

Research Article

Exact Solution of Impulse Response to a Class of Fractional Oscillators and Its Stability

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Oscillator of single-degree-freedom is a typical model in system analysis. Oscillations resulted from differential equations with fractional order attract the interests of researchers since such a type of oscillations may appear dramatic behaviors in system responses. However, a solution to the impulse response of a class of fractional oscillators studied in this paper remains unknown in the field. In this paper, we propose the solution in the closed form to the impulse response of the class of fractional oscillators. Based on it, we reveal the stability behavior of this class of fractional oscillators as follows. A fractional oscillator in this class may be strictly stable, nonstable, or marginally stable, depending on the ranges of its fractional order.

1. Introduction

Fractional systems gain increasing attention in applied sciences, ranging from mechanical engineering to electrical engineering, see, for example, [1–11]. Recall that stability is an essential property of systems, see for example [12], for the stability of conventional systems of integer order and [13–15] for fractional systems.

One of the typical models used in system analysis is the oscillator of single-degree-freedom [16]. It is given by

$$m \frac{d^2 Y(t)}{dt^2} + c \frac{dY(t)}{dt} + kY(t) = e(t), \quad (1.1)$$

where $m > 0$ is the mass, c the damping constant, $k > 0$ the stiffness and $e(t)$ the forcing function. Let $2b = c/m \geq 0$ and $\omega_0 = \sqrt{k/m}$. Then, we rewrite the above by

$$\frac{d^2Y(t)}{dt^2} + 2b\frac{dY(t)}{dt} + \omega_0^2Y(t) = e(t). \quad (1.2)$$

The parameter b is called damping coefficient and ω_0 is inherent frequency.

Let $g(t)$ be the impulse response to the above equation. It is the solution to (1.2) for $e(t) = \delta(t)$ (the Dirac delta function) with zero initial conditions and is given by

$$g(t) = \frac{1}{\omega} e^{-bt} \sin(\omega t), \quad (1.3)$$

where $\omega = \sqrt{\omega_0^2 - b^2}$ is angular frequency. Equation (1.3) implies a damped oscillation.

In the case of $b = 0$, that is, the zero damping, (1.2) reduces to

$$\left(\frac{d^2}{dt^2} + \omega_0^2 \right) Y(t) = e(t), \quad \omega_0 > 0. \quad (1.4)$$

The impulse response to the above system is

$$g(t) = \frac{1}{\omega_0} \sin(\omega_0 t), \quad (1.5)$$

which corresponds to a free oscillation.

Recently, research on fractional oscillators has attracted considerable interests, see for example, [17–22]. In the stability analysis of fractional oscillators, the authors in [17–19] studied a class of fractional oscillators expressed by

$$\frac{d^{2-\varepsilon}}{dt^{2-\varepsilon}} x(t) + \omega_0^2 x(t) = e(t), \quad 0 < \varepsilon < 1. \quad (1.6)$$

They concluded that the above oscillator may be strictly stable as if it is a damped oscillator. In this paper, we focus on another class of fractional oscillators that were first introduced by Lim and Muniandy [23]. It satisfies

$$\left(\frac{d^2}{dt^2} + \omega_0^2 \right)^\beta x(t) = e(t), \quad \beta > 0. \quad (1.7)$$

The solutions to (1.7) for $e(t)$ being a white noise in time domain and frequency domain are obtained in [21], which are further extended to the oscillator with two fractional indexes in [22, 24, 25]. However, its stability remains an unsolved issue. This paper shows that this type of fractional oscillator is strictly stable when $0 < \beta < 1$, nonstable when $\beta > 1$, and marginally stable for $\beta = 1$.

The remainder of the paper is organized as follows. In Section 2, the impulse response of the fractional system (1.7) in the closed form is proposed. Stability analysis is given in Section 3. Discussions and conclusions are in Section 4.

2. Impulse Response

For $t > 0$ and $\nu > 0$, denote by D_t^ν the fractional derivative of Caputo type [26] defined by

$$D_t^\nu f(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t \frac{f^{(n)}(u) du}{(t-u)^{\nu-n+1}}, \quad n-1 \leq \alpha \leq n, \quad (2.1)$$

where Γ is the Gamma function. For simplicity, we write ${}_0D_t^\nu$ by D^ν below. One can regard $(D^2 + \omega_0^2)^\beta$ as a shifted fractional derivative of $D^{2\nu}$. By taking the binomial expansion, one gets

$$(D^2 + \omega_0^2)^\beta = \sum_{n=0}^{\infty} (\beta)_n \omega_0^{2n} D^{2(\beta-n)}, \quad (2.2)$$

where $(\beta)_m$ is the Pochhammer symbol, that is,

$$(\beta)_m = \frac{\beta!}{(\beta-m)!} \quad (2.3)$$

The fractional oscillator satisfying $(D^2 + \omega_0^2)^\beta x(t) = e(t)$, $\beta > 0$ has the impulse response function $h(t)$, which is the solution to $(D^2 + \omega_0^2)^\beta h(t) = \delta(t)$. Denote the Laplace transform of $h(t)$ by $H(s)$. Then, we have

$$H(s) = L[h(t)] = \frac{1}{(s^2 + \omega_0^2)^\beta} = \frac{1}{\omega_0^{2\beta} (1 + s^2/\omega_0^2)^\beta}, \quad (2.4)$$

where L is the operator of the Laplace transform. Expanding the right side of the above using the binomial series yields

$$\frac{1}{\omega_0^{2\beta} (1 + s^2/\omega_0^2)^\beta} = \frac{1}{\omega_0^{2\beta}} \sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} (-1)^m s^m, \quad \left| \frac{s}{\omega_0} \right| < 1. \quad (2.5)$$

Therefore, the impulse response to (1.7) is given by

$$h(t) = \frac{1}{\omega_0^{2\beta}} L^{-1} \left[\sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} (-1)^m s^m \right] = \frac{1}{\omega_0^{2\beta}} \sum_{m=0}^{\infty} \frac{(\beta)_m}{m!} (-1)^m \delta^{(m)}(t), \quad t \geq 0, \quad (2.6)$$

where L^{-1} is the operator of the inverse Laplace transform.

Another form of $h(t)$, which may be more convenient for the stability analysis, is expressed below. Considering the Laplace transform pair given in [27], we have

$$\mathcal{L}^{-1} \left[\frac{1}{(s^2 + \omega_0^2)^\beta} \right] = \frac{\sqrt{\pi}}{\Gamma(\beta) (2\omega_0)^{\beta-1/2}} t^{\beta-1/2} J_{\beta-1/2}(\omega_0 t), \quad (2.7)$$

where $J_{\beta-1/2}(\omega_0 t)$ is the Bessel function of the first kind of order $\beta - 1/2$. Therefore, we have the impulse response given by

$$h(t) \triangleq h(t; \beta) = \frac{\sqrt{\pi}}{\Gamma(\beta) (2\omega_0)^{\beta-1/2}} t^{\beta-1/2} J_{\beta-1/2}(\omega_0 t), \quad \beta > 0, t \geq 0. \quad (2.8)$$

3. Stability Analysis

The above discussion allows us to obtain the following results concerning the stability of the fractional oscillator under consideration. Before we discuss these stability properties, we first recall the criteria of stability based on the principle of bounded-input and bounded-output (BIBO). A system is said to be stable if

$$\int_0^\infty |h(t)| dt = \text{constant}, \quad (3.1)$$

which implies $\lim_{t \rightarrow \infty} h(t) = 0$ and poles of $L[h(t)]$ are located on the left-hand portion of the s -plane. A system is said to be nonstable if $h(t)$ is increasing, and accordingly poles of $L[h(t)]$ are located on the right-hand portion of the s -plane. One says that a system is neutral if poles of $L[h(t)]$ are on the complex $j\omega$ -axis [28, 29].

Note that [27]

$$J_\nu(t) = \frac{(t/2)^\nu}{\Gamma(\nu + 1/2)\Gamma(1/2)} \int_{-1}^1 (1 - u^2)^{\nu-1/2} \cos(tu) du, \quad \text{Re } \nu > -1/2. \quad (3.2)$$

Thus, according to (3.2) for $\nu = 1/2$ and considering (2.8) for $\beta = 1$, $h(t; \beta)$ in (2.8) reduces to the simple case with impulse response corresponds to the free oscillator. That is,

$$h(t; \beta) \Big|_{\beta=1} = \frac{\sin \omega_0 t}{\omega_0}. \quad (3.3)$$

This leads to the following remark.

Remark 3.1. $h(t; \beta)$ reduces to the impulse response to the ordinary oscillator for $\beta = 1$. Figure 1 indicates the plot of $h(t; 1)$ for $\omega_0 = 1$. The ordinary oscillator is neutral.

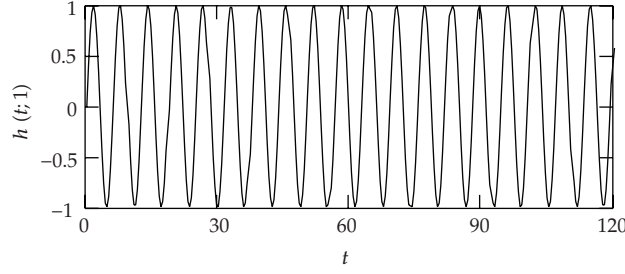


Figure 1: Plot of the impulse response of the ordinary oscillator.

One notes that $h(t; \beta)$ is unbounded if $\beta > 1$. As a matter of fact, from (3.2), we have

$$\frac{-(t/2)^v}{\Gamma(v+1/2)\Gamma(1/2)} \int_{-1}^1 (1-u^2)^{v-1/2} du \leq J_v(t) \leq \frac{(t/2)^v}{\Gamma(v+1/2)\Gamma(1/2)} \int_{-1}^1 (1-u^2)^{v-1/2} du. \quad (3.4)$$

Since $\beta > 1$ implies $v > 1/2$, one immediately sees that both the right side and the left one on the above expression are, respectively, unbounded as $t \rightarrow \infty$. Figure 2 shows the oscillations for various values of β for $\omega_0 = 1$. Thus, we have the following remark.

Remark 3.2. For $\beta > 1$, the fractional oscillator (1.7) is non stable.

The other interesting thing is that the $h(t; \beta)$ becomes an oscillation with decreasing amplitude if $0 < \beta < 1$. In fact,

$$J_v(t) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m+v+1)} \left(\frac{t}{2}\right)^{2m+v}. \quad (3.5)$$

Thus,

$$J_v(t) \sim \frac{1}{\sqrt{t}} \quad \text{for } t \rightarrow \infty. \quad (3.6)$$

Consequently, we have

$$\lim_{t \rightarrow \infty} h(t; \beta) = \lim_{t \rightarrow \infty} \frac{\sqrt{\pi}}{\Gamma(\beta) (2\omega_0)^{\beta-1/2}} t^{\beta-1/2} J_{\beta-1/2}(\omega_0 t) = 0, \quad 0 < \beta < 1, \quad (3.7)$$

which implies that the poles of $L[h(t; \beta)]$ for $0 < \beta < 1$ are located on the left of the s -plane. Therefore, the following remark is an obvious consequence. Figure 3 gives plots of $h(t; \beta)$ for several values of β for $\omega_0 = 1$.

Remark 3.3. The system (1.7) is strictly stable for $0 < \beta < 1$.

The stability of the fractional oscillators expressed by (1.7) is summarized in Table 1.

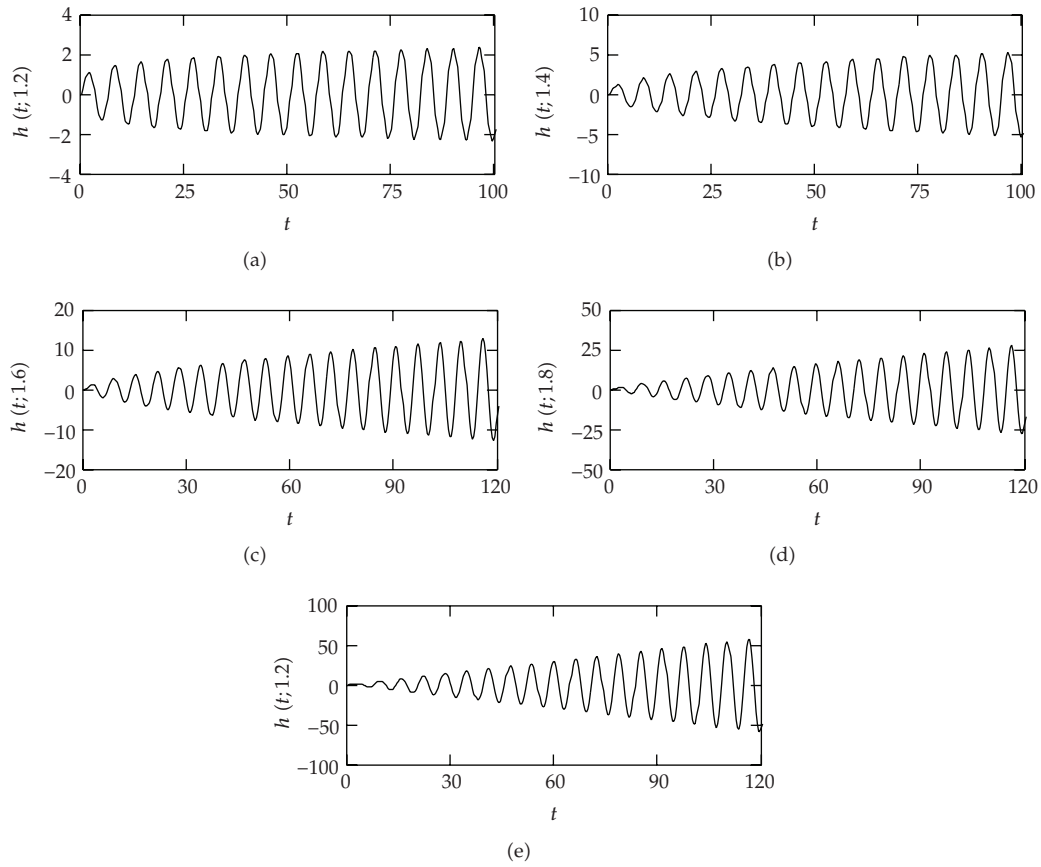


Figure 2: Increasing oscillations. (a) $\beta = 1.2$. (b) $\beta = 1.4$. (c) $\beta = 1.6$. (d) $\beta = 1.8$. (e) $\beta = 1.2$.

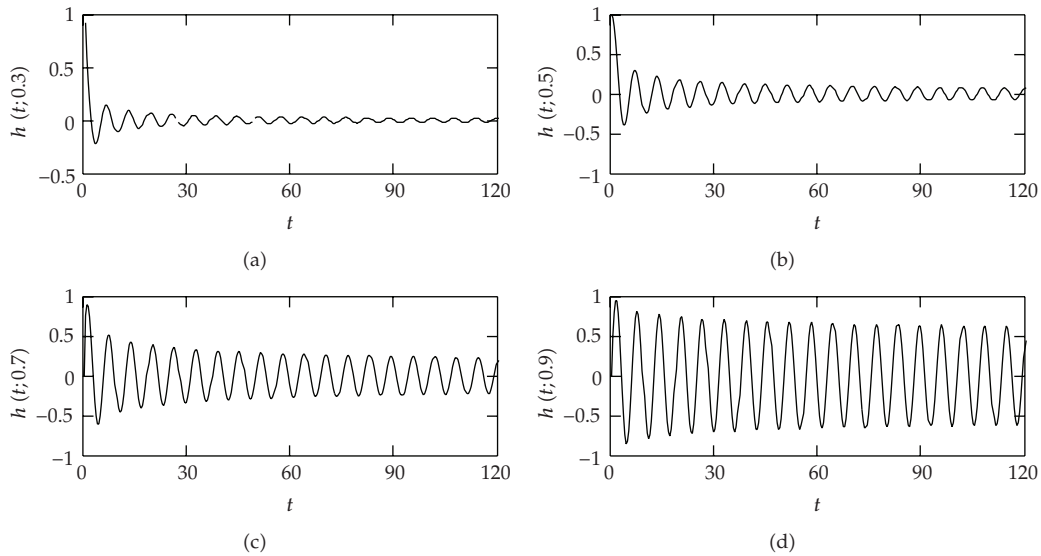


Figure 3: Oscillations. (a) $\beta = 0.3$ (b) $\beta = 0.5$ (c) $\beta = 0.7$ (d) $\beta = 0.9$.

Table 1: Stability performances of (1.7).

Value of β	Type of stability
$0 < \beta < 1$	Strictly stable
$\beta > 1$	Non stable
$\beta = 1$	Neutral

4. Discussions And Conclusions

As previously noted in [25], a fractal time series can be considered as a solution to a fractional differential equation driven by a white noise. Thus, there may be research niche for other series, for example, those discussed in [30–43]. In this paper, we have presented two forms, that is, (2.6) and (2.8), of the impulse response to the fractional system expressed by (1.7). Such a type of fractional oscillators has dramatic performances in its stability. We have revealed that the system with (1.7) contains three subclasses of oscillators. Ordinary free oscillators are a special case of (1.7) for $\beta = 1$. It corresponds to fractional oscillators of strictly stable for $0 < \beta < 1$ and non stable if $\beta > 1$. Note that there is no damping term in the discussed oscillators in form. However, in the case of $0 < \beta < 1$, they are strictly stable, as though they were ordinary oscillators equipped with a certain damping. On the other side, for $\beta > 1$, they are non stable.

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