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Research Article

Approximation for the Finite-Time Ruin Probability of a General Risk Model with Constant Interest Rate and Extended Negatively Dependent Heavy-Tailed Claims

Yang Yang,[1,2] Xin Ma,[3] and Jin-guan Lin[2]

[1] *School of Mathematics and Statistics, Nanjing Audit University, Nanjing 210029, China*
[2] *Department of Mathematics, Southeast University, Nanjing 210096, China*
[3] *Golden Audit College, Nanjing Audit University, Nanjing 210029, China*

Correspondence should be addressed to Yang Yang, yyangmath@gmail.com

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We propose a general continuous-time risk model with a constant interest rate. In this model, claims arrive according to an arbitrary counting process, while their sizes have dominantly varying tails and fulfill an extended negative dependence structure. We obtain an asymptotic formula for the finite-time ruin probability, which extends a corresponding result of Wang (2008).

1. The Dependent General Risk Model

In this paper, we consider the finite-time ruin probability with constant interest rate in a dependent general risk model. In this model, the claim sizes $\{X_n, n \geq 1\}$ form a sequence of identically distributed, not necessarily independent, and nonnegative random variables (r.v.s) with common distribution F such that $\overline{F}(x) = 1 - F(x) = \mathrm{P}(X_1 > x) > 0$ for all $x > 0$; the claim arrival process $\{N(t), t \geq 0\}$ is a general counting process, namely, a nonnegative, nondecreasing, right continuous, and integer-valued stochastic process with $0 < \mathrm{E}N(t) = \lambda(t) < \infty$ for all large $t > 0$. The times of the successive claims are denoted by $\{\tau_n, n \geq 1\}$. The total amount of premiums accumulated up to time $t \geq 0$, denoted by $C(t)$ with $C(0) = 0$ and $C(t) < \infty$ almost surely for every $t > 0$, is another nonnegative and nondecreasing stochastic process. Assume that $\{X_n, n \geq 1\}$, $\{N(t), t \geq 0\}$ and $\{C(t), t \geq 0\}$ are mutually independent. Let $\delta > 0$ be the constant interest rate (i.e., after time t one dollar becomes $e^{\delta t}$ dollars), and let $x \geq 0$ be the initial capital reserve of an insurance company. Then, the total discounted reserve up to time $t \geq 0$, denoted by $D(t, x)$, can be written as

$$D(t, x) = x + \int_0^t e^{-\delta s} C(\mathrm{d}s) - \sum_{n=1}^{N(t)} X_n e^{-\delta \tau_n}. \tag{1.1}$$

For a finite time $T > 0$, the finite-time ruin probability is defined by

$$\begin{aligned}\Psi(x, T) &= \mathrm{P}(D(t, x) < 0, \text{ for some } 0 \leq t \leq T) \\ &= \mathrm{P}\left(\sup_{t \in [0,T]} \left(\sum_{n=1}^{N(t)} X_n e^{-\delta \tau_n} - \int_0^t e^{-\delta s} C(\mathrm{d}s) \right) > x \right),\end{aligned} \tag{1.2}$$

while the ultimate ruin probability is defined by

$$\Psi(x) = \Psi(x, \infty) = \mathrm{P}(D(t, x) < 0, \text{ for some } t \geq 0). \tag{1.3}$$

If the claim sizes $\{X_n, n \geq 1\}$ are independent r.v.s, the model is called the independent general risk model, which was introduced by Wang [1]. In particular, if $C(t) = ct$, $t \geq 0$, with $c > 0$ a deterministic constant and $\{N(t), t \geq 0\}$ is a Poisson process, then the model reduces to the classical one.

2. Introduction and Main Result

Hereafter, all limit relationships hold for x tending to ∞ unless otherwise stated. For two positive functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$; write $f(x) \lesssim g(x)$ if $\limsup f(x)/g(x) \leq 1$ and $f(x) = o(g(x))$ if $\lim f(x)/g(x) = 0$. The indicator function of an event A is denoted by $\mathbf{1}_A$.

In risk theory, heavy-tailed distributions are often used to model large claim amounts. They play a key role in insurance and finance. We will restrict the claim-size distribution F to be heavy tailed. A distribution V is said to be dominatedly varying tailed, denoted by $V \in \mathfrak{D}$, if $\limsup \overline{V}(xy)/\overline{V}(x) < \infty$ for any $y > 0$. A distribution V is said to be long tailed, denoted by $V \in \mathcal{L}$, if $\lim \overline{V}(x + y)/\overline{V}(x) = 1$ for any $y > 0$. A distribution V is said to be subexponential, denoted by $V \in \mathcal{S}$, if $\overline{V^{n*}}(x) \sim n\overline{V}(x)$ for any $n \geq 2$, where V^{n*} denotes the n-fold convolution of itself. A distribution V is said to be regularly varying tailed, denoted by $\mathcal{R}_{-\alpha}$, $\alpha > 0$, if $\lim \overline{V}(xy)/\overline{V}(x) = y^{-\alpha}$ for any $y \geq 1$. A proper inclusion relationship holds that

$$\mathcal{R}_{-\alpha} \subset \mathcal{L} \cap \mathfrak{D} \subset \mathcal{S} \subset \mathcal{L}, \tag{2.1}$$

see, for example, Cline [2] or Embrechts and Omey [3]. For a distribution V, denote the upper Matuszewska index of the distribution V by

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y} \quad \text{with } \overline{V}_*(y) = \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)}, \quad y > 1. \tag{2.2}$$

In the terminology of Bingham et al. [4], the quantity J_V^+ is actually the upper Matuszewska index of the function $1/\overline{V}(x)$, $x \geq 0$, as also pointed out in Tang and Tsitsiashvili [5]. Additionally, denote $L_V = \lim_{y\searrow 1}\overline{V}_*(y)$ (clearly, $0 \leq L_V \leq 1$). The presented definitions yield that the following assertions are equivalent:

$$\text{(i) } V \in \mathcal{D}, \qquad \text{(ii) } \overline{V}_*(y) > 0 \quad \text{for some } y > 1, \qquad \text{(iii) } L_V > 0, \qquad \text{(iv) } J_V^+ < \infty. \tag{2.3}$$

The asymptotic behavior of the ruin probability in the classical risk model has been extensively investigated in the literature. Klüppelberg and Stadtmüller [6] considered the ultimate ruin probability for the case of regularly-varying-tailed claim sizes. Using the reflected random walk theory, Asmussen [7] extended the study to a larger class of heavy-tailed distributions; see Corollary 4.1(ii) of his paper. Complementary discussions on the ultimate ruin probability can be found in Kalashnikov and Konstantinides [8], Konstantinides et al. [9], Tang [10], among others.

In this paper, we are interested in the finite-time ruin probability. In this aspect, Tang [11] established an asymptotic result in the classical risk model: under the condition $F \in \mathcal{S}$, he obtained that for every $T > 0$ for which $\lambda(T) > 0$,

$$\Psi(x,T) \sim \int_{0-}^{T} \overline{F}\left(xe^{\delta t}\right)\lambda(\mathrm{d}t). \tag{2.4}$$

Recently, Wang [1] derived some important and interesting results in two independent risk models. One is the delayed renewal risk model, in which (2.4) holds if $F \in \mathcal{S}$; another is the general risk model, in which (2.4) also holds if $F \in \mathcal{L} \cap \mathcal{D}$. We are interested in the latter, for example, the general risk model, and restate Theorem 2.2 of Wang [1] here.

Theorem 2.1. *In the independent general risk model introduced in Section 1, assume that the claim sizes $\{X_n, n \geq 1\}$ are independent and identically distributed nonnegative r.v.s with common distribution $F \in \mathcal{L} \cap \mathcal{D}$. Assume that for any $T > 0$ with $\lambda(T) - \lambda(0) > 0$, there exists some constant $\eta = \eta(T) > 0$ such that*

$$\mathrm{E}(1+\eta)^{N(T)} < \infty. \tag{2.5}$$

Then, (2.4) *holds.*

In the present paper, we aim to deal with the extended negatively dependent general risk model to get a similar result under $F \in \mathcal{D}$. Simultaneously, the condition (2.5) can be weakened to (2.8) below.

We call r.v.s $\{\xi_n,\ n \geq 1\}$ are extended negatively dependent (END) if there exists some positive constant M such that both

$$P\left(\bigcap_{k=1}^{n}\{\xi_k > y_k\}\right) \leq M\prod_{k=1}^{n}P(\xi_k > y_k), \tag{2.6}$$

$$P\left(\bigcap_{k=1}^{n}\{\xi_k \leq y_k\}\right) \leq M\prod_{k=1}^{n}P(\xi_k \leq y_k) \tag{2.7}$$

hold for each $n \geq 1$ and all $y_1, \ldots, y_n$. This dependence structure was introduced by Liu [12]. Recall that r.v.s $\{\xi_n,\ n \geq 1\}$ are called upper negatively dependent (UND) if (2.6) holds with $M = 1$, they are called lower negatively dependent (LND) if (2.7) holds with $M = 1$, and they are called negatively dependent (ND) if both (2.6) and (2.7) hold with $M = 1$. These negative dependence structures were introduced by Ebrahimi and Ghosh [13] and Block et al. [14]. Clearly, ND r.v.s must be END r.v.s., and Example 4.1 of Liu [12] shows that the END structure also includes some other dependence structures.

Motivated by the work of Wang [1], under the END structure, we formulate our main result as follows.

Theorem 2.2. *In the dependent general risk model introduced in Section 1, assume that the claim sizes $\{X_n, n \geq 1\}$ are END nonnegative r.v.s with common distribution $F \in \mathfrak{D}$ and finite mean μ. Assume that for any $T > 0$ with $\lambda(T) - \lambda(0) > 0$, there exists some constant $p > J_F^+$ such that*

$$E(N(T))^p < \infty. \tag{2.8}$$

Then, it holds that

$$L_F \int_{0-}^{T} \overline{F}\left(xe^{\delta t}\right)\lambda(dt) \lesssim \Psi(x,T) \lesssim L_F^{-1} \int_{0-}^{T} \overline{F}\left(xe^{\delta t}\right)\lambda(dt). \tag{2.9}$$

Furthermore, if $F \in \mathcal{L} \cap \mathfrak{D}$, then (2.4) holds.

The rest of the present paper consists of two sections. We give some lemmas and the proof of Theorem 2.2 in Section 3. In Section 4, we perform some numerical calculations to verify the approximate relationship in our main result.

3. Proof of Main Result and Some Lemmas

In the sequel, M and a always represent some finite and positive constants whose values may vary in different places. In this section, we start by giving some lemmas to show some properties of the class $\mathfrak{D}$ and the END structure. The first lemma is a combination of Proposition 2.2.1 of Bingham et al. [4] and Lemma 3.5 of Tang and Tsitsiashvili [15].

Lemma 3.1. *If a distribution $V \in \mathfrak{D}$, then*

(i) *for any $\gamma > J_V^+$, there exist positive constants a and b such that $\overline{V}(y)/\overline{V}(x) \leq a(y/x)^{-\gamma}$ holds for all $x \geq y \geq b$ and*

(ii) *it holds for every $\gamma > J_V^+$ that $x^{-\gamma} = o(\overline{V}(x))$.*

By direct verification, END r.v.s have the following properties similar to those of ND r.v.s; see Lemma 3.1 of Liu [12]. For some refined properties of END r.v.s, one can refer to Chen et al. [16]. The following lemma can also be found in Lemma 2.2 of Chen et al. [16].

Lemma 3.2. (i) *If r.v.s* $\{\xi_n,\ n \geq 1\}$ *are nonnegative and END, then for any* $n \geq 1$*, there exists some positive constant* M *such that* $\mathrm{E}(\prod_{k=1}^{n}\xi_k) \leq M\prod_{k=1}^{n}\mathrm{E}\xi_k$.

(ii) *If r.v.s* $\{\xi_n,\ n \geq 1\}$ *are END and* $\{f_n(\cdot),\ n \geq 1\}$ *are either all monotone increasing or all monotone decreasing, then* $\{f_n(\xi_n),\ n \geq 1\}$ *are still END.*

The following two lemmas play important roles in the proof of our main result.

Lemma 3.3. *Let* $\{\xi_n,\ n \geq 1\}$ *be identically distributed and END r.v.s with common distribution* V *and* $\mu_V^+ = \mathrm{E}\xi_1\mathbf{1}_{\{\xi_1 \geq 0\}} < \infty$*. Then, for any* $\theta > 0$*,* $x > 0$ *and* $n \geq 1$*, there exists some positive constant* M *such that*

$$\mathrm{P}\left(\sum_{k=1}^{n}\xi_k > x\right) \leq n\overline{V}(\theta x) + M\left(\frac{e\mu_V^+ n}{x}\right)^{\theta^{-1}}. \tag{3.1}$$

Proof. Following the proof of Lemma 2.3 of Tang [17], we employ a standard truncation argument to prove this lemma. For simplicity, we write $S_n^{\xi} = \sum_{k=1}^{n}\xi_k$, $n \geq 1$. If $\mu_V^+ = 0$, then ξ_n is almost surely nonpositive for each $n \geq 1$, implying $\mathrm{P}(S_n^{\xi} > x) = 0$ for any positive x, and thus (3.1) holds.

Let, in the following, $\mu_V^+ > 0$. For any fixed $\theta > 0$ and positive integer n, define

$$\begin{aligned} \widetilde{\xi}_n &= \min\{\xi_n, \theta x\}, \\ \widetilde{\xi}_n^+ &= \max\left\{\widetilde{\xi}_n, 0\right\} = \xi_n\mathbf{1}_{\{0 \leq \xi_n \leq \theta x\}} + \theta x\mathbf{1}_{\{\xi_n > \theta x\}}. \end{aligned} \tag{3.2}$$

According to Lemma 3.2(ii), $\{\widetilde{\xi}_n, n \geq 1\}$ and $\{\widetilde{\xi}_n^+, n \geq 1\}$ are still END r.v.s, respectively. Denote $\widetilde{S}_n^{\xi} = \sum_{k=1}^{n}\widetilde{\xi}_k$, $n \geq 1$. Clearly,

$$\begin{aligned} \mathrm{P}\left(S_n^{\xi} > x\right) &= \mathrm{P}\left(S_n^{\xi} > x, \max_{1 \leq k \leq n}\xi_k > \theta x\right) + \mathrm{P}\left(S_n^{\xi} > x, \max_{1 \leq k \leq n}\xi_k \leq \theta x\right) \\ &\leq n\overline{V}(\theta x) + \mathrm{P}\left(\widetilde{S}_n^{\xi} > x\right). \end{aligned} \tag{3.3}$$

It remains to estimate the second summand in (3.3). For a positive h, by Lemma 3.2(ii), $\{e^{h\widetilde{\xi}_n^+}, n \geq 1\}$ are END nonnegative r.v.s. Hence, using identity

$$\mathrm{E}e^{h\widetilde{\xi}_1^+} = \int_0^{\theta x}\left(e^{hu} - 1\right)V(\mathrm{d}u) + \left(e^{h\theta x} - 1\right)\overline{V}(\theta x) + 1, \tag{3.4}$$

by Markov inequality and Lemma 3.2(i) we have

$$\begin{aligned} P\left(\widetilde{S}_n^{\xi} > x\right) &\le e^{-hx} E e^{h\widetilde{S}_n^{\xi}} \\ &\le e^{-hx} E e^{h\sum_{k=1}^{n}\widetilde{\xi}_k^{+}} \\ &\le e^{-hx} M \left(E e^{h\widetilde{\xi}_1^{+}}\right)^n \\ &= M e^{-hx}\left(\int_0^{\theta x}\left(e^{hu}-1\right)V(du) + \left(e^{h\theta x}-1\right)\overline{V}(\theta x)+1\right)^n. \end{aligned} \tag{3.5}$$

Since $1+u \le e^u$ for all $u \in \mathbb{R}$ and $(e^{hu}-1)/u$ is strictly increasing in $u > 0$, from (3.5), we obtain

$$\begin{aligned} P\left(\widetilde{S}_n^{\xi} > x\right) &\le M\exp\left\{n\int_0^{\theta x}\frac{e^{hu}-1}{u}uV(du) + n\left(e^{h\theta x}-1\right)\overline{V}(\theta x) - hx\right\} \\ &\le M\exp\left\{n\ \frac{e^{h\theta x}-1}{\theta x}\left(\int_0^{\theta x}uV(du) + \theta x\overline{V}(\theta x)\right) - hx\right\} \\ &\le M\exp\left\{n\ \frac{e^{h\theta x}-1}{\theta x}\mu_V^{+} - hx\right\}. \end{aligned} \tag{3.6}$$

Choose $h = (\theta x)^{-1}\log(x(\mu_V^{+} n)^{-1}+1)$, which is positive. For such h, by (3.6), we have

$$\begin{aligned} P\left(\widetilde{S}_n^{\xi} > x\right) &\le M\exp\left\{\frac{1}{\theta} - \frac{1}{\theta}\log\left(\frac{x}{\mu_V^{+} n}+1\right)\right\} \\ &\le M\exp\left\{\frac{1}{\theta}\log\frac{e\mu_V^{+} n}{x}\right\}. \end{aligned} \tag{3.7}$$

The last estimate and (3.3) imply the desired estimate (3.1). The lemma is proved. □

Lemma 3.4. *In the dependent general risk model introduced in Section 1, assume that the claim sizes $\{X_n,\ n \ge 1\}$ are END nonnegative r.v.s with common distribution $F \in \mathfrak{D}$. Let Z be an arbitrary nonnegative r.v. and assume that $\{X_n,\ n \ge 1\}$, $\{N(t),\ t \ge 0\}$ and Z are mutually independent. Then, for any $T > 0$ and any positive integer n_0,*

$$\begin{aligned} L_F\sum_{k=1}^{n_0}\sum_{j=1}^{k} P\left(X_j e^{-\delta\tau_j} > x, N(T)=k\right) &\lesssim \sum_{k=1}^{n_0} P\left(\sum_{j=1}^{k} X_j e^{-\delta\tau_j} > x+Z, N(T)=k\right) \\ &\lesssim L_F^{-1}\sum_{k=1}^{n_0}\sum_{j=1}^{k} P\left(X_j e^{-\delta\tau_j} > x, N(T)=k\right). \end{aligned} \tag{3.8}$$

Furthermore, if $F \in \mathcal{L} \cap \mathfrak{D}$, then

$$\sum_{k=1}^{n_0} P\left(\sum_{j=1}^{k} X_j e^{-\delta\tau_j} > x + Z, N(T) = k\right) \sim \sum_{k=1}^{n_0}\sum_{j=1}^{k} P\left(X_j e^{-\delta\tau_j} > x, N(T) = k\right). \tag{3.9}$$

We remark that if F is consistently varying tailed (see the definition in Chen and Yuen [18]), then by conditioning (3.9) easily follows from Theorem 3.2 of Chen and Yuen [18]. Note that this case is in a broader scope, since there is no need to assume independence between $(\tau_1, \ldots, \tau_{n_0})$ and Z.

Proof. We follow the line of the proof of Lemma 3.6 of Wang [1] with some modifications in relation to the properties of the class $\mathfrak{D}$ and the END structure. Clearly, for each $k = 1, \ldots, n_0$,

$$\begin{aligned} &\mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta\tau_j} > x + Z, N(T) = k\right) \\ &\quad = \int_{\{0 \le t_1 \le \cdots \le t_k \le T, t_{k+1} > T\}} \int_{0-}^{\infty} \mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta t_j} > x + z\right) \times \mathrm{P}(Z \in \mathrm{d}z, \tau_1 \in \mathrm{d}t_1, \ldots, \tau_{k+1} \in \mathrm{d}t_{k+1}). \end{aligned} \tag{3.10}$$

We first show the upper bound. For any fixed $l > 0$,

$$\begin{aligned} \mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta t_j} > x + z\right) &\le \mathrm{P}\left(\bigcup_{j=1}^{k}\left\{X_j e^{-\delta t_j} > x + z - l\right\}\right) \\ &\quad + \mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta t_j} > x + z, \max_{1\le j\le k} X_j e^{-\delta t_j} \le x + z - l\right) := I_1 + I_2. \end{aligned} \tag{3.11}$$

By $F \in \mathfrak{D}$, for any $0 < \theta < 1$ and each $k = 1, \ldots, n_0$, we have uniformly for all $t_1, \ldots, t_k \in [0, T]$ and $z \in [0, \infty)$,

$$I_1 \le \sum_{j=1}^{k} \overline{F}\left(\theta(x+z)e^{\delta t_j}\right) \lesssim L_F^{-1} \sum_{j=1}^{k} \overline{F}\left((x+z)e^{\delta t_j}\right), \tag{3.12}$$

by firstly letting $x \to \infty$ then $\theta \nearrow 1$. We note that $\{X_n, n \geq 1\}$ are END r.v.s. Then, by $F \in \mathfrak{D}$, there exists some positive constant $M = M(n_0)$ such that for sufficiently large x, each $k = 1, \ldots, n_0$, all $t_1, \ldots, t_k \in [0, T]$ and $z \in [0, \infty)$,

$$\begin{aligned} I_2 &= \mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta t_j} > x + z, \frac{x+z}{k} < \max_{1 \leq j \leq k} X_j e^{-\delta t_j} \leq x + z - l\right) \\ &\leq \mathrm{P}\left(\bigcup_{i=1}^{k}\left\{\sum_{j \neq i} X_j e^{-\delta t_j} > l, X_i e^{-\delta t_i} > \frac{x+z}{k}\right\}\right) \\ &\leq \sum_{i=1}^{k}\sum_{j \neq i} \mathrm{P}\left(X_j e^{-\delta t_j} > \frac{l}{k-1}, X_i e^{-\delta t_i} > \frac{x+z}{k}\right) \\ &\leq M \sum_{i=1}^{k}\sum_{j \neq i} \overline{F}\left(\frac{l e^{\delta t_j}}{k-1}\right) \overline{F}\left(\frac{(x+z)e^{\delta t_i}}{k}\right) \\ &\leq M \overline{F}\left(\frac{l}{n_0 - 1}\right) \sum_{j=1}^{k} \overline{F}\left((x+z)e^{\delta t_j}\right). \end{aligned} \tag{3.13}$$

Since l can be arbitrarily large, it follows that

$$\limsup_{l \to \infty} \limsup_{x \to \infty} \sup_{t_1, \ldots, t_k \in [0,T],\, z \in [0,\infty)} \frac{I_2}{\sum_{j=1}^{k} \overline{F}\left((x+z)e^{\delta t_j}\right)} = 0. \tag{3.14}$$

Hence, from (3.10)–(3.14), we obtain for each $k = 1, \ldots, n_0$,

$$\begin{aligned} \mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta \tau_j} > x + Z, N(T) = k\right) &\lesssim L_F^{-1} \sum_{j=1}^{k} \int_{\{0 \leq t_1 \leq \cdots \leq t_k \leq T, t_{k+1} > T\}} \int_{0-}^{\infty} \overline{F}\left((x+z)e^{\delta t_j}\right) \\ &\quad \times \mathrm{P}(Z \in \mathrm{d}z, \tau_1 \in \mathrm{d}t_1, \ldots, \tau_{k+1} \in \mathrm{d}t_{k+1}) \\ &= L_F^{-1} \sum_{j=1}^{k} \mathrm{P}\left(X_j e^{-\delta \tau_j} > x + Z, N(T) = k\right) \\ &\leq L_F^{-1} \sum_{j=1}^{k} \mathrm{P}\left(X_j e^{-\delta \tau_j} > x, N(T) = k\right). \end{aligned} \tag{3.15}$$

As for the lower bound for (3.10), since $\{X_n, n \geq 1\}$ are END r.v.s, we have for sufficiently large x and each $k = 1, \ldots, n_0$,

$$\begin{aligned}
P\left(\sum_{j=1}^{k} X_j e^{-\delta t_j} > x + z\right) &\geq P\left(\bigcup_{j=1}^{k}\left\{X_j e^{-\delta t_j} > x + z\right\}\right) \\
&\geq \sum_{j=1}^{k} \overline{F}\left((x+z)e^{\delta t_j}\right) - \sum_{1 \leq i < j \leq k} P\left(X_i e^{-\delta t_i} > x + z, X_j e^{-\delta t_j} > x + z\right) \\
&\geq \sum_{j=1}^{k} \overline{F}\left((x+z)e^{\delta t_j}\right) - M \sum_{1 \leq i < j \leq k} \overline{F}\left((x+z)e^{\delta t_i}\right)\overline{F}\left((x+z)e^{\delta t_j}\right) \\
&= (1 - o(1))\sum_{j=1}^{k} \overline{F}\left((x+z)e^{r t_j}\right)
\end{aligned} \tag{3.16}$$

holds uniformly for all $t_1, \ldots, t_k \in [0, T]$ and $z \in [0, \infty)$. By $F \in \mathfrak{D}$ and Fatou's lemma, we have for any $\tilde{\theta} > 1$ and all $j = 1, 2, \ldots$,

$$\begin{aligned}
\liminf \frac{1}{\overline{F}(x)} P\left(X_j > x + Ze^{\delta T}\right) &= \liminf \int_{0-}^{\infty} \frac{\overline{F}\left(x + ze^{\delta T}\right)}{\overline{F}(x)} P(Z \in dz) \\
&\geq \int_{0-}^{\infty} \liminf \frac{\overline{F}\left(\tilde{\theta} x\right)}{\overline{F}(x)} P(Z \in dz) \\
&= \overline{F_*}\left(\tilde{\theta}\right) \longrightarrow L_F, \quad \tilde{\theta} \searrow 1,
\end{aligned} \tag{3.17}$$

which means

$$P\left(X_j > x + Ze^{\delta T}\right) \gtrsim L_F \overline{F}(x). \tag{3.18}$$

Similar to (3.15), from (3.10), (3.16), and (3.18), we obtain for each $k = 1, \ldots, n_0$,

$$\begin{aligned}
&P\left(\sum_{j=1}^{k} X_j e^{-\delta \tau_j} > x + Z, N(T) = k\right) \\
&\quad \gtrsim \sum_{j=1}^{k} P\left(X_j e^{-\delta \tau_j} > x + Z, N(T) = k\right)
\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{j=1}^{k} \int_{\{0\leq t_1\leq \ldots \leq t_k \leq T, t_{k+1}>T\}} \mathrm{P}\left(X_j > xe^{\delta t_j} + Ze^{\delta T}\right) \mathrm{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{k+1} \in \mathrm{d}t_{k+1}) \\
&\gtrsim L_F \sum_{j=1}^{k} \int_{\{0\leq t_1\leq \ldots \leq t_k \leq T, t_{k+1}>T\}} \overline{F}\left(xe^{\delta t_j}\right) \mathrm{P}(\tau_1 \in \mathrm{d}t_1, \ldots, \tau_{k+1} \in \mathrm{d}t_{k+1}) \\
&= L_F \sum_{j=1}^{k} \mathrm{P}\left(X_j e^{-\delta \tau_j} > x, N(T) = k\right).
\end{aligned} \tag{3.19}$$

The desired relation (3.8) follows now from (3.15) and (3.19).

If $F \in \mathcal{L} \cap \mathfrak{D}$, (3.9) follows by using the properties of the class $\mathcal{L}$ to establish analogies of relations (3.12) and (3.17). This ends the proof of the lemma. □

Proof of Theorem 2.2. We use the idea in the proof of Theorem 2.2 of Wang [1] (e.g., Theorem 2.1 of this paper) to prove this result. Clearly, $F \in \mathfrak{D}$ and $\mu < \infty$ imply $J_F^+ \geq 1$. By (2.8), we have for any $\epsilon > 0$, there exists some positive integer $n_1 = n_1(T, \epsilon)$ such that

$$\mathrm{E}(N(T))^p \mathbf{1}_{\{N(T)>n_1\}} \leq \epsilon. \tag{3.20}$$

To estimate the upper bound of $\Psi(x, T)$, we split it into two parts as

$$\begin{aligned}
\Psi(x,T) &\leq \mathrm{P}\left(\sum_{j=1}^{N(T)} X_j e^{-\delta\tau_j} > x\right) \\
&= \left(\sum_{k=1}^{n_1} + \sum_{k=n_1+1}^{\infty}\right) \mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta\tau_j} > x, N(T) = k\right) := I_3 + I_4.
\end{aligned} \tag{3.21}$$

According to Lemma 3.4 of this paper and Lemma 3.5 of Wang [1], we have for sufficiently large x,

$$\begin{aligned}
I_3 &\leq (1+\epsilon) L_F^{-1} \sum_{k=1}^{n_1} \sum_{j=1}^{k} \mathrm{P}\left(X_j e^{-\delta\tau_j} > x, N(T) = k\right) \\
&\leq (1+\epsilon) L_F^{-1} \sum_{j=1}^{\infty} \mathrm{P}\left(X_j e^{-\delta\tau_j} > x, N(T) \geq j\right) \\
&= (1+\epsilon) L_F^{-1} \sum_{j=1}^{\infty} \mathrm{P}\left(X_j e^{-\delta\tau_j} > x, \tau_j \leq T\right) \\
&= (1+\epsilon) L_F^{-1} \int_{0-}^{T} \overline{F}\left(xe^{\delta t}\right) \lambda(\mathrm{d}t).
\end{aligned} \tag{3.22}$$

By Lemma 3.3, $F \in \mathfrak{D}$, Lemma 3.1(ii), (3.20), and $p > J_F^+ \geq 1$, there exists some positive constant M such that for sufficiently large x,

$$\begin{aligned} I_4 &\leq \sum_{k=n_1+1}^{\infty} \mathrm{P}\left(\sum_{j=1}^{k} X_j > x\right)\mathrm{P}(N(T)=k) \\ &\leq \overline{F}\left(p^{-1}x\right)\sum_{k=n_1+1}^{\infty} k\mathrm{P}(N(T)=k) + M(e\mu)^p x^{-p}\sum_{k=n_1+1}^{\infty} k^p\mathrm{P}(N(T)=k) \\ &\leq M\overline{F}(x)\left(\mathrm{E}N(T)\mathbf{1}_{\{N(T)>n_1\}} + \mathrm{E}(N(t))^p\mathbf{1}_{\{N(T)>n_1\}}\right) \\ &= M\epsilon\overline{F}(x). \end{aligned} \tag{3.23}$$

By Lemma 3.1(i), for any $\gamma > J_F^+$, there exists some positive constant a such that for sufficiently large x,

$$\begin{aligned} \int_{0-}^{T} \overline{F}\left(xe^{\delta t}\right)\lambda(\mathrm{d}t) &\geq a^{-1}\overline{F}(x)\int_{0-}^{T} e^{-\gamma\delta t}\lambda(\mathrm{d}t) \\ &\geq a^{-1}e^{-\gamma\delta T}(\lambda(T)-\lambda(0))\overline{F}(x), \end{aligned} \tag{3.24}$$

which, combining (3.23) and $\lambda(T)-\lambda(0)>0$, implies

$$I_4 \leq M\epsilon\int_{0-}^{T} \overline{F}\left(xe^{\delta t}\right)\lambda(\mathrm{d}t). \tag{3.25}$$

From (3.21), (3.22), and (3.25), we derive the right-hand side of (2.9).

As for the lower bound of $\Psi(x,T)$, by Lemma 3.4, we have for the above given $\epsilon > 0$ and sufficiently large x,

$$\begin{aligned} \Psi(x,T) &\geq \mathrm{P}\left(\sum_{j=1}^{N(T)} X_j e^{-\delta\tau_j} > x + \int_0^T e^{-\delta s}C(\mathrm{d}s)\right) \\ &\geq \sum_{k=1}^{n_1}\mathrm{P}\left(\sum_{j=1}^{k} X_j e^{-\delta\tau_j} > x + \int_0^T e^{-\delta s}C(\mathrm{d}s), N(T)=k\right) \end{aligned}$$

$$
\begin{aligned}
&\geq (1-\epsilon) L_F \sum_{k=1}^{n_1} \sum_{j=1}^{k} \mathrm{P}\left(X_j e^{-\delta \tau_j} > x, N(T) = k\right) \\
&= (1-\epsilon) L_F \left(\sum_{j=1}^{\infty} \mathrm{P}\left(X_j e^{-\delta \tau_j} > x, \tau_j \leq T\right) - \sum_{k=n_1+1}^{\infty} \sum_{j=1}^{k} \mathrm{P}\left(X_j e^{-\delta \tau_j} > x, N(T) = k\right) \right) \\
&:= (1-\epsilon) L_F \left(\int_{0-}^{T} \overline{F}\left(x e^{\delta t}\right) \lambda(\mathrm{d}t) - I_5 \right).
\end{aligned}
\tag{3.26}
$$

Analogously to the estimate for I_4, we have for sufficiently large x,

$$
\begin{aligned}
I_5 &\leq \overline{F}(x) \mathrm{E} N(T) \mathbf{1}_{\{N(T) > n_1\}} \\
&\leq M\epsilon \int_{0-}^{T} \overline{F}\left(x e^{\delta t}\right) \lambda(\mathrm{d}t).
\end{aligned}
\tag{3.27}
$$

From (3.26) and (3.27), we obtain the left-hand side of (2.9).

If $F \in \mathcal{L} \cap \mathfrak{D}$, then (2.4) follows by using (3.9) in the proof of (3.22) and (3.26). □

4. Numerical Calculations

In this section, we perform some numerical calculations to check the accuracy of the asymptotic relations obtained in Theorem 2.2. The main work is to estimate the finite-time ruin probability defined in (1.2).

We assume that the claim sizes $\{X_n, n \geq 1\}$ come from the common Pareto distribution with parameter $\kappa = 1, \beta = 2$,

$$
F(x; \kappa, \beta) = 1 - \left(\frac{\kappa}{\kappa + x} \right)^{\beta}, \quad x \geq 0,
\tag{4.1}
$$

which belongs to the class $\mathcal{L} \cap \mathfrak{D}$, and $\{(X_{2n-1}, X_{2n}), n \geq 1\}$ are independent replications of (X_1, X_2) with the joint distribution

$$
F_{X_1, X_2}(x, y) = -\frac{1}{\alpha} \ln\left(1 + \frac{\left(e^{-\alpha F(x)} - 1\right)\left(e^{-\alpha F(y)} - 1\right)}{e^{-\alpha} - 1} \right),
\tag{4.2}
$$

with parameter $\alpha = 1$, where the joint distribution $F_{X_1, X_2}(x, y)$ is constructed according to the Frank Copula. It has been proved in Example 4.2 of Liu [12] that X_1 and X_2 are END r.v.s. Since $\{(X_{2n-1}, X_{2n}), n \geq 1\}$ are independent copies of (X_1, X_2), the r.v.s $\{X_n, n \geq 1\}$ are END as well.

Assume that the claim arrival process $N(t)$ is the homogeneous Poisson process with intensity parameter λ. Clearly, such an integer-valued process $N(t)$ satisfies the condition (2.8). Choose $\lambda = 0.1$. The total amount of premiums is simplified as $C(t) = ct$ with

Table 1: Comparison between the analog value and the theoretical result in Theorem 2.2.

x ($\times 10^3$)	Theoretical result	Analog value
0.5	$3.2846e-6$	$3.8120e-6$ (16.1%)
1	$8.2270e-7$	$9.1100e-7$ (10.7%)
2	$2.0586e-7$	$2.2300e-7$ (8.3%)
5	$3.2956e-8$	$3.5000e-8$ (6.2%)

the premium rate $c = 500$, and the constant interest rate $\delta = 0.02$. Here, we set the time T as $T = 10$ and the initial capital reserve $x = 500, 10^3, 2 \times 10^3, 5 \times 10^3$, respectively. We aim to verify the accuracy of relation (2.4). The procedure of the computation of the finite-time ruin probability $\Psi(x, T)$ in Theorem 2.2 is listed here.

Step 1. Assign a value for the variable x and set $l = 0$.

Step 2. Divide the close interval $[0, T]$ into $m = 1000$ pieces, and denote each time point as t_i, $i = 1, \ldots, m$.

Step 3. For each t_i, generate a random number n_i from the Poisson distribution $P(\lambda t_i)$, and set n_i as the sample size of the claims.

Step 4. Generate the accident arrival time $\{\tau_k^i,\ k = 1, \ldots, n_i\}$ from the uniform distribution $U(0, t_i)$ and the claim sizes $\{X_k^i,\ k = 1, \ldots, n_i\}$ from (4.1) and (4.2).

Step 5. Calculate the expression D below for each t_i and denote them as $\{D_i,\ i = 1, \ldots, m\}$:

$$D_i = \sum_{k=1}^{n_i} X_k^i e^{-r\tau_k^i} - \int_0^{t_i} e^{-rs} C(ds), \quad i = 1, \ldots, m, \tag{4.3}$$

where r and $C(t)$ have been defined and their values have also been assigned.

Step 6. Select the maximum value from $\{D_i,\ i = 1, \ldots, m\}$, and denote it as D^*, compare D^* with x; if $D^* > x$, then the value of l increases 1.

Step 7. Repeat Step 2 through Step 6, $N = 10^9$ times.

Step 8. Calculate the moment estimate of the finite-time ruin probability, l/N.

Step 9. Repeat Step 1 through Step 8 ten times and get ten estimates. Then, choose the median of the ten estimates as the analog value of the finite-time ruin probability.

For different value of x, the analog value and the theoretical result of the finite-time ruin probability are presented in Table 1, and the percentage of the error relative to the theoretical result is also presented in the bracket behind the analog value. It can be found that from Table 1, the larger x becomes, the smaller the difference between the analog value and the theoretical result is. Therefore, the approximate relationship in Theorem 2.2 is reasonable.

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