + x = 0. \tag{5.38}$$

Introducing the variables $z_1 = x$ and $z_2 = \dot{x}$, this equation can be written in the form:

$$\frac{dz_1}{dt} = z_2, \qquad \frac{dz_2}{dt} = -z_1 - \varepsilon z_1^3. \tag{5.39}$$

Thus,

$$Z^{(1)} = \begin{bmatrix} 0 \\ -z_1^3 \end{bmatrix}. \tag{5.40}$$

Following the simplified algorithm I and repeating the procedure described in Section 5.1.1, the first-order terms $Z^{*(1)}$ and $T^{(1)}$ are obtained as follows. Introducing (4.21) into (5.40), and computing the secular part, one gets

$$\left\langle \Delta_\zeta^{-1} Z^{(1)} \right\rangle = \begin{bmatrix} 0 \\ \frac{3}{8}c^2 \end{bmatrix}. \tag{5.41}$$

Thus, from (4.15) and (5.41), it follows that $Z^{*(1)}$ is given by

$$Z^{*(1)} = \begin{bmatrix} -\frac{3}{8}c^3 \sin(t+\theta) \\ -\frac{3}{8}c^3 \cos(t+\theta) \end{bmatrix}. \tag{5.42}$$

In view of (4.21), $Z^{*(1)}$ can be written explicitly in terms of the new variables ζ_1 and ζ_2:

$$Z^{*(1)} = \begin{bmatrix} \frac{3}{8}\zeta_2\left(\zeta_1^2 + \zeta_2^2\right) \\ -\frac{3}{8}\zeta_1\left(\zeta_1^2 + \zeta_2^2\right) \end{bmatrix}. \tag{5.43}$$

Calculating the indefinite integral $\int \lfloor \Delta_\zeta^{-1} Z^{(1)} - \langle \Delta_\zeta^{-1} Z^{(1)} \rangle \rfloor dt$, one finds

$$\int \left[\Delta_\zeta^{-1} Z^{(1)} - \left\langle \Delta_\zeta^{-1} Z^{(1)} \right\rangle \right] dt = \begin{bmatrix} -\frac{1}{32}c^3 (4\cos 2(t+\theta) + \cos 4(t+\theta)) \\ \frac{1}{32}c^2 (8\sin 2(t+\theta) + \sin 4(t+\theta)) \end{bmatrix}. \tag{5.44}$$

Multiplying this result by Δ_ζ, it follows, according to (4.18), that $T^{(1)}$ is given by

$$T^{(1)} = \begin{bmatrix} -\frac{3}{16}c^3\cos(t+\theta) + \frac{1}{32}c^3\cos 3(t+\theta) \\ -\frac{3}{16}c^3\sin(t+\theta) - \frac{3}{32}c^3\sin 3(t+\theta) \end{bmatrix} + \Delta_\zeta D^{(1)}. \tag{5.45}$$

Taking $D^{(1)}$ as a null vector and using (4.21), $T^{(1)}$ can be written explicitly in terms of the new variables ζ_1 and ζ_2 as follows:

$$T^{(1)} = \begin{bmatrix} -\frac{5}{32}\zeta_1^3 - \frac{9}{32}\zeta_1\zeta_2^2 \\ \frac{15}{32}\zeta_1^2\zeta_2 + \frac{3}{32}\zeta_2^3 \end{bmatrix}. \tag{5.46}$$

In the second-order approximation, one finds after lengthy calculations using MAPLE software:

$$\Psi^{(2)} = \begin{bmatrix} -\frac{69}{256}\zeta_1^4\zeta_2 + \frac{27}{128}\zeta_1^2\zeta_2^3 + \frac{27}{256}\zeta_2^5 \\ \frac{165}{256}\zeta_1^5 + \frac{69}{128}\zeta_1^3\zeta_2^2 - \frac{27}{256}\zeta_2^4\zeta_1 \end{bmatrix}. \tag{5.47}$$

Repeating the procedure described in the above paragraphs, one finds

$$\left\langle \Delta_\zeta^{-1}\Psi^{(2)} \right\rangle = \begin{bmatrix} 0 \\ -\frac{51}{256}c^4 \end{bmatrix},$$
$$Z^{*(2)} = \begin{bmatrix} -\frac{51}{256}\zeta_2\left(\zeta_1^2+\zeta_2^2\right)^2 \\ \frac{51}{256}\zeta_1\left(\zeta_1^2+\zeta_2^2\right)^2 \end{bmatrix}. \tag{5.48}$$

Taking $D^{(2)}$ as a null vector, it follows that

$$T^{(2)} = \begin{bmatrix} \frac{19}{256}\zeta_1^5 + \frac{13}{32}\zeta_1^3\zeta_2^2 + \frac{65}{256}\zeta_1\zeta_2^4 \\ -\frac{95}{256}\zeta_2\zeta_1^4 - \frac{13}{32}\zeta_1^2\zeta_2^3 - \frac{13}{256}\zeta_2^5 \end{bmatrix}. \tag{5.49}$$

The new system of differential equations and the generating function are given, up to the second-order of the small parameter ε, by

$$\frac{d\zeta_1}{dt} = \zeta_2 + \varepsilon\frac{3}{8}\zeta_2\left(\zeta_1^2 + \zeta_2^2\right) - \varepsilon^2\frac{51}{256}\zeta_2\left(\zeta_1^2 + \zeta_2^2\right)^2,$$
$$\frac{d\zeta_2}{dt} = -\zeta_1 - \varepsilon\frac{3}{8}\zeta_1\left(\zeta_1^2 + \zeta_2^2\right) + \varepsilon^2\frac{51}{256}\zeta_1\left(\zeta_1^2 + \zeta_2^2\right)^2, \tag{5.50}$$

$$T_1 = -\varepsilon\left(\frac{5}{32}\zeta_1^3 + \frac{9}{32}\zeta_1\zeta_2^2\right) + \varepsilon^2\left(\frac{19}{256}\zeta_1^5 + \frac{13}{32}\zeta_1^3\zeta_2^2 + \frac{65}{256}\zeta_1\zeta_2^4\right),$$
$$T_2 = \varepsilon\left(\frac{15}{32}\zeta_1^2\zeta_2 + \frac{3}{32}\zeta_2^3\right) + \varepsilon^2\left(-\frac{95}{256}\zeta_2\zeta_1^4 - \frac{13}{32}\zeta_1^2\zeta_2^3 - \frac{13}{256}\zeta_2^5\right). \tag{5.51}$$

The Lagrange variational equations for the new system of differential equations, defined by (5.50), are given by

$$\frac{dc}{dt} = 0,$$
$$\frac{d\theta}{dt} = \varepsilon\frac{3}{8}c^2 - \varepsilon^2\frac{51}{256}c^4. \tag{5.52}$$

These differential equations are the well-known equations obtained through Krylov-Bogoliubov method [5].

As described in Section 5.1.1, the solution of the new system of differential equations, defined by (5.50), can be obtained by introducing the solution of the above variational equations into (4.21).

The original variables x and $\dot{x}$ are calculated through (2.6), which provides the following second-order asymptotic solution:

$$x = \zeta_1 - \varepsilon\frac{1}{32}\zeta_1\left(5\zeta_1^2 + 9\zeta_2^2\right) + \varepsilon^2\frac{1}{2048}\left(227\zeta_1^5 + 742\zeta_1^3\zeta_2^2 + 547\zeta_1\zeta_2^4\right),$$
$$\dot{x} = \zeta_2 + \varepsilon\frac{3}{32}\zeta_2\left(5\zeta_1^2 + \zeta_2^2\right) - \varepsilon^2\frac{1}{2048}\left(77\zeta_2^5 + 922\zeta_2^3\zeta_1^2 + 685\zeta_2\zeta_1^4\right). \tag{5.53}$$

These equations are in agreement with the solution obtained through the canonical version of the Hori method [6].

5.2. Determination of Asymptotic Solutions through Simplified Algorithm II

5.2.1. Van der Pol Equation

For the van der Pol equation, the function $f(x, \dot{x})$ is written in terms of the variables $z_1 = c'$ and $z_2 = \theta'$ by

$$f(x, \dot{x}) = f(z_1\cos(t + z_2), -z_1\sin(t + z_2)) = -\left(z_1^2\cos^2(t + z_2) - 1\right)z_1\sin(t + z_2). \tag{5.54}$$

Thus, it follows from (4.30) that

$$Z^{(1)} = \begin{bmatrix} \frac{1}{2}z_1\left(1 - \frac{1}{4}z_1^2 - \cos 2(t+z_2) + \frac{1}{4}z_1^2 \cos 4(t+z_2)\right) \\ \frac{1}{2}\left(1 - \frac{1}{2}z_1^2\right)\sin 2(t+z_2) - \frac{1}{8}z_1^2 \sin 4(t+z_2) \end{bmatrix}. \tag{5.55}$$

As mentioned before, two different choices of $D^{(m)}$ will be made, and two generating functions will be determined.

(1) First Asymptotic Solution

Following the simplified algorithm II defined by (4.41) and (4.42), the first-order terms $Z^{*(1)}$ and $T^{(1)}$ are calculated as follows.

Taking the secular part of $Z^{(1)}$, with ζ replacing z, one finds

$$Z^{*(1)} = \begin{bmatrix} \frac{1}{2}\zeta_1\left(1 - \frac{1}{4}\zeta_1^2\right) \\ 0 \end{bmatrix}, \tag{5.56}$$

and, integrating the remaining part,

$$T^{(1)} = \begin{bmatrix} -\frac{1}{4}\zeta_1 \sin 2(t+\zeta_2) + \frac{1}{32}\zeta_1^3 \sin 4(t+\zeta_2) \\ -\frac{1}{4}\left(1 - \frac{1}{2}\zeta_1^2\right)\cos 2(t+\zeta_2) + \frac{1}{32}\zeta_1^2 \cos 4(t+\zeta_2) \end{bmatrix} + D^{(1)}, \tag{5.57}$$

with $D_i^{(1)} = D_i^{(1)}(\zeta_1)$, $i = 1, 2$.

Following the algorithm of the Hori method described in Section 2, the second-order equation, (2.11), involves the term $\Psi^{(2)}$ given by

$$\Psi^{(2)} = \frac{1}{2}\left[Z^{(1)} + Z^{*(1)}, T^{(1)}\right]. \tag{5.58}$$

After tedious lengthy calculations using MAPLE software, one finds

$$\begin{aligned}
\Psi_1^{(2)} &= \left(\frac{7}{64}\zeta_1^3 - \frac{3}{128}\zeta_1^5\right)\sin 2(t+\zeta_2) \\
&\quad - \frac{1}{32}\zeta_1^3 \sin 4(t+\zeta_2) - \frac{1}{128}\zeta_1^5 \sin 6(t+\zeta_2) \\
&\quad + D_1^{(1)}\left(\frac{1}{2} - \frac{3}{8}\zeta_1^2 - \frac{1}{4}\cos 2(t+\zeta_2) + \frac{3}{16}\zeta_1^2 \cos 4(t+\zeta_2)\right) \\
&\quad + D_2^{(1)}\left(\frac{1}{2}\zeta_1 \sin 2(t+\zeta_2) - \frac{1}{4}\zeta_1^3 \sin 4(t+\zeta_2)\right) \\
&\quad + \frac{dD_1^{(1)}}{d\zeta_1}\left(-\frac{1}{2}\zeta_1 + \frac{1}{8}\zeta_1^3 + \frac{1}{4}\zeta_1 \cos 2(t+\zeta_2) - \frac{1}{16}\zeta_1^3 \cos 4(t+\zeta_2)\right), \\
\Psi_2^{(2)} &= -\frac{1}{8} + \frac{3}{16}\zeta_1^2 - \frac{11}{256}\zeta_1^4 - \left(\frac{1}{32}\zeta_1^2 + \frac{1}{64}\zeta_1^4\right)\cos 2(t+\zeta_2) \\
&\quad + \left(-\frac{1}{32}\zeta_1^2 + \frac{1}{128}\zeta_1^4\right)\cos 4(t+\zeta_2) - \frac{1}{128}\zeta_1^4 \cos 6(t+\zeta_2) \\
&\quad + D_1^{(1)}\left(-\frac{1}{4}\zeta_1 \sin 2(t+\zeta_2) - \frac{1}{8}\zeta_1 \sin 4(t+\zeta_2)\right) \\
&\quad + D_2^{(1)}\left(\left(\frac{1}{2} - \frac{1}{4}\zeta_1^2\right)\cos 2(t+\zeta_2) - \frac{1}{4}\zeta_1^2 \cos 4(t+\zeta_2)\right) \\
&\quad + \frac{dD_2^{(1)}}{d\zeta_1}\left(-\frac{1}{2}\zeta_1 + \frac{1}{8}\zeta_1^3 + \frac{1}{4}\zeta_1 \cos 2(t+\zeta_2) - \frac{1}{16}\zeta_1^3 \cos 4(t+\zeta_2)\right).
\end{aligned} \tag{5.59}$$

In order to obtain the same averaged Lagrange variational equations given by (5.28a) and (5.28b), $D^{(1)}$ must be taken as

$$D^{(1)} = \begin{bmatrix} 0 \\ -\frac{1}{4} + \frac{1}{16}\zeta_1^2 \end{bmatrix}. \tag{5.60}$$

Thus, it follows that

$$Z^{*(2)} = \begin{bmatrix} 0 \\ -\frac{1}{8} + \frac{1}{8}\zeta_1^2 - \frac{7}{256}\zeta_1^4 \end{bmatrix}. \tag{5.61}$$

In view of the choice of $D^{(1)}$, $T^{(1)}$ is then given by

$$T^{(1)} = \begin{bmatrix} -\frac{1}{4}\zeta_1 \sin 2(t+\zeta_2) + \frac{1}{32}\zeta_1^3 \sin 4(t+\zeta_2) \\ -\frac{1}{4} + \frac{1}{16}\zeta_1^2 - \frac{1}{4}\left(1 - \frac{1}{2}\zeta_1^2\right)\cos 2(t+\zeta_2) + \frac{1}{32}\zeta_1^2 \cos 4(t+\zeta_2) \end{bmatrix}. \tag{5.62}$$

Repeating the procedure for the third-order approximation, and, taking

$$D^{(2)} = \begin{bmatrix} -\frac{1}{16}\zeta_1 + \frac{15}{256}\zeta_1^3 - \frac{7}{512}\zeta_1^5 \\ 0 \end{bmatrix}, \tag{5.63}$$

one finds

$$\begin{aligned} T_1^{(2)} &= \frac{1}{1536}\left(-96\zeta_1 + 90\zeta_1^3 - 21\zeta_1^5\right) + \left(\frac{1}{16}\zeta_1 - \frac{9}{128}\zeta_1^3 + \frac{9}{768}\zeta_1^5\right)\cos 2(t+\zeta_2) \\ &\quad + \left(-\frac{1}{128}\zeta_1^3 + \frac{1}{256}\zeta_1^5\right)\cos 4(t+\zeta_2) + \frac{1}{768}\zeta_1^5 \cos 6(t+\zeta_2), \\ T_2^{(2)} &= \left(-\frac{1}{16} + \frac{3}{64}\zeta_1^2 - \frac{1}{64}\zeta_1^4\right)\sin 2(t+\zeta_2) + \left(\frac{1}{128}\zeta_1^2 - \frac{1}{256}\zeta_1^4\right)\sin 4(t+\zeta_2) \\ &\quad - \frac{1}{768}\zeta_1^4 \sin 6(t+\zeta_2). \end{aligned} \tag{5.64}$$

The new system of differential equations is given, up to the second-order of the small parameter ε, by

$$\frac{d\zeta_1}{dt} = \varepsilon\frac{1}{2}\zeta_1\left(1 - \frac{1}{4}\zeta_1^2\right), \tag{5.65a}$$

$$\frac{d\zeta_2}{dt} = \varepsilon^2\left(-\frac{1}{8} + \frac{1}{8}\zeta_1^2 - \frac{7}{256}\zeta_1^4\right). \tag{5.65b}$$

These differential equations are exactly the same equations given by (5.28a) and (5.28b).

The generating function is obtained from (5.62), (5.64), and it is given, up to the second-order of the small parameter ε, by

$$\begin{aligned} T_1 &= \varepsilon\left(-\frac{1}{4}\zeta_1 \sin 2(t+\zeta_2) + \frac{1}{32}\zeta_1^3 \sin 4(t+\zeta_2)\right) \\ &\quad + \varepsilon^2\left(\frac{1}{1536}\left(-96\zeta_1 + 90\zeta_1^3 - 21\zeta_1^5\right) + \left(\frac{1}{16}\zeta_1 - \frac{9}{128}\zeta_1^3 + \frac{9}{768}\zeta_1^5\right)\cos 2(t+\zeta_2)\right. \\ &\qquad \left. + \left(-\frac{1}{128}\zeta_1^3 + \frac{1}{256}\zeta_1^5\right)\cos 4(t+\zeta_2) + \frac{1}{768}\zeta_1^5 \cos 6(t+\zeta_2)\right), \\ T_2 &= \varepsilon\left(\frac{1}{16}\zeta_1^2 - \frac{1}{4}\left(1 - \frac{1}{2}\zeta_1^2\right)\cos 2(t+\zeta_2) + \frac{1}{32}\zeta_1^2 \cos 4(t+\zeta_2)\right) \\ &\quad + \varepsilon^2\left(\left(-\frac{1}{16} + \frac{3}{64}\zeta_1^2 - \frac{1}{64}\zeta_1^4\right)\sin 2(t+\zeta_2) + \left(\frac{1}{128}\zeta_1^2 - \frac{1}{256}\zeta_1^4\right)\right. \\ &\qquad \left. \times \sin 4(t+\zeta_2) - \frac{1}{768}\zeta_1^4 \sin 6(t+\zeta_2)\right). \end{aligned} \tag{5.66}$$

The original variables x and $\dot{x}$ are calculated through (2.7), which provides, up to the second-order of the small parameter, the following solution:

$$\begin{aligned}
x &= \zeta_1 \cos(t+\zeta_2) - \varepsilon \frac{1}{32}\zeta_1^3 \sin 3(t+\zeta_2) \\
&\quad + \varepsilon^2 \left\{ \left[\frac{3}{256}\zeta_1^3 - \frac{9}{2048}\zeta_1^5\right] \cos(t+\zeta_2) - \left[\frac{1}{128}\zeta_1^3 + \frac{1}{1024}\zeta_1^5\right] \cos 3(t+\zeta_2) \right. \\
&\qquad \left. - \frac{5}{3072}\zeta_1^5 \cos 5(t+\zeta_2) \right\}, \\
\dot{x} &= -\zeta_1 \sin(t+\zeta_2) + \varepsilon \left\{ \left[\frac{1}{2}\zeta_1 - \frac{1}{8}\zeta_1^3\right] \cos(t+\zeta_2) - \frac{3}{32}\zeta_1^3 \cos 3(t+\zeta_2) \right\} \\
&\quad + \varepsilon^2 \left\{ \left[\frac{1}{8}\zeta_1 - \frac{35}{256}\zeta_1^3 + \frac{65}{2048}\zeta_1^5\right] \sin(t+\zeta_2) - \left[\frac{3}{128}\zeta_1^3 - \frac{15}{1024}\zeta_1^5\right] \right. \\
&\qquad \left. \times \sin 3(t+\zeta_2) + \frac{25}{3072}\zeta_1^5 \sin 5(t+\zeta_2) \right\},
\end{aligned} \tag{5.67}$$

with ζ_1 and ζ_2 given by the solution of (5.65a) and (5.65b).

Note that (5.29) and (5.67) give the same second-order asymptotic solution for the van der Pol equation. Recall that ζ_1 and ζ_2 have different meaning in these equations, but they are related through an equation similar to (4.21).

(2) Second Asymptotic Solution

Now, let us to take $D^{(1)}$ and $D^{(2)}$ as null vectors. Equations (5.59) simplifies, and $Z^{*(2)}$ is given by

$$Z^{*(2)} = \begin{bmatrix} 0 \\ -\frac{1}{8} + \frac{3}{16}\zeta_1^2 - \frac{11}{256}\zeta_1^4 \end{bmatrix}. \tag{5.68}$$

The functions $T^{(1)}$ and $T^{(2)}$ are then given by

$$\begin{aligned}
T^{(1)} &= \begin{bmatrix} -\frac{1}{4}\zeta_1 \sin 2(t+\zeta_2) + \frac{1}{32}\zeta_1^3 \sin 4(t+\zeta_2) \\ -\frac{1}{4}\left(1 - \frac{1}{2}\zeta_1^2\right) \cos 2(t+\zeta_2) + \frac{1}{32}\zeta_1^2 \cos 4(t+\zeta_2) \end{bmatrix}, \\
T_1^{(2)} &= -\left(\frac{7}{128}\zeta_1^3 - \frac{3}{256}\zeta_1^5\right) \cos 2(t+\zeta_2) + \frac{1}{128}\zeta_1^3 \cos 4(t+\zeta_2) \\
&\quad + \frac{1}{768}\zeta_1^5 \cos 6(t+\zeta_2), \\
T_2^{(2)} &= -\left(\frac{1}{64}\zeta_1^2 + \frac{1}{128}\zeta_1^4\right) \sin 2(t+\zeta_2) + \left(-\frac{1}{128}\zeta_1^2 + \frac{1}{512}\zeta_1^4\right) \sin 4(t+\zeta_2) \\
&\quad - \frac{1}{768}\zeta_1^4 \sin 6(t+\zeta_2).
\end{aligned} \tag{5.69}$$

The new system of differential equations is obtained from (5.56) and (5.68), and it is given, up to the second-order of the small parameter, by

$$\frac{d\zeta_1}{dt} = \varepsilon\frac{1}{2}\zeta_1\left(1-\frac{1}{4}\zeta_1^2\right),$$
$$\frac{d\zeta_2}{dt} = \varepsilon^2\left(-\frac{1}{8}+\frac{3}{16}\zeta_1^2-\frac{11}{256}\zeta_1^4\right). \tag{5.70}$$

These differential equations are exactly the same equations given by (5.36a) and (5.36b).

The generating function is obtained from (5.69), and it is given, up to the second-order of the small parameter ε, by

$$\begin{aligned}
T_1 &= \varepsilon\left(-\frac{1}{4}\zeta_1\sin 2(t+\zeta_2)+\frac{1}{32}\zeta_1^3\sin 4(t+\zeta_2)\right)\\
&\quad+\varepsilon^2\left(-\left(\frac{7}{128}\zeta_1^3-\frac{3}{256}\zeta_1^5\right)\cos 2(t+\zeta_2)+\frac{1}{128}\zeta_1^3\cos 4(t+\zeta_2)+\frac{1}{768}\zeta_1^5\cos 6(t+\zeta_2)\right),\\
T_2 &= \varepsilon\left(-\frac{1}{4}\left(1-\frac{1}{2}\zeta_1^2\right)\cos 2(t+\zeta_2)+\frac{1}{32}\zeta_1^2\cos 4(t+\zeta_2)\right)\\
&\quad+\varepsilon^2\left(-\left(\frac{1}{64}\zeta_1^2+\frac{1}{128}\zeta_1^4\right)\sin 2(t+\zeta_2)+\left(-\frac{1}{128}\zeta_1^2+\frac{1}{512}\zeta_1^4\right)\sin 4(t+\zeta_2)\right.\\
&\qquad\left.-\frac{1}{768}\zeta_1^4\sin 6(t+\zeta_2)\right).
\end{aligned} \tag{5.71}$$

Equations (5.71) are in agreement with the solution obtained by Ahmed and Tapley [7] through a different integration theory for the Hori method.

A second-order asymptotic solution for the original variables x and $\dot{x}$ is calculated through (2.7), and it is given by

$$\begin{aligned}
x &= \zeta_1\cos(t+\zeta_2)+\varepsilon\left\{\left[-\frac{1}{4}\zeta_1+\frac{1}{16}\zeta_1^3\right]\sin(t+\zeta_2)-\frac{1}{32}\zeta_1^3\sin 3(t+\zeta_2)\right\}\\
&\quad+\varepsilon^2\left\{\left[\frac{1}{32}\zeta_1-\frac{1}{32}\zeta_1^3+\frac{15}{2048}\zeta_1^5\right]\cos(t+\zeta_2)+\left[-\frac{1}{32}\zeta_1^3+\frac{5}{1024}\zeta_1^5\right]\right.\\
&\qquad\left.\times\cos 3(t+\zeta_2)-\frac{5}{3072}\zeta_1^5\cos 5(t+\zeta_2)\right\},\\
\dot{x} &= -\zeta_1\sin(t+\zeta_2)+\varepsilon\left\{\left[\frac{1}{4}\zeta_1-\frac{1}{16}\zeta_1^3\right]\cos(t+\zeta_2)-\frac{3}{32}\zeta_1^3\cos 3(t+\zeta_2)\right\}\\
&\quad+\varepsilon^2\left\{\left[-\frac{1}{32}\zeta_1-\frac{1}{32}\zeta_1^3+\frac{25}{2048}\zeta_1^5\right]\sin(t+\zeta_2)+\left[\frac{3}{64}\zeta_1^3-\frac{3}{1024}\zeta_1^5\right]\right.\\
&\qquad\left.\times\sin 3(t+\zeta_2)+\frac{25}{3072}\zeta_1^5\sin 5(t+\zeta_2)\right\},
\end{aligned} \tag{5.72}$$

with ζ_1 and ζ_2 given by the solution of (5.70).

As before, note that (5.37) and (5.72) give the same second-order asymptotic solution for the van der Pol equation. Recall that ζ_1 and ζ_2 have different meaning in these equations, but they are related through an equation similar to (4.21).

5.2.2. Duffing Equation

For the Duffing equation, the function $f(x,\dot{x})$ is written in terms of the variables $z_1 = c'$ and $z_2 = \theta'$, by

$$f(x,\dot{x}) = f(z_1\cos(t+z_2), -z_1\sin(t+z_2)) = -z_1^3\cos^3(t+z_2). \tag{5.73}$$

Thus, it follows from (4.30) that

$$Z^{(1)} = \begin{bmatrix} \dfrac{1}{4}z_1^3\sin 2(t+z_2) + \dfrac{1}{8}z_1^3\sin 4(t+z_2) \\ \dfrac{3}{8}z_1^2 + \dfrac{1}{2}z_1^2\cos 2(t+z_2) + \dfrac{1}{8}z_1^2\cos 4(t+z_2) \end{bmatrix}. \tag{5.74}$$

Following the simplified algorithm II and repeating the procedure described in Section 5.2.1, the first-order terms $Z^{*(1)}$ and $T^{(1)}$ are obtained as follows. Taking the secular part of $Z^{*(1)}$, with ζ replacing z, and, integrating the remaining part, one finds

$$Z^{*(1)} = \begin{bmatrix} 0 \\ \dfrac{3}{8}\zeta_1^2 \end{bmatrix}, \tag{5.75}$$

$$T^{(1)} = \begin{bmatrix} -\dfrac{1}{32}\zeta_1^3(4\cos 2(t+\zeta_2) + \cos 4(t+\zeta_2)) \\ \dfrac{1}{32}\zeta_1^2(8\sin 2(t+\zeta_2) + \sin 4(t+\zeta_2)) \end{bmatrix}. \tag{5.76}$$

In the second-order approximation, one finds

$$\Psi^{(2)} = \begin{bmatrix} \dfrac{1}{256}\zeta_1^5(-33\sin 2(t+\zeta_2) - 12\sin 4(t+\zeta_2) + 3\sin 6(t+\zeta_2)) \\ \dfrac{1}{256}\zeta_1^4(-51 - 99\cos 2(t+\zeta_2) - 18\cos 4(t+\zeta_2) + 3\cos 6(t+\zeta_2)) \end{bmatrix}. \tag{5.77}$$

Taking the secular part of $\Psi^{(2)}$, and, integrating the remaining part, one finds

$$Z^{*(2)} = \begin{bmatrix} 0 \\ -\dfrac{51}{256}\zeta_1^4 \end{bmatrix}, \tag{5.78}$$

$$T^{(2)} = \begin{bmatrix} \dfrac{1}{512}\zeta_1^5(33\cos 2(t+\zeta_2) + 6\cos 4(t+\zeta_2) - \cos 6(t+\zeta_2)) \\ \dfrac{1}{512}\zeta_1^4(-99\sin 2(t+\zeta_2) - 9\sin 4(t+\zeta_2) + \sin 6(t+\zeta_2)) \end{bmatrix}. \tag{5.79}$$

The new system of differential equations is given, up to the second-order of the small parameter ε, by

$$\frac{d\zeta_1}{dt} = 0,$$
$$\frac{d\zeta_2}{dt} = \varepsilon\frac{3}{8}\zeta_1^2 - \varepsilon^2\frac{51}{256}\zeta_1^4. \tag{5.80}$$

These differential equations are exactly the same equations given by (5.52).

The generating function is obtained from (5.76) and (5.79), and it is given, up to the second-order of the small parameter ε, by

$$\begin{aligned} T_1 &= -\varepsilon\frac{1}{32}\zeta_1^3(4\cos 2(t+\zeta_2) + \cos 4(t+\zeta_2)) \\ &\quad + \varepsilon^2\frac{1}{512}\zeta_1^5(33\cos 2(t+\zeta_2) + 6\cos 4(t+\zeta_2) - \cos 6(t+\zeta_2)), \\ T_2 &= \varepsilon\frac{1}{32}\zeta_1^2(8\sin 2(t+\zeta_2) + \sin 4(t+\zeta_2)) \\ &\quad + \varepsilon^2\frac{1}{512}\zeta_1^4(-99\sin 2(t+\zeta_2) - 9\sin 4(t+\zeta_2) + \sin 6(t+\zeta_2)). \end{aligned} \tag{5.81}$$

A second-order asymptotic solution for the original variables x and $\dot{x}$ is calculated through (2.7), and it is given by

$$\begin{aligned} x &= \zeta_1\cos(t+\zeta_2) + \varepsilon\frac{1}{32}\zeta_1^3(-6\cos(t+\zeta_2) + \cos 3(t+\zeta_2)) \\ &\quad + \varepsilon^2\frac{1}{2048}\zeta_1^5(303\cos(t+\zeta_2) - 78\cos 3(t+\zeta_2) + 2\cos 5(t+\zeta_2)), \\ \dot{x} &= -\zeta_1\sin(t+\zeta_2) - \varepsilon\frac{1}{32}\zeta_1^3(6\sin(t+\zeta_2) + 3\sin 3(t+\zeta_2)) \\ &\quad + \varepsilon^2\frac{1}{2048}\zeta_1^5(249\sin(t+\zeta_2) + 162\sin 3(t+\zeta_2) - 10\sin 5(t+\zeta_2)). \end{aligned} \tag{5.82}$$

These equations are in agreement with the solution obtained through the canonical version of the Hori method [6]. Note that (5.53) and (5.82) give the same second-order asymptotic solution for the Duffing equation. Recall that ζ_1 and ζ_2 have different meaning in these equations, but they are related through an equation similar to (4.21).

6. Conclusions

In this paper, the Hori method for noncanonical systems is applied to theory of nonlinear oscillations. Two different simplified algorithms are derived from the general algorithm proposed by Sessin. It has been shown that the mth-order terms $T_j^{(m)}$ and $Z_j^{*(m)}$ that define the near-identity transformation and the new system of differential equations, respectively, are not uniquely determined, since the algorithms involve at each order arbitrary functions of the constants of integration of the general solution of the undisturbed system. This arbitrariness is an intrinsic characteristic of perturbation methods, since some kind of averaging principle must be applied to determine these functions. The simplified algorithms are then applied in determining second-order asymptotic solutions of two well-known equations in the theory of nonlinear oscillations: van der Pol and Duffing equations. For van der Pol equation, the appropriate use of the arbitrary functions allows the determination of the solution presented by Hori. This solution defines a new system of differential equations with a different frequency for the phase in comparison with the solution obtained by Ahmed and Tapley, who used a different approach for determining the near-identity transformation and the new system of differential equations for the Hori method, and, with the solution obtained by Nayfeh through the method of averaging. For the Duffing equation, only one generating function is determined, and the second simplified algorithm gives the same generating function obtained through Krylov-Bogoliubov method.

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