Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 629760, 13 pages
doi:10.1155/2011/629760

Research Article

Travelling Wave Solutions to the Generalized Pochhammer-Chree (PC) Equations Using the First Integral Method

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Received 11 August 2011; Accepted 26 September 2011

Academic Editor: Anuar Ishak

By using the first integral method, the traveling wave solutions for the generalized Pochhammer-Chree (PC) equations are constructed. The obtained results include complex exponential function solutions, complex traveling solitary wave solutions, complex periodic wave solutions, and complex rational function solutions. The power of this manageable method is confirmed.

1. Introduction

In this paper, we study the generalized Pochhammer-Chree (PC) equations:

$$u_{tt} - u_{ttxx} - \left(\alpha u + \beta u^{n+1} + \nu u^{2n+1}\right)_{xx} = 0, \quad n \geq 1, \tag{1.1}$$

where α, β, and ν are constants. Equation (1.1) represents a nonlinear model of longitudinal wave propagation of elastic rods [1–14].The model for $\alpha = 1, \beta = 1/(n+1)$, and $\nu = 0$ was studied in [4, 7, 8] where solitary wave solutions for this model were obtained for $n = 1, 2$, and 4. A second model for $\alpha = 0,\ \beta = -1/2$, and $\nu = 0$ was studied by [9], and solitary wave solutions were obtained as well.

However, a third model was investigated in [10–13] for $n = 1, 2$ where explicit solitary wave solutions and kinks solutions were derived.

It is the objective of this work to further complement studies on a generalized PC equations in [1–14].

The first integral method, which is based on the ring theory of commutative algebra, was first proposed by Feng [15]. This method was further developed by the same author in [16–21] and some other mathematicians [22–26]. Our first interest in the present work is to implement the first integral method to stress its power in handling nonlinear equations, so that one can apply it for solving various types of nonlinearity. The next interest is in the determination of exact traveling wave solutions for the generalized PC equations. The remaining structure of this paper is organized as follows: Section 2 is a brief introduction to the first integral method. In Section 3, by implementing the first integral method, new exact traveling wave solutions to the generalized PC equations are reported with the aid of mathematical software Mathematica 8.0. This describes the ability and reliability of the method. A conclusion is given in Section 4.

2. The First Integral Method

Consider a general nonlinear partial differential equation in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, u_{xxx}, \ldots) = 0. \tag{2.1}$$

Using the wave variable $\xi = x - ct$ carries (2.1) into the following ordinary differential equation (ODE):

$$Q(U, U', U'', U''', \ldots) = 0, \tag{2.2}$$

where prime denotes the derivative with respect to the same variable ξ.

Next, we introduce new independent variables $x = u,\ y = u_\xi$ which change (2.2) to a system of ODEs:

$$\begin{aligned} x' &= y, \\ y' &= f(x, y). \end{aligned} \tag{2.3}$$

According to the qualitative theory of differential equations [27], if one can find the first integrals to System (2.3) under the same conditions, the analytic solutions to (2.3) can be solved directly. However, in general, it is difficult to realize this even for a single first integral, because for a given plane autonomous system, there is no general theory telling us how to find its first integrals in a systematic way. A key idea of this approach here to find the first integral is to utilize the Division Theorem. For convenience, first let us recall the Division Theorem for two variables in the complex domain C [15].

Division Theorem

Suppose that $P(x, y)$ and $Q(x, y)$ are polynomials of two variables x and y in $C\ [x, y]$ and $P(x, y)$ is irreducible in $C\ [x, y]$. If $Q(x, y)$ vanishes at all zero points of $P(x, y)$, then there exists a polynomial $G(x, y)$ in $C[x, y]$ such that

$$Q(x, y) = P(x, y)G(x, y). \tag{2.4}$$

3. The Generalized PC Equations

We conduct our analysis by examining all possible cases of ν for the generalized PC equations (1.1).

Case 1.

$$\beta \neq 0, \qquad \nu \neq 0. \tag{3.1}$$

Using the wave variable $\xi = x - ct$ and integrating twice, we obtain

$$\left(c^2 - \alpha\right)u - c^2 u'' - \beta u^{n+1} - \nu u^{2n+1} = 0, \tag{3.2}$$

where prime denotes the derivative with respect to the same variable ξ. Making the following transformation:

$$v = u^n, \tag{3.3}$$

then (3.2) becomes

$$\left(c^2 - \alpha\right) n^2 v^2 - nc^2 v v'' - c^2(1-n)\left(v'\right)^2 - n^2\beta v^3 - n^2 \nu v^4 = 0, \tag{3.4}$$

where $v'$and v'' denote $dv/d\xi$ and $d^2v/d\xi^2$, respectively. Equation (3.4) is a nonlinear ODE, and we can rewrite it as

$$v'' - av + b\frac{(v')^2}{v} + dv^2 + fv^3 = 0, \tag{3.5}$$

where

$$a = \left(1 - \frac{\alpha}{c^2}\right)n, \qquad b = \frac{1-n}{n}, \qquad d = \frac{n\beta}{c^2}, \qquad f = \frac{n\nu}{c^2}. \tag{3.6}$$

Let $x = v$, let $y = dv/d\xi$, and let (3.5) be equivalent to the following two-dimensional autonomous system

$$\begin{aligned} \frac{dx}{d\xi} &= y, \\ \frac{dy}{d\xi} &= ax - b\frac{y^2}{x} - dx^2 - fx^3. \end{aligned} \tag{3.7}$$

Assume that

$$d\tau = \frac{d\xi}{x}, \tag{3.8}$$

thus system (3.7) becomes

$$\frac{dx}{d\tau} = xy, \qquad \frac{dy}{d\tau} = ax^2 - by^2 - dx^3 - fx^4. \tag{3.9}$$

Now, we are applying the Division Theorem to seek the first integral to system (3.9). Suppose that $x = x(\tau),\ y = y(\tau)$ are the nontrivial solutions to (3.9), and $p(x,y) = \sum_{i=0}^{m} a_i(x)y^i$ is an irreducible polynomial in $C[x,y]$, such that

$$p[x(\tau), y(\tau)] = \sum_{i=0}^{m} a_i(x(\tau))y(\tau)^i = 0, \tag{3.10}$$

where $a_i(x)$ $(i = 0,1,\ldots,m)$ are polynomials of x and $a_m(x) \neq 0$. We call (3.10) the first integral of polynomial form to system (3.9). We start our study by assuming $m = 1$ in (3.10). Note that $dp/d\tau$ is a polynomial in x and y, and $p[x(\tau),\ y(\tau)] = 0$ implies $dp/d\tau|_{(3.9)} = 0$. According to the Division Theorem, there exists a polynomial $H(x,y) = h(x) + g(x)y$ in $C[x,\ y]$ such that

$$\begin{aligned}\left.\frac{dp}{d\tau}\right|_{(3.9)} &= \left.\left(\frac{\partial p}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \tau}\right)\right|_{(3.9)} \\ &= \sum_{i=0}^{1}\left(a_i'(x)y^i \cdot xy\right) + \sum_{i=0}^{1}\left(ia_i(x)y^{i-1}\cdot\left[ax^2 - by^2 - dx^3 - fx^4\right]\right) \\ &= \left(h(x) + g(x)y\right)\left(\sum_{i=0}^{1} a_i(x)y^i\right),\end{aligned} \tag{3.11}$$

where prime denotes differentiation with respect to the variable x. On equating the coefficients of y^i $(i = 2,1,0)$ on both sides of (3.11), we have

$$xa_1'(x) - ba_1(x) = g(x)a_1(x), \tag{3.12}$$

$$xa_0'(x) = h(x)a_1(x) + g(x)a_0(x), \tag{3.13}$$

$$a_1(x)\left[ax^2 - dx^3 - fx^4\right] = h(x)a_0(x). \tag{3.14}$$

Since, $a_1(x)$ is a polynomial of x, from (3.12) we conclude that $a_1(x)$ is a constant and $g(x) = -b$. For simplicity, we take $a_1(x) = 1$, and balancing the degrees of $h(x)$ and $a_0(x)$ we conclude that $\deg(h(x)) = 2$ and $\deg(a_0(x)) = 2$ only. Now suppose that

$$h(x) = A_2x^2 + A_1x + A_0, \quad a_0(x) = B_2x^2 + B_1x + B_0 \quad (A_2 \neq 0, B_2 \neq 0), \tag{3.15}$$

where A_i, B_i, $(i = 0, 1, 2)$ are all constants to be determined. Substituting (3.15) into (3.13), we obtain

$$h(x) = ((b+2)B_2)x^2 + ((b+1)B_1)x + bB_0. \tag{3.16}$$

Substituting $a_0(x), a_1(x)$, and $h(x)$ in (3.14) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations, and by, solving it, we obtain the following solutions:

$$d = -\frac{\sqrt{a}(3+2b)\sqrt{f}}{\sqrt{-2-b}\sqrt{1+b}}, \qquad B_0 = 0, \qquad B_1 = -\frac{\sqrt{a}}{\sqrt{1+b}}, \qquad B_2 = -\frac{\sqrt{f}}{\sqrt{-2-b}}, \tag{3.17}$$

$$d = -\frac{\sqrt{a}(3+2b)\ \sqrt{f}}{\sqrt{-2-b}\sqrt{1+b}}, \qquad B_0 = 0, \qquad B_1 = \frac{\sqrt{a}}{\sqrt{1+b}}, \qquad B_2 = \frac{\sqrt{f}}{\sqrt{-2-b}}, \tag{3.18}$$

$$d = \frac{\sqrt{a}(3+2b)\sqrt{f}}{\sqrt{-\ 2-b}\sqrt{1+b}}, \qquad B_0 = 0, \qquad B_1 = -\frac{\sqrt{a}}{\sqrt{1+b}}, \qquad B_2 = \frac{\sqrt{f}}{\sqrt{-2-b}}, \tag{3.19}$$

$$d = \frac{\sqrt{a}(3+2b)\sqrt{f}}{\sqrt{-\ 2-b}\sqrt{1+b}}, \qquad B_0 = 0, \qquad B_1 = \frac{\sqrt{a}}{\sqrt{1+b}}, \qquad B_2 = -\frac{\sqrt{f}}{\sqrt{-2-b}}. \tag{3.20}$$

Setting (3.17) and (3.18) in (3.10), we obtain that System (3.9) has one first integral

$$y \mp \left(\frac{\sqrt{f}}{\sqrt{-2-b}}x^2 + \frac{\sqrt{a}}{\sqrt{1+b}}x \right) = 0, \tag{3.21}$$

respectively. Combining this first integral with (3.9), the second-order differential equation (3.5) can be reduced to

$$\frac{dv}{d\xi} = \pm\left(\frac{\sqrt{f}}{\sqrt{-2-b}}v^2 + \frac{\sqrt{a}}{\sqrt{1+b}}v \right). \tag{3.22}$$

Solving (3.22) directly and changing to the original variables, we obtain the following complex exponential function solutions to (1.1):

$$u_1(x,t) = \left(\frac{iR}{\exp\left(-n\sqrt{1-\alpha/c^2}(x-ct) - iRc_1\right) - \sqrt{\nu}/c} \right)^{1/n}, \tag{3.23}$$

$$u_2(x,t) = \left(\frac{iR\exp(iRc_1)}{\exp\left(n\sqrt{1-\alpha/c^2}(x-ct)\right) - (\sqrt{\nu}/c)\exp(iRc_1)} \right)^{1/n}. \tag{3.24}$$

Similarly, for the cases of (3.19) and (3.20), we have another complex exponential function solutions:

$$u_3(x,t) = \left(\frac{iR}{-\exp(-n\ \sqrt{1-(\alpha/c^2)}\quad (x-ct) - iRc_1) + (\sqrt{\nu}/c)}\right)^{1/n}, \tag{3.25}$$

$$u_4(x,t) = \left(\frac{iR\exp(iRc_1)}{-\exp\left(n\sqrt{1-\alpha/c^2}(x-ct)\right) + \sqrt{\nu}/c\exp(iRc_1)}\right)^{1/n}, \tag{3.26}$$

where, $R = \sqrt{1-\alpha/c^2}\sqrt{1+n}$, c_1 is an arbitrary constant. These solutions are all new exact solutions. Now we assume that $m = 2$ in (3.10). By the Division Theorem, there exists a polynomial $H(x,y) = h(x) + g(x)y$ in $C\ [x,y]$ such that

$$\begin{aligned} \left.\frac{dp}{d\tau}\right|_{(3.9)} &= \left.\left(\frac{\partial p}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \tau}\right)\right|_{(3.9)} \\ &= \sum_{m=1}^{2}\left(a_i'(x)y^i \cdot xy\right) + \sum_{m=1}^{2}\left(ia_i(x)y^{i-1}\cdot\left[ax^2 - by^2 - dx^3 - fx^4\right]\right) \\ &= \left(h(x) + g(x)y\right)\left(\sum_{m=1}^{2}a_i(x)y^i\right), \end{aligned} \tag{3.27}$$

On equating the coefficients of y^i $(i = 3,2,1,0)$ on both sides of (3.11), we have

$$xa_2'(x) - 2ba_2(x) = g(x)a_2(x), \tag{3.28}$$

$$xa_1'(x) - ba_1(x) = h(x)a_2(x) + g(x)a_1(x), \tag{3.29}$$

$$xa_0'(x) + 2a_2(x)\left[ax^2 - dx^3 - fx^4\right] = h(x)a_1(x) + g(x)a_0(x), \tag{3.30}$$

$$a_1(x)\left[ax^2 - dx^3 - fx^4\right] = h(x)a_0(x). \tag{3.31}$$

Since $a_2(x)$ is a polynomial of x, from (3.28) we conclude that $a_2(x)$ is a constant and $g(x) = -2b$. For simplicity, we take $a_2(x) = 1$, and balancing the degrees of $h(x)$, $a_0(x)$, and $a_1(x)$ we conclude that $\deg(h(x)) = 2$ and $\deg(a_1(x)) = 2$. In this case, we assume that

$$h(x) = A_2x^2 + A_1x + A_0, \quad a_1(x) = B_2x^2 + B_1x + B_0 \quad (A_2 \neq 0, B_2 \neq 0), \tag{3.32}$$

where A_i, B_i $(i = 0, 1, 2)$ are constants to be determined. Substituting (3.32) into (3.29) and (3.30), we have

$$\begin{aligned} h(x) &= ((2+b)B_2)x^2 + ((1+b)B_1)x + bB_0, \\ a_0(x) &= \left(\frac{2f + (2+b)B_2^2}{2(2+b)}\right)x^4 + \left(\frac{2d + (3+2b)B_1B_2}{3+2b}\right)x^3 \\ &\quad + \left(\frac{-2a + (1+b)B_1^2 + 2(1+b)B_0B_2}{2(1+b)}\right)x^2 + B_0B_1x + \frac{B_0^2}{2} + Fx^{-2b}, \end{aligned} \tag{3.33}$$

where F is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$, and $h(x)$ in (3.31) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations, and by solving it we obtain

$$F = 0, \qquad a = \frac{4(1+b)d^2}{(3+2\,b)^2B_2^2}, \qquad f = -\frac{1}{4}(2+b)B_2^2, \qquad B_0 = 0, \qquad B_1 = -\frac{4d}{(3+2b)B_2}. \tag{3.34}$$

Setting (3.34) in (3.10), we obtain

$$y = \frac{4dx - (3+2b)B_2^2x^2}{2(3+2b)B_2}. \tag{3.35}$$

Using this first integral, the second-order ODE (3.5) reduces to

$$\frac{dv}{d\xi} = \frac{4dv - (3+2b)B_2^2v^2}{2(3+2b)B_2}. \tag{3.36}$$

Similarly, solving (3.36) and changing to the original variables, we obtain the exponential function solutions:

$$u_5(x,t) = \left(\frac{2\beta(2+n)B_2S}{n\exp(\beta[2B_2(1+2/n)c_1 - x + ct]S) + (2+n)B_2^2}\right)^{1/n}, \tag{3.37}$$

where $S = 2n^2/(2+n)c^2B_2$, c_1 is an arbitrary constant. These solutions are all new exact solutions.

Case 2.

$$\beta = 0, \qquad \nu \neq 0. \tag{3.38}$$

We now investigate the generalized PC equation (1.1) for $\beta = 0$, then, we obtain

$$\left(c^2 - \alpha\right)u - c^2u'' - \nu u^{2n+1} = 0, \tag{3.39}$$

where prime denotes the derivative with respect to ξ. Similarly as in Case 1, making then the following transformation:

$$v = u^n, \tag{3.40}$$

then (3.39) becomes

$$\left(c^2 - \alpha\right)n^2v^2 - nc^2vv'' - c^2(1-n)\left(v'\right)^2 - n^2\nu v^4 = 0, \tag{3.41}$$

where $v'$and v'' denote $dv/d\xi$ and $d^2v/d\xi^2$, respectively. Let us rewrite (3.41) as

$$v'' - av + b\frac{(v')^2}{v} + fv^3 = 0, \tag{3.42}$$

where a, b, f are as given in (3.6). Let $x = v$, let $y = dv/d\xi$, and (3.42) become the following two-dimensional autonomous system:

$$\begin{aligned} \frac{dx}{d\xi} &= y, \\ \frac{dy}{d\xi} &= ax - b\frac{y^2}{x} - fx^3. \end{aligned} \tag{3.43}$$

Assume that

$$d\tau = \frac{d\xi}{x}, \tag{3.44}$$

thus system (3.43) becomes

$$\begin{aligned} \frac{dx}{d\tau} &= xy, \\ \frac{dy}{d\tau} &= ax^2 - by^2 - fx^4. \end{aligned} \tag{3.45}$$

Following the same procedures as in Case 1, so we are applying the Division Theorem to seek the first integral to system (3.45). Suppose that $x = x(\tau)$ and $y = y(\tau)$ are the nontrivial solutions to (3.45), and $p(x,y) = \sum_{i=0}^{m} a_i(x)y^i$ is an irreducible polynomial in $C[x,y]$, such that

$$p\left[x(\tau), y(\tau)\right] = \sum_{i=0}^{m} a_i(x(\tau))y(\tau)^i = 0, \tag{3.46}$$

where $a_i(x)$ $(i = 0, 1, \ldots, m)$ are polynomials of x and $a_m(x) \neq 0$. We call (3.46) the first integral of polynomial form to system (3.45). We start by assuming $m = 1$ in (3.46). Note that $dp/d\tau$

is a polynomial in x and y, and $p[x(\tau),\ y(\tau)] = 0$ implies $dp/d\tau|_{(3.44)} = 0$. According to the Division Theorem, there exists a polynomial $H(x,y) = h(x) + g(x)y$ in $C[x,y]$ such that

$$\begin{aligned} \left.\frac{dp}{d\tau}\right|_{(3.45)} &= \left.\left(\frac{\partial p}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \tau}\right)\right|_{(3.45)} \\ &= \sum_{i=0}^{1}\left(a_i'(x)y^i \cdot xy\right) + \sum_{i=0}^{1}\left(ia_i(x)y^{i-1} \cdot \left[ax^2 - by^2 - fx^4\right]\right) \\ &= \left(h(x) + g(x)y\right)\left(\sum_{i=0}^{1} a_i(x)y^i\right), \end{aligned} \tag{3.47}$$

where prime denotes differentiation with respect to the variable x. On equating the coefficients of y^i $(i = 2,1,0)$ on both sides of (3.47), we have

$$xa_1'(x) - ba_1(x) = g(x)a_1(x), \tag{3.48}$$

$$xa_0'(x) = h(x)a_1(x) + g(x)a_0(x), \tag{3.49}$$

$$a_1(x)\left[ax^2 - fx^4\right] = h(x)a_0(x). \tag{3.50}$$

Since, $a_1(x)$ is a polynomial of x, from (3.48) we conclude that $a_1(x)$ is a constant and $g(x) = -b$. For simplicity, we take $a_1(x) = 1$, and balancing the degrees of $h(x)$ and $a_0(x)$ we conclude that $\deg(h(x)) = 2$ and $\deg(a_0(x)) = 2$ only. Now suppose that

$$h(x) = A_2x^2 + A_1x + A_0, \quad a_0(x) = B_2x^2 + B_1x + B_0 \quad (A_2 \neq 0, B_2 \neq 0), \tag{3.51}$$

where $A_i, B_i, (i = 0,1,2)$ are constants to be determined. Substituting (3.51) into (3.49), we have

$$h(x) = ((2+b)B_2)x^2 + ((1+b)B_1)x + bB_0. \tag{3.52}$$

Substituting $a_0(x), a_1(x)$, and $h(x)$ in (3.50) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations, and, by solving it, we obtain the following solutions:

$$\begin{aligned} &a = -\frac{2\sqrt{f}B_0}{\sqrt{-2-b}}, \qquad B_2 = -\frac{\sqrt{f}}{\sqrt{-2-b}}, \qquad B_1 = 0, \\ &a = \frac{2\sqrt{f}B_0}{\sqrt{-2-b}}, \qquad B_2 = \frac{\sqrt{f}}{\sqrt{-2-b}}, \qquad B_1 = 0. \end{aligned} \tag{3.53}$$

Thus, by the similar procedure explained above in Case 1, the complex traveling solitary wave and the complex periodic wave solutions to the generalized PC equations in this Case 2 are given, respectively, by

$$u_1(x,t) = \left(-\frac{q\sqrt{B_0}\sqrt{c}\tanh\left[p\left(x - ct - i\sqrt{1+1/n}c_1\right)\sqrt{B_0}\ /(q\sqrt{c})\right]}{p} \right)^{1/n},$$
$$u_2(x,t) = \left(-\frac{q\sqrt{B_0}\sqrt{c}\tan\left[p\left(x - ct - i\sqrt{1+1/n}c_1\right)\sqrt{B_0}\ /(q\sqrt{c})\right]}{p} \right)^{1/n}, \tag{3.54}$$

where $p = n^{1/4}\nu^{1/4}$, $q = i^{1/4}(1+1/n)^{1/4}, c_1$ is an arbitrary constant. These solutions are all new exact solutions. Now we assume that $m = 2$ in (3.46). By the Division Theorem, there exists a polynomial $H(x,y) = h(x) + g(x)y$ in $C[x,y]$ such that

$$\begin{aligned} \left.\frac{dp}{d\tau}\right|_{(3.45)} &= \left.\left(\frac{\partial p}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial p}{\partial y}\frac{\partial y}{\partial \tau}\right)\right|_{(3.45)} \\ &= \sum_{i=0}^{2}\left(a_i'(x)y^i \cdot xy\right) + \sum_{i=0}^{2}\left(ia_i(x)y^{i-1}\cdot\left[ax^2 - by^2 - fx^4\right]\right) \\ &= (h(x) + g(x)y)\left(\sum_{i=0}^{2} a_i(x)y^i\right) \end{aligned} \tag{3.55}$$

On equating the coefficients of y^i $(i = 3,2,1,0)$ on both sides of (3.55), we have

$$xa_2'(x) - 2ba_2(x) = g(x)a_2(x), \tag{3.56}$$

$$xa_1'(x) - ba_1(x) = h(x)a_2(x) + g(x)a_1(x), \tag{3.57}$$

$$xa_0'(x) + 2a_2(x)\left[ax^2 - fx^4\right] = h(x)a_1(x) + g(x)a_0(x), \tag{3.58}$$

$$a_1(x)\left[ax^2 - fx^4\right] = h(x)a_0(x). \tag{3.59}$$

Since $a_2(x)$ is a polynomial of x, from (3.56) we conclude that $a_2(x)$ is a constant and $g(x) = -2b$. For simplicity, we take $a_2(x) = 1$, and balancing the degrees of $h(x)$, $a_0(x)$ and $a_1(x)$ we conclude that $\deg(h(x)) = 1$, $\deg(a_1(x)) = 1$ and $\deg(h(x)) = 2$, $\deg(a_1(x)) = 2$.

Subcase 2.1. $\deg(h(x)) = 1$ and $\deg(a_1(x)) = 1$. In this case, we assume that

$$h(x) = A_1x + A_0, \quad a_1(x) = B_1x + B_0 \quad (A_1 \neq 0, B_1 \neq 0), \tag{3.60}$$

where A_i, B_i $(i = 0, 1)$ are constants to be determined. Inserting (3.60) into (3.57) and (3.58), we deduce that

$$h(x) = ((1+b)B_1)x + bB_0$$
$$a_0(x) = \left(\frac{f}{2+b}\right)x^4 + \left(\frac{-2a + (1+b)B_1^2}{2(1+b)}\right)x^2 + B_0B_1x + \frac{B_0^2}{2} + Fx^{-2b}, \tag{3.61}$$

where F is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$, and $h(x)$ in (3.59) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations, and by solving it we obtain

$$a = \frac{1}{4}B_1^2(1+b), \qquad F = 0, \qquad B_0 = 0. \tag{3.62}$$

Then, by the similar procedure explained above, we get the complex exponential function solutions which can be expressed as

$$u_3(x,t) = \left(\frac{iK\exp(KB_1c_1)}{-\exp((B_1/2)(x-ct)) + (2K/c)\sqrt{v}\sqrt{n}\ \exp(KB_1c_1)}\right)^{1/n},$$
$$u_4(x,t) = \left(-\frac{iK\exp(KB_1c_1)}{-\exp((B_1/2)(x-ct)) + (2K/c)\sqrt{v}\sqrt{n}\ \exp(KB_1c_1)}\right)^{1/n}, \tag{3.63}$$

where $K = \sqrt{1+1/n}$. These solutions are all new exact solutions.

Subcase 2.2. $\deg(h(x)) = 2$ and $\deg(a_1(x)) = 2$. Now suppose that

$$h(x) = A_2x^2 + A_1x + A_0, \quad a_1(x) = B_2x^2 + B_1x + B_0 \quad (A_2 \neq 0, B_2 \neq 0), \tag{3.64}$$

where, $A_i, B_i, (i = 0, 1, 2)$ are constants to be determined. Substituting (3.64) into (3.57) and (3.58), we have

$$h(x) = ((2+b)B_2)x^2 + ((1+b)B_1)x + bB_0 \tag{3.65}$$

$$a_0(x) = \left(\frac{2f + (2+b)B_2^2}{2\ (2+b)}\right)x^4 + B_1B_2x^3$$
$$+ \left(\frac{-2a + (1+b)B_1^2\ + 2(1+b)B_0B_2}{2(1+b)}\right)x^2 + B_0B_1x + \frac{B_0^2}{2} + Fx^{-2b}, \tag{3.66}$$

where F is an arbitrary integration constant. Substituting $a_0(x)$, $a_1(x)$, and $h(x)$ in (3.59) and setting all the coefficients of powers x to be zero, we obtain a system of nonlinear algebraic equations, and, by solving it, we obtain the following solutions:

$$F = 0, \quad a = 0, \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = -\frac{2\sqrt{f}}{\sqrt{-2-b}},$$
$$F = 0, \quad a = 0, \quad B_0 = 0, \quad B_1 = 0, \quad B_2 = \frac{2\sqrt{f}}{\sqrt{-2-b}}. \tag{3.67}$$

Thus, as above, we obtain the complex rational function solutions which can be written as

$$u_5(x,t) = \left(\frac{iK}{\sqrt{n}\sqrt{\nu}\left(\mp x/\sqrt{\alpha}+t\right) - iKc_1}\right)^{1/n},$$
$$u_6(x,t) = \left(-\frac{iK}{\sqrt{n}\sqrt{\nu}\left(\mp x/\sqrt{\alpha}+t\right) + iKc_1}\right)^{1/n}, \tag{3.68}$$

where K as defined above. These solutions are all new exact solutions.

Notice that the results in this paper are based on the assumption of $m = 1, 2$ for the generalized PC equations. For the cases of $m = 3, 4$ for these equations, the discussions become more complicated and involves the irregular singular point theory and the elliptic integrals of the second kind and the hyperelliptic integrals. Some solutions in the functional form cannot be expressed explicitly. One does not need to consider the cases $m \geq 5$ because it is well known that an algebraic equation with the degree greater than or equal to 5 is generally not solvable.

4. Conclusion

In this work, we are concerned with the generalized PC equations for seeking their traveling wave solutions. We first transform each equation into an equivalent two-dimensional planar autonomous system then use the first integral method to find one first integral which enables us to reduce the generalized PC equations to a first-order integrable ordinary differential equations. Finally, a class of traveling wave solutions for the considered equations are obtained. These solutions include complex exponential function solutions, complex traveling solitary wave solutions, complex periodic wave solutions, and complex rational function solutions. We believe that this method can be applied widely to many other nonlinear evolution equations, and this will be done in a future work.

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