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Research Article

Existence Results for a Nonlinear Semipositone Telegraph System with Repulsive Weak Singular Forces

Fanglei Wang,[1] Yunhai Wang,[2] and Yukun An[3]

[1] *College of Science, Hohai University, Nanjing 210098, China*
[2] *College of Aeronautics and Astronautics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*
[3] *Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China*

Correspondence should be addressed to Fanglei Wang, leizi1123@163.com

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Using the fixed point theorem of cone expansion/compression, we consider the existence results of positive solutions for a nonlinear semipositone telegraph system with repulsive weak singular forces.

1. Introduction

In this paper, we are concerned with the existence of positive solutions for the nonlinear telegraph system:

$$\begin{aligned} u_{tt} - u_{xx} + c_1 u_t + a_1(t,x)u &= f(t,x,v), \\ v_{tt} - v_{xx} + c_2 v_t + a_2(t,x)v &= g(t,x,u), \end{aligned} \tag{1.1}$$

with doubly periodic boundary conditions

$$\begin{aligned} u(t+2\pi,x) = u(t,x+2\pi) = u(t,x), \ \ (t,x) \in R^2, \\ v(t+2\pi,x) = v(t,x+2\pi) = v(t,x), \ \ (t,x) \in R^2. \end{aligned} \tag{1.2}$$

In particular, the function $f(t,x,v)$ may be singular at $v = 0$ or superlinear at $v = +\infty$, and $g(t,x,u)$ may be singular at $u = 0$ or superlinear at $u = +\infty$.

In the latter years, the periodic problem for the semilinear singular equation

$$x'' + a(t)x = \frac{b(t)}{x^{\lambda}} + c(t), \tag{1.3}$$

with $a, b, c \in L^1[0,T]$ and $\lambda > 0$, has received the attention of many specialists in differential equations. The main methods to study (1.3) are the following three common techniques:

(i) the obtainment of a priori bounds for the possible solutions and then the applications of topological degree arguments;

(ii) the theory of upper and lower solutions;

(iii) some fixed point theorems in a cone.

We refer the readers to see [1–7] and the references therein.

Equation (1.3) is related to the stationary version of the telegraph equation

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t, x, u), \tag{1.4}$$

where $c > 0$ is a constant and $\lambda \in R$. Because of its important physical background, the existence of periodic solutions for a single telegraph equation or telegraph system has been studied by many authors; see [8–16]. Recently, Wang utilize a weak force condition to enable the achievement of new existence criteria for positive doubly periodic solutions of nonlinear telegraph system through a basic application of Schauder's fixed point theorem in [17]. Inspired by these papers, here our interest is in studying the existence of positive doubly periodic solutions for a semipositone nonlinear telegraph system with repulsive weak singular forces by using the fixed point theorem of cone expansion/compression.

Lemma 1.1 (see [18]). *Let E be a Banach space, and let $K \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_2$; *or*

(ii) $\|Tu\| \ge \|u\|,\ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \le \|u\|,\ u \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

This paper is organized as follows: in Section 2, some preliminaries are given; in Section 3, we give the main results.

2. Preliminaries

Let T^2 be the torus defined as

$$\mathrm{T}^2 = \left(\frac{R}{2\pi Z}\right) \times \left(\frac{R}{2\pi Z}\right). \tag{2.1}$$

Doubly 2π-periodic functions will be identified to be functions defined on T^2. We use the notations

$$L^p\left(\mathbb{T}^2\right), C\left(\mathbb{T}^2\right), C^\alpha\left(\mathbb{T}^2\right), D\left(\mathbb{T}^2\right) = C^\infty\left(\mathbb{T}^2\right), \dots \tag{2.2}$$

to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space $D'(\mathbb{T}^2)$ denotes the space of distributions on $\mathbb{T}^2$.

By a doubly periodic solution of (1.1)-(1.2) we mean that a $(u, v) \in L^1(\mathbb{T}^2) \times L^1(\mathbb{T}^2)$ satisfies (1.1)-(1.2) in the distribution sense; that is,

$$\begin{aligned} \int_{\mathbb{T}^2} u\left(\varphi_{tt} - \varphi_{xx} - c_1\varphi_t + a_1(t,x)\varphi\right) dt\, dx &= \int_{\mathbb{T}^2} f(t,x,v)\varphi\, dt\, dx, \\ \int_{\mathbb{T}^2} v\left(\varphi_{tt} - \varphi_{xx} - c_2\varphi_t + a_2(t,x)\varphi\right) dt\, dx &= \int_{\mathbb{T}^2} g(t,x,u)\varphi\, dt\, dx, \end{aligned} \quad \forall \varphi \in D\left(\mathbb{T}^2\right). \tag{2.3}$$

First, we consider the linear equation

$$u_{tt} - u_{xx} + c_i u_t - \lambda_i u = h_i(t,x), \quad \text{in } D'\left(\mathbb{T}^2\right), \tag{2.4}$$

where $c_i > 0$, $\lambda_i \in R$, and $h_i(t,x) \in L^1(\mathbb{T}^2)$, $(i = 1, 2)$.

Let $\pounds_{\lambda_i}$ be the differential operator

$$\pounds_{\lambda_i} = u_{tt} - u_{xx} + c_i u_t - \lambda_i u, \tag{2.5}$$

acting on functions on $\mathbb{T}^2$. Following the discussion in [14], we know that if $\lambda_i < 0$, then $\pounds_{\lambda_i}$ has the resolvent R_{λ_i}:

$$R_{\lambda_i} : L^1\left(\mathbb{T}^2\right) \longrightarrow C\left(\mathbb{T}^2\right), \quad h_i \longmapsto u_i, \tag{2.6}$$

where u_i is the unique solution of (2.4), and the restriction of R_{λ_i} on $L^p(\mathbb{T}^2)$ $(1 < p < \infty)$ or $C(\mathbb{T}^2)$ is compact. In particular, $R_{\lambda_i} : C(\mathbb{T}^2) \to C(\mathbb{T}^2)$ is a completely continuous operator.

For $\lambda_i = -c_i^2/4$, the Green function $G_i(t,x)$ of the differential operator $\pounds_{\lambda_i}$ is explicitly expressed; see lemma 5.2 in [14]. From the definition of $G_i(t,x)$, we have

$$\begin{aligned} \underline{G_i} &:= \operatorname{ess\ inf} G_i(t,x) = \frac{e^{-3c_i\pi/2}}{(1 - e^{-c_i\pi})^2}, \\ \overline{G_i} &:= \operatorname{ess\ sup} G_i(t,x) = \frac{(1 + e^{-c_i\pi})}{2(1 - e^{-c_i\pi})^2}. \end{aligned} \tag{2.7}$$

Let E denote the Banach space $C(\mathbb{T}^2)$ with the norm $\|u\| = \max_{(t,x)\in\mathbb{T}^2} |u(t,x)|$, then E is an ordered Banach space with cone

$$K_0 = \left\{ u \in E \mid u(t,x) \geq 0,\ \forall (t,x) \in \mathbb{T}^2 \right\}. \tag{2.8}$$

For convenience, we assume that the following condition holds throughout this paper:

(H1) $a_i(t,x) \in C(\mathbb{T}^2, R^+)$, $0 < a_i(t,x) \leq c_i^2/4$ for $(t,x) \in \mathbb{T}^2$, and $\int_{\mathbb{T}^2} a_i(t,x)\, dt\, dx > 0$.

Next, we consider (2.4) when $-\lambda_i$ is replaced by $a_i(t,x)$. In [10], Li has proved the following unique existence and positive estimate result.

Lemma 2.1. *Let* $h_i(t,x) \in L^1(\mathrm{T}^2)$; *E is the Banach space* $C(\mathrm{T}^2)$. *Then;* (2.4) *has a unique solution* $u_i = P_i h_i$; $P_i : L^1(\mathrm{T}^2) \to C(\mathrm{T}^2)$ *is a linear bounded operator with the following properties;*

(i) $P_i : C(\mathrm{T}^2) \to C(\mathrm{T}^2)$ *is a completely continuous operator;*

(ii) *if* $h_i(t,x) > 0$, *then a.e.* $(t,x) \in \mathrm{T}^2$, $P_i[h_i(t,x)]$ *has the positive estimate*

$$\underline{G_i}\|h_i\|_{L^1} \leq P_i[h_i(t,x)] \leq \frac{\overline{G_i}}{\underline{G_i}\|a_i\|_{L^1}}\|h_i\|_{L^1}. \tag{2.9}$$

3. Main Result

In this section, we establish the existence of positive solutions for the telegraph system

$$\begin{gathered} v_{tt} - v_{xx} + c_1 v_t + a_1(t,x)v = f(t,x,u), \\ v_{tt} - v_{xx} + c_2 v_t + a_2(t,x)v = g(t,x,u). \end{gathered} \tag{3.1}$$

where $a_i \in C(R^2, R^+)$ and $f(t,x,v)$ may be singular at $v = 0$. In particular, $f(t,x,v)$ may be negative or superlinear at $v = +\infty$. $g(t,x,u)$ has the similar assumptions. Our interest is in working out what weak force conditions of $f(t,x,v)$ at $v = 0$, $g(t,x,u)$ at $u = 0$ and what superlinear growth conditions of $f(t,x,v)$ at $v = +\infty$, $g(t,x,u)$ at $u = +\infty$ are needed to obtain the existence of positive solutions for problem (1.1)-(1.2).

We assume the following conditions throughout.

(H2) $f,\ g : \mathrm{T}^2 \times (0,\infty) \to R$ is continuous, and there exists a constant $M > 0$ such that

$$f_1(t,x,u) + M \geq 0, \quad f_2(t,x,u) + M \geq 0, \quad \forall (t,x) \in \mathrm{T}^2 \text{ and } u, v \in (0,\infty). \tag{3.2}$$

(H3) $F(t,x,v) = f(t,x,v) + M \leq j_1(v) + h_1(v)$ for $(t,x,v) \in \mathrm{T}^2 \times (0,\infty)$ with $j_1 > 0$ continuous and nonincreasing on $(0,\infty)$, $h_1 \geq 0$ continuous on $(0,\infty)$ and h_1/j_1 nondecreasing on $(0,\infty)$.

$G(t,x,u) = g(t,x,u) + M \leq j_2(u) + h_2(u)$ for $(t,x,u) \in \mathrm{T}^2 \times (0,\infty)$ with $j_2 > 0$ continuous and nonincreasing on $(0,\infty)$, $h_2 \geq 0$ continuous on $(0,\infty)$ and h_2/j_2 nondecreasing on $(0,\infty)$.

(H4) $F(t,x,v) = f(t,x,v) + M \geq j_3(v) + h_3(v)$ for all $(t,x,v) \in \mathrm{T}^2 \times (0,\infty)$ with $j_3 > 0$ continuous and nonincreasing on $(0,\infty)$, $h_3 \geq 0$ continuous on $(0,\infty)$ with h_3/j_3 nondecreasing on $(0,\infty)$;

$G(t,x,u) = g(t,x,u) + M \geq j_4(u) + h_4(u)$ for all $(t,x,u) \in \mathrm{T}^2 \times (0,\infty)$ with $j_4 > 0$ continuous and nonincreasing on $(0,\infty)$, $h_4 \geq 0$ continuous on $(0,\infty)$ with h_4/j_4 nondecreasing on $(0,\infty)$.

(H5) There exists

$$r > \frac{M\|\omega_1\|}{\delta_1}, \tag{3.3}$$

such that

$$r \geq \frac{4\pi^2 \overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} I_1 \cdot I_2, \tag{3.4}$$

here

$$\begin{aligned} I_1 &= j_1\left(\underline{G_2} j_4(r)\left\{1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right\}4\pi^2 - M\|\omega_2\|\right), \\ I_2 &= 1 + \frac{h_1\left(\left(4\pi^2\overline{G_2}/\underline{G_2}\|a_2\|_{L^1}\right) j_2(\delta_1 r - M\|\omega_1\|)\{1 + h_2(r)/j_2(r)\}\right)}{j_1\left(\left(4\pi^2\overline{G_2}/\underline{G_2}\|a_2\|_{L^1}\right) j_2(\delta_1 r - M\|\omega_1\|)\{1 + h_2(r)/j_2(r)\}\right)}, \end{aligned} \tag{3.5}$$

where $\delta_i = (\underline{G_i}^2\|a_i\|_{L^1}/\overline{G_i}) \in (0,1)$, and $\omega_i(t,x)$ is the unique solution to problem:

$$\begin{aligned} u_{tt} - u_{xx} + c_i u_t + a_i(t,x)u &= 1, \\ u(t+2\pi, x) = u(t, x+2\pi) &= u(t,x), \end{aligned} \qquad (t,x) \in R^2. \tag{3.6}$$

(H6) There exists $R > r$, such that

$$\begin{gathered} 4\pi^2 \underline{G_1} I_3 \cdot I_4 \geq R, \\ \delta_2 j_4(R)\left\{1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right\} > M, \end{gathered} \tag{3.7}$$

where

$$\begin{aligned} I_3 &= \underline{G_1} j_3\left(\frac{4\pi^2\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|)\left\{1 + \frac{h_2(R)}{j_2(R)}\right\}\right), \\ I_4 &= 1 + \frac{h_3\left(\underline{G_2} j_4(R)\{1 + h_4(\delta_1 R - M\|\omega_1\|)/j_4(\delta_1 R - M\|\omega_1\|)\}4\pi^2 - M\|\omega_2\|\right)}{j_3\left(\underline{G_2} j_4(R)\{1 + h_4(\delta_1 R - M\|\omega_1\|)/j_4(\delta_1 R - M\|\omega_1\|)\}4\pi^2 - M\|\omega_2\|\right)}. \end{aligned} \tag{3.8}$$

Theorem 3.1. *Assume that (H1)–(H6) hold. Then, the problem* (1.1)-(1.2) *has a positive doubly periodic solution* (u,v).

Proof. To show that (1.1)-(1.2) has a positive solution, we will proof that

$$\begin{aligned} u_{tt} - u_{xx} + c_1 u_t + a_1(t,x)u &= F(t,x,v - M\omega_2), \\ v_{tt} - v_{xx} + c_2 v_t + a_2(t,x)v &= G(t,x,u - M\omega_1) \end{aligned} \tag{3.9}$$

has a solution $(\tilde{u},\tilde{v}) = (u+M\omega_1, v+M\omega_2)$ with $\tilde{u} > M\omega_1, \tilde{v} > M\omega_2$ for $(t,x) \in \mathbb{T}^2$. In addition, by Lemma 2.1, it is clear to see that $(u,v) \in C^2(\mathbb{T}^2) \times C^2(\mathbb{T}^2)$ is a solution of (3.9) if and only if $(u,v) \in C(\mathbb{T}^2) \times C(\mathbb{T}^2)$ is a solution of the following system:

$$\begin{aligned} u &= P_1(F(t,x,v - M\omega_2)), \\ v &= P_2(G(t,x,u - M\omega_1)). \end{aligned} \tag{3.10}$$

Evidently, (3.10) can be rewritten as the following equation:

$$u = P_1(F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2)). \tag{3.11}$$

Define a cone $K \subset E$ as

$$K = \{u \in E : u \geq 0, u \geq \delta_1 \|u\|\}. \tag{3.12}$$

We define an operator $T : E \to K$ by

$$(Tu)(t,x) = P_1(F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2)) \tag{3.13}$$

for $u \in E$ and $(t,x) \in \mathbb{T}^2$. We have the conclusion that $T : E \to E$ is completely continuous and $T(K) \subseteq K$. The complete continuity is obvious by Lemma 2.1. Now, we show that $T(K) \subseteq K$.

For any $u \in K$, we have

$$Tu = P_1(F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2)). \tag{3.14}$$

From (H1)–(H3) and Lemma 2.1, we have

$$\begin{aligned} Tu &= P_1(F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2)) \\ &\geq \underline{G_1} \|F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2)\|_{L^1}, \\ \|Tu\| &= \|P(F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2))\| \\ &\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} \|F(t,x,P_2(G(t,x,u - M\omega_1)) - M\omega_2)\|_{L^1}. \end{aligned} \tag{3.15}$$

So, we get

$$Tu \geq \frac{\underline{G_1}^2 \|a_1\|_{L^1}}{\overline{G_1}} \|Tu\| \geq \delta_1 \|Tu\|, \tag{3.16}$$

namely, $T(K) \subseteq K$.

Let

$$\Omega_r = \{u \in E : \|u\| < r\}, \qquad \Omega_R = \{u \in E : \|u\| < R\}. \tag{3.17}$$

Since $r \leq \|u\| \leq R$ for any $u \in K \cap (\overline{\Omega_R} \setminus \Omega_r)$, we have $0 < \delta_1 r - M\|\omega\| \leq u - M\omega_1 \leq R$.

First, we show

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_r. \tag{3.18}$$

In fact, if $u \in K \cap \partial\Omega_r$, then $\|u\| = r$ and $u \geq \delta_1 r > M\|\omega_1\|$ for$(t,x) \in \mathbb{T}^2$. By (H3) and (H4), we have

$$\begin{aligned} P_2(G(t,x,u-M\omega_1)) &\leq \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}}\|G(t,x,u-M\omega_1)\|_{L^1} \\ &\leq \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}}\left\|j_2(u-M\omega_1)\left(1+\frac{h_2(u-M\omega_1)}{j_2(u-M\omega_1)}\right)\right\|_{L^1} \\ &\leq \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}}j_2(\delta_1 r - M\|\omega_1\|)\left\{1+\frac{h_2(r)}{j_2(r)}\right\}4\pi^2, \end{aligned} \tag{3.19}$$

$$\begin{aligned} P_2(G(t,x,u-M\omega_1)) &\geq \underline{G_2}\|G(t,x,u-M\omega_1)\|_{L^1} \\ &\geq \underline{G_2}\left\|j_4(u-M\omega_1)\left(1+\frac{h_4(u-M\omega_1)}{j_4(u-M\omega_1)}\right)\right\|_{L^1} \\ &\geq \underline{G_2}j_4(r)\left\{1+\frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right\}4\pi^2. \end{aligned} \tag{3.20}$$

In addition, we also have

$$\begin{aligned} P_2(G(t,x,u-M\omega_1)) &\geq \underline{G_2}j_4(r)\left\{1+\frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right\}4\pi^2 \\ &\geq \underline{G_2}j_4(R)\left\{1+\frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right\}4\pi^2 \\ &> \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}}M4\pi^2 \\ &\geq M\omega_2, \end{aligned} \tag{3.21}$$

by (H5), (H6), and (3.20).

So, we have

$$\begin{aligned} Tu &= P_1(F(t,x,v-M\omega_2)) \\ &\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}}\|F(t,x,v-M\omega_2)\|_{L^1} \\ &\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}}\left\|j_1(v-M\omega_2)\left\{1+\frac{h_1(v-M\omega_2)}{j_1(v-M\omega_2)}\right\}\right\|_{L^1} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} \Big\| j_1(P_2(G(t,x,u-M\omega_1)) - M\omega_2) \\
&\quad \times \left\{1 + \frac{h_1(P_2(G(t,x,u-M\omega_1)) - M\omega_2)}{j_1(P_2(G(t,x,u-M\omega_1)) - M\omega_2)}\right\} \Big\|_{L^1} \\
&\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} j_1\left(\underline{G_2} j_4(r)\left\{1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)}\right\}4\pi^2 - M\|\omega_2\|\right) \\
&\quad \times \left\{1 + \frac{h_1\left(\left(\overline{G_2}/\underline{G_2}\|a_2\|_{L^1}\right) j_2(\delta_1 r - M\|\omega_1\|)\{1 + h_2(r)/j_2(r)\}4\pi^2\right)}{j_1\left(\left(\overline{G_2}/\underline{G_2}\|a_2\|_{L^1}\right) j_2(\delta_1 r - M\|\omega_1\|)\{1 + h_2(r)/j_2(r)\}4\pi^2\right)}\right\}4\pi^2 \\
&\leq r = \|u\|
\end{aligned}
\tag{3.22}
$$

for $(t,x) \in \mathrm{T}^2$, since $\delta_1 r - M\|\omega_1\| \leq u - M\omega_1 \leq r$.

This implies that $\|Tu\| \leq \|u\|$; that is, (3.18) holds.

Next, we show

$$
\|Tu\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_R. \tag{3.23}
$$

If $u \in K \cap \partial\Omega_R$, then $\|u\| = R$ and $u \geq \delta R > M\|\omega_1\|$ for $(t,x) \in \mathrm{T}^2$. From (H4) and (H6), we have

$$
\begin{aligned}
Tu &= P_1(F(t,x,v-M\omega_1)) \\
&\geq \underline{G_1} \left\| j_3(v - M\omega_2)\left\{1 + \frac{h_3(v-M\omega_2)}{j_3(v-M\omega_2)}\right\}\right\|_{L^1} \\
&\geq \underline{G_1} \left\| j_3(P_2(G(t,x,u-M\omega_1)) - M\omega_2) \times \left\{1 + \frac{h_3(P_2(G(t,x,u-M\omega_1)) - M\omega_2)}{j_3(P_2(G(t,x,u-M\omega_1)) - M\omega_2)}\right\}\right\|_{L^1} \\
&\geq \underline{G_1} \Bigg\| j_3\left(\frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|)\left\{1 + \frac{h_2(R)}{j_2(R)}\right\}4\pi^2\right) \\
&\qquad \times \left\{1 + \frac{h_3\left(\underline{G_2} j_4(R)\{1 + h_4(\delta_1 R - M\|\omega_1\|)/j_4(\delta_1 R - M\|\omega_1\|)\}4\pi^2 - M\|\omega_2\|\right)}{j_3\left(\underline{G_2} j_4(R)\{1 + h_4(\delta_1 R - M\|\omega_1\|)/j_4(\delta_1 R - M\|\omega_1\|)\}4\pi^2 - M\|\omega_2\|\right)}\right\}\Bigg\|_{L^1} \\
&\geq R = \|u\|
\end{aligned}
\tag{3.24}
$$

for $(t,x) \in \mathrm{T}^2$, since $\delta_1 R - M\|\omega_1\| \leq u - M\omega_1 \leq R$.

This implies that $Tu \geq \|u\|$; that is, (3.23) holds.

Finally, (3.18), (3.23), and Lemma 1.1 guarantee that T has a fixed point $u \in K \cap \overline{\Omega_R} \setminus \Omega_r$ with $r \leq \|u\| \leq R$. Clearly, $u > M\omega_1$.

Since

$$\begin{aligned} P_2(G(t,x,u-M\omega_1)) &\geq \underline{G_2}\|G(t,x,M\omega_1)\|_{L^1} \\ &\geq \underline{G_2}\left\| j_4(u-M\omega_1)\left(1+\frac{h_4(u-M\omega_1)}{j_4(u-M\omega_1)}\right)\right\|_{L^1} \\ &\geq \underline{G_2}j_4(R)\left\{1+\frac{h_4(\delta_1 r-M\|\omega_1\|)}{j_4(\delta_1 r-M\|\omega_1\|)}\right\}4\pi^2 \\ &> \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}}M4\pi^2 \\ &\geq M\omega_2, \end{aligned} \tag{3.25}$$

then we have a doubly periodic solution (u,v) of (3.9) with $u > M\omega_1$, $v > M\omega_2$, namely, $(u-M\omega_1, v-M\omega_2) > (0,0)$ is a positive solution of (1.1) with (1.2). □

Similarly, we also obtain the following result.

Theorem 3.2. *Assume that (H1)–(H4) hold. In addition, we assume the following.*

(H7) *There exists*

$$r > \frac{M\|\omega_2\|}{\delta_2}, \tag{3.26}$$

such that

$$r \geq \frac{4\pi^2\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}}I_5\cdot I_6, \tag{3.27}$$

here

$$\begin{aligned} I_5 &= j_2\left(4\pi^2\underline{G_1}j_3(r)\left\{1+\frac{h_3(\delta_2 r-M\|\omega_2\|)}{j_3(\delta_2 r-M\|\omega_2\|)}\right\}-M\|\omega_1\|\right), \\ I_6 &= 1+\frac{h_2\left(\left(4\pi^2\overline{G_1}/\underline{G_1}\|a_1\|_{L^1}\right)j_1(\delta_2 r-M\|\omega_2\|)\{1+h_1(r)/j_1(r)\}\right)}{j_2\left(\left(4\pi^2\overline{G_1}/\underline{G_1}\|a_1\|_{L^1}\right)j_1(\delta_2 r-M\|\omega_2\|)\{1+h_1(r)/j_1(r)\}\right)}. \end{aligned} \tag{3.28}$$

(H8) *There exists $R > r$, such that*

$$\begin{aligned} &4\pi^2\underline{G_2}I_7\cdot I_8 \geq R, \\ &\delta_1 j_3(R)\left\{1+\frac{h_3(\delta_2 r-M\|\omega_2\|)}{j_3(\delta_2 r-M\|\omega_2\|)}\right\} > M, \end{aligned} \tag{3.29}$$

where

$$I_7 = j_4\left(\frac{4\pi^2\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} j_1(\delta_2 R - M\|\omega_2\|)\left\{1 + \frac{h_1(R)}{j_1(R)}\right\}\right),$$
$$I_8 = 1 + \frac{h_4\left(4\pi^2\underline{G_1} j_3(R)\{1 + h_3(\delta_2 R - M\|\omega_2\|)/j_3(\delta_2 R - M\|\omega_2\|)\} - M\|\omega_1\|\right)}{j_4\left(4\pi^2\underline{G_1} j_3(R)\{1 + h_3(\delta_2 R - M\|\omega_2\|)/j_3(\delta_2 R - M\|\omega_2\|)\} - M\|\omega_1\|\right)}. \tag{3.30}$$

Then, problem (1.1)-(1.2) *has a positive periodic solution.*

4. An Example

Consider the following system:

$$\begin{gathered}
u_{tt} - u_{xx} + 2u_t + \sin^2(t+x)u = \mu\left(v^{-\alpha} + v^{\beta} + k_1(t,x)\right),\\
v_{tt} - v_{xx} + 2v_t + \cos^2(t+x)v = \lambda\left(u^{-\tau} + u^{\sigma} + k_2(t,x)\right),\\
u(t+2\pi,x) = u(t,x+2\pi) = u(t,x), \quad (t,x)\in R^2,\\
v(t+2\pi,x) = v(t,x+2\pi) = v(t,x), \quad (t,x)\in R^2,
\end{gathered} \tag{4.1}$$

where $c_1 = c_2 = 2$, $\mu,\ \lambda > 0$, $\alpha,\ \tau > 0$, $\beta,\ \sigma > 1$, $a_1(t,x) = \sin^2(t+x)$, $a_2(t,x) = \cos^2(t+x) \in C(\mathrm{T}^2, R^+)$, $k_i : \mathrm{T}^2 \to R$ is continuous. When μ is chosen such that

$$\mu < \sup_{u\in((M\|\omega_1\|)/\delta_1,\infty)} \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2}\frac{I^1}{I^2}, \tag{4.2}$$

here we denote

$$\begin{aligned}
I^1 &= u\left(\underline{G}\lambda u^{-\tau}\{1 + (\delta_1 u - M\|\omega_1\|)^{\ \sigma+\tau}\}4\pi^2 - M\|\omega_2\|\right)^{\alpha},\\
I^2 &= 1 + \left(\frac{\overline{G}}{\underline{G}\|a_2\|_{L^1}}\lambda(\delta_1 u - M\|\omega_1\|)^{-\tau}(1 + u^{\sigma+\tau} + 2Hu^{\tau})4\pi^2\right)^{\beta+\alpha}\\
&\quad + 2H\left(\frac{\overline{G}}{\underline{G}\|a_2\|_{L^1}}\lambda(\delta_1 u - M\|\omega_1\|)^{-\tau}(1 + u^{\sigma+\tau} + 2Hu^{\tau})4\pi^2\right),
\end{aligned} \tag{4.3}$$

where $H = \max\{\|k_1\|, \|k_2\|\}$ and the Green function $G_1 = G_2 = G$. Then, problem (4.1) has a positive solution.

To verify the result, we will apply Theorem 3.1 with $M = \max\{\mu H, \lambda H\}$ and

$$j_1(v) = j_3(v) = \mu v^{-\alpha},\ h_1(v) = \mu\left(v^{\beta} + 2H\right),\ h_3(v) = \mu v^{\beta},$$
$$j_2(u) = j_4(u) = \lambda u^{-\tau},\ h_2(u) = \mu(u^{\sigma} + 2H),\ h_4(u) = \mu u^{\sigma}. \tag{4.4}$$

Clearly, (H1)–(H4) are satisfied.

Set

$$T(u) = \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2}\frac{I^1}{I^2}, \quad u \in \left(\frac{(M\|\omega_1\|)}{\delta_1}, +\infty\right). \tag{4.5}$$

Obviously, $T((M\|\omega_1\|)/\delta_1) = 0$, $T(\infty) = 0$, then there exists $r \in ((M\|\omega_1\|)/\delta_1, +\infty)$ such that

$$T(r) = \sup_{u\in((M\|\omega_1\|)/\delta_1,\infty)} \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2}\frac{I^1}{I^2}. \tag{4.6}$$

This implies that there exists

$$r \in \left(\frac{(M\|\omega_1\|)}{\delta_1}, +\infty\right), \tag{4.7}$$

such that

$$\mu < \sup_{u\in((M\|\omega_1\|)/\delta_1,\infty)} \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2}\frac{I^1}{I^2}. \tag{4.8}$$

So, (H5) is satisfied.

Finally, since

$$\frac{R\left(\left(\overline{G}/\underline{G}\|a_2\|_{L^1}\right)\lambda(\delta_1 R - M\|\omega_1\|)^{-\tau}(1 + R^{\sigma+\tau} + 2HR^{\tau})4\pi^2\right)^{\alpha}}{\mu\underline{G}\left[1 + \left(\underline{G}\lambda R^{-\tau}\{1 + (\delta_1 R - M\|\omega_1\|)^{\sigma+\tau}\}4\pi^2 - M\|\omega_2\|\right)^{\alpha+\beta}\right]} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty, \tag{4.9}$$

this implies that there exists R. In addition, for fixed r, R, choosing λ sufficiently large, we have

$$\delta_2\lambda R^{-\tau}\{1 + (\delta_1 r - M\|\omega_1\|)^{\sigma+\tau}\} > M. \tag{4.10}$$

Thus, (H6) is satisfied. So, all the conditions of Theorem 3.1 are satisfied.

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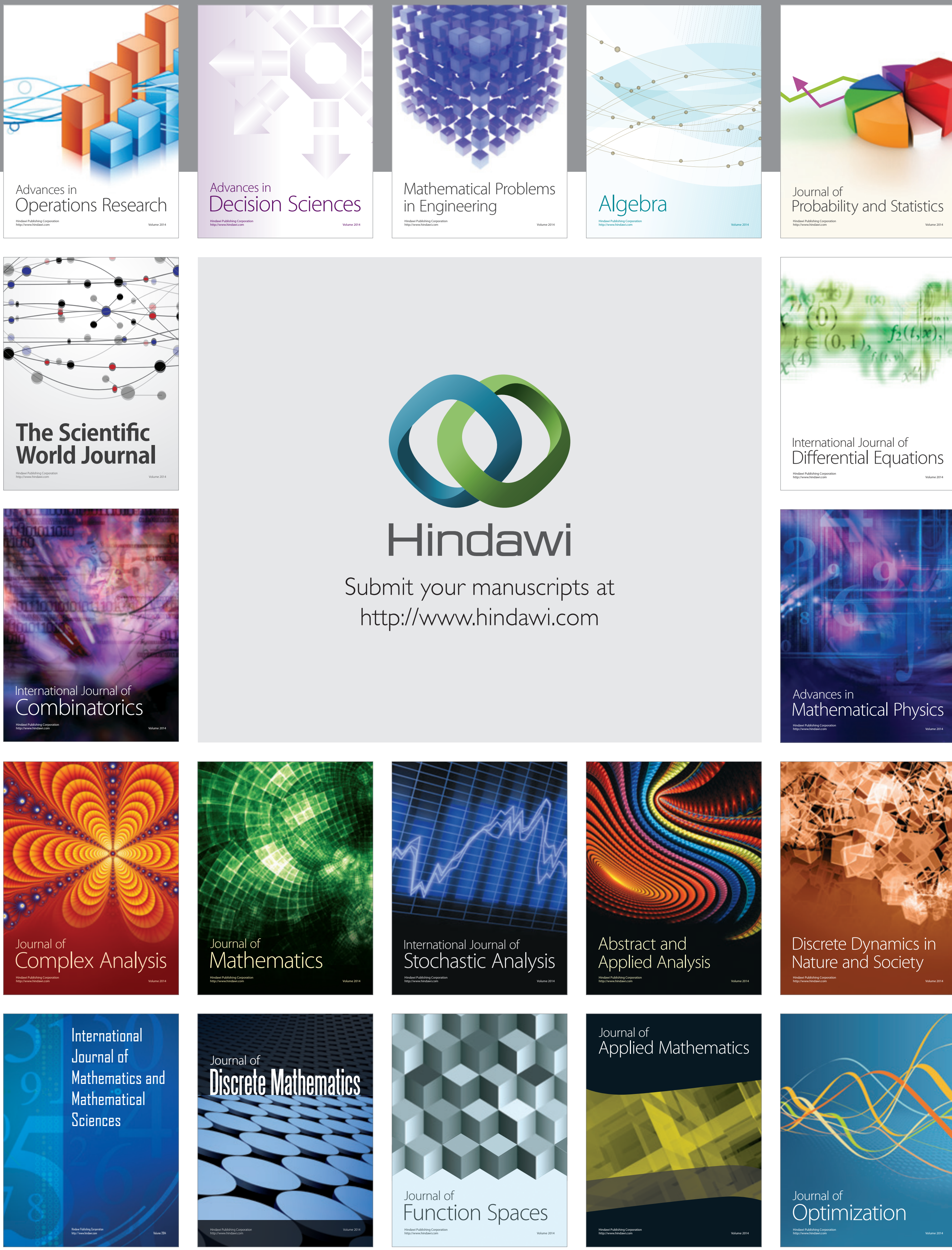
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