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Research Article

Suppressing Chaos of Duffing-Holmes System Using Random Phase

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The effect of random phase for Duffing-Holmes equation is investigated. We show that as the intensity of random noise properly increases the chaotic dynamical behavior will be suppressed by the criterion of top Lyapunov exponent, which is computed based on the Khasminskii's formulation and the extension of Wedig's algorithm for linear stochastic systems. Then, the obtained results are further verified by the Poincaré map analysis, phase plot, and time evolution on dynamical behavior of the system, such as stability, bifurcation, and chaos. Thus excellent agrement between these results is found.

1. Introduction

For the past ten years, there has been a great deal of interest in the chaos control's research which has become one of nonlinear scientific field hot spot issues. After OGY methods were proposed by Ott et al. [1], various methods for chaos control's have been given which are composed of the feedback control and the nonfeedback control. The feedback control methods [1–3], which can exploit the chaos control's characteristic: the sensitivity to initial condition, use some weak feedback control to make the chaotic trajectory approach and settle down finally to a desired stabilized periodic orbit, formerly unstably embedded in the chaotic manifold. The nonfeedback methods [4–9] can eliminate chaos by using a period adjustment coming from a out period incentive for coupling system variables. Because noise is ubiquitous in actual environment, the research into the influence of noise on the system is very important.

Stochastic forces or random noise have been greatly used in studying the control of chaos. For example, Ramesh and Narayanan [10] explored the robustness in nonfeedback chaos control in presence of uniform noise and found that the system would lose control while noise intensity was raised to a threshold level. Wei and Leng [11] studied the chaotic behavior in Duffing oscillator in presence of white noise by the Lyapunov exponent. Liu et al. [12] investigated the generation of chaos in a kind of Hamiltonian system subject to bounded noise by the criterion of stochastic Melnikov function and Lyapunov exponent. Qu et al. [13] further applied weak harmonic excitations to investigate the chaos control of nonautonomous systems and especially observed that the phase control in weak harmonic excitation may greatly affect taming nonautonomous chaos, and Lei et al. [14] have investigated the control of chaos with effect of proper random phase. Recently, much work for suppressing chaos by random excitation [15–18] is carried out.

Duffing equation is the reduced form of lots of practical system model, for example, swinging pendulum model and financial model. Duffing system is a typical nonlinear vibration system; in engineering, lots of mathematical models of nonlinear vibration problems could be transformed to this equation, for example, ship's weaving, structural vibration, destruction of chemical bond, and so forth. Disturbed axial tensile force model of lateral wave equation and dynamics equation of rotor bearing are also the same as Duffing system. To some extent, Duffing system is the basis of lots of complicated dynamics; it has not only theoretical significance but also important actual value. This paper focuses on the study of the influences of random phase on the behaviors of Duffing-Holmes dynamics and shows that the random phase methods can actualize the chaos control. Since the Lyapunov index is an important symbol to describe chaos system, by using the Khasminskii [19] spherical coordinates and Wedig [20] algorithm, we can figure out the top of Lyapunov index. Furthermore, we can ascertain the vanishing of chaos by checking the sign of average value of the Lyapunov index. Finally, we show that the random phase can control the chaos behaviors by combining the Poincar section and the time history.

2. Chaotic Behavior of Duffing-Holmes System

Consider the following Duffing-Holmes system [21–23]

$$\ddot{x} + \delta\dot{x} - x + x^3 = f\sin(\omega t), \tag{2.1}$$

where δ is damping coefficient, f is excitation amplitude, and ω is excitation frequency. Equation (2.1) can be reformulated to one-order nonautonomous equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y + x - x^3 + f\sin(\omega t). \end{aligned} \tag{2.2}$$

The linear format of (2.2) is

$$\begin{aligned} \dot{x}_1 &= y_1, \\ \dot{y}_1 &= \left(1 - 3x^2\right)x_1 - \delta y_1. \end{aligned} \tag{2.3}$$

Denote top Lyapunov exponent [24] as

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\log\frac{\|Y(t)\|}{\|Y(0)\|}, \tag{2.4}$$

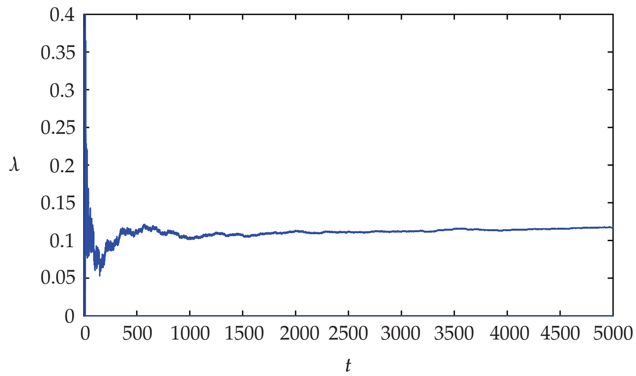

Figure 1: Lyapunov exponents.

where $\|Y(t)\| = \sqrt{x_1^2 + y_1^2}$, the symbol of top Lyapunov exponent is always used to identify the motion state of system, when $\lambda \geq 0$, movement of system is chaotic, and when $\lambda < 0$, it is robust.

Select parameters $\delta = 0.25$, $f = 0.27$, $\omega = 1.0$; the initial condition is $x = 1.0$, $y = 0.0$; we use Runge-Kutta-Verner method of sixth-order to solve system (2.2) and (2.3); the analysis of top Lyapunov exponent in (2.1) is shown in Figure 1.

It is seen from Figure 1 that robust top Lyapunov exponent symbol is positive ($\lambda \approx 0.12$); this illustrate that the system is chaotic. The phase map and time-history map are shown in Figures 2(a) and 2(b).

Let

$$\begin{aligned} \theta &: \mathbb{R}^1 \longrightarrow S^1, \\ t &\longmapsto \theta(t) = \omega t, \quad \bmod 2\pi. \end{aligned} \tag{2.5}$$

Equation (2.2) is reformulated as

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y + x - x^3 + f\sin(\theta), \\ \dot{\theta} &= \omega. \end{aligned} \tag{2.6}$$

Denote cross section as

$$\Sigma^{\theta_0} = \left\{ (x, \theta) \in \mathbb{R}^n \times S^1 \mid \theta = \theta_0 \in (0, 2\pi] \right\}. \tag{2.7}$$

Poincaré cross section is shown in Figure 2(c).

From Figures 2(a) and 2(b), we find that the phase portrait is chaotic and the time history is not regular. From Figure 2(c), the Poincaré surface of section is a chaotic attractor. The conclusion in our paper illustrates that the system is chaos.

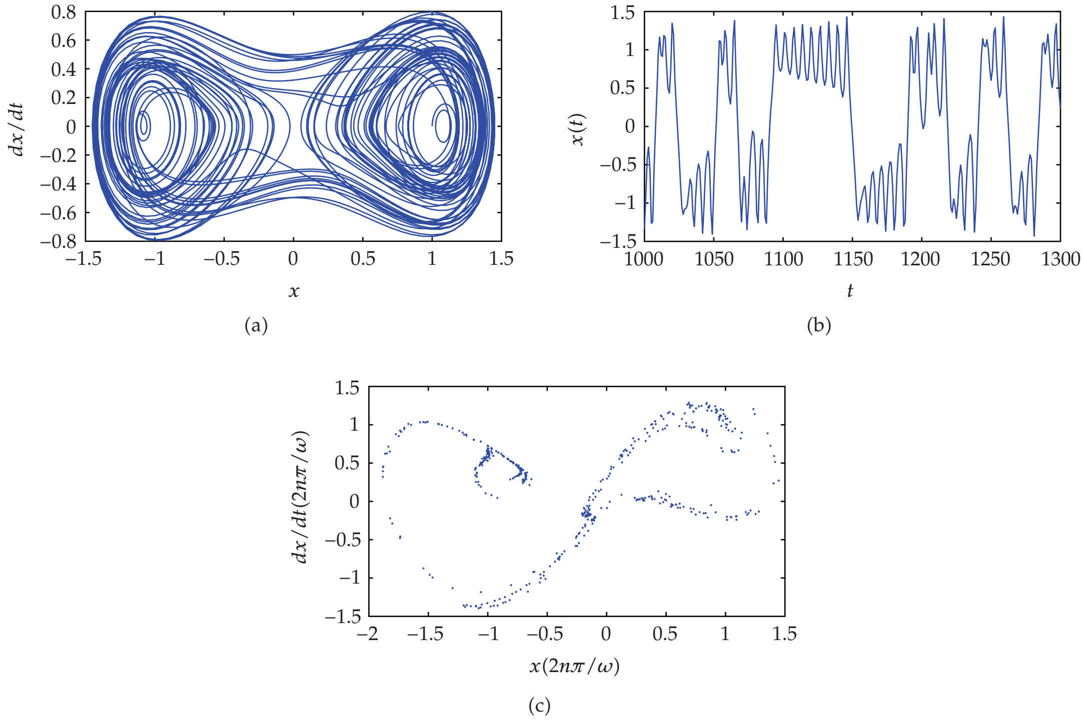

Figure 2: (a) Phase portrait, (b) time history, and (c) Poincaré surface of section.

3. Suppressing Chaos of the Duffing-Holmes System Using a Random Phase

We plug a random phase into (2.2)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -\delta y + x - x^3 + f \sin(\omega t + \sigma \xi(t)), \end{aligned} \tag{3.1}$$

where $\xi(t)$ denotes a standard Gaussian white noise, σ is an intensity, and $\xi(t)$ satisfies $E\xi(t) = 0$, $E\xi(t)\xi(t+\tau) = \zeta(\tau)$, where $\zeta(\tau)$ is the Dirac-Delta function, that is,

$$\zeta(\tau) = \begin{cases} 0, & \tau \neq 0, \\ 1, & \tau = 0. \end{cases} \tag{3.2}$$

Equation (3.1) is linearized as follows:

$$\begin{aligned} \dot{x}_1 &= y_1, \\ \dot{y}_1 &= \left(1 - 3x^2\right) x_1 - \delta y_1. \end{aligned} \tag{3.3}$$

Let

$$A = (A_{ij})_{2\times 2}\begin{bmatrix} 0 & 1 \\ 0 & -\delta \end{bmatrix}, \qquad F(t) = (f_{ij})_{2\times 2} = \begin{bmatrix} 0 & 0 \\ -3x^2 & 0 \end{bmatrix}, \qquad Y = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{3.4}$$

We have

$$\dot{Y} = [A + F(t)]Y. \tag{3.5}$$

Assume that f_{ij}, i, $j = 1, 2$, are ergodic and $E\|b + F(t)\| < \infty$, where the norm $\|A\|$ is defined as the square root of the largest eigenvalue of the matrix $A^T A$. By Ossledec multiple ergodic theorem [25], there exist two real numbers λ_1, λ_2 and two random subspaces $E_1, E_2 (E_1 \oplus E_2 = u_\delta(0) \subset R^2$, and $U_\delta(0)$ denotes the neighborhood of $O(0,0))$, such that,

$$\lambda_i = \lim_{t\to+\infty}\frac{1}{t}\log\|Y(t)\| \quad \text{iff } y_0 \in E_i \setminus \{0\},\ i = 1, 2, \tag{3.6}$$

where $\|Y(t)\| = \sqrt{x_1^2 + y_1^2}$, λ_i ($i = 1$, or 2) is the Lyapunov exponent, representing the rate of exponential convergence or divergence of nearby orbits in a specific direction in E_i. The Ossledec multiple ergodic theorem states that for almost all random initial values in random subset $U_\delta(0)$ there holds $\lambda = \max_i \lambda_i = \lim_{t\to+\infty}(1/t)\log\|Y(t)\|$ and λ is defined as the largest (or top) Lyapunov exponent.

Using Khasminskii's [19] technique, the computation of the top Lyapunov exponent of system (3.5) can be presented as follows.

Let

$$s_1 = \frac{x_1}{a}, \qquad s_2 = \frac{y_1}{a}, \quad a = \|Y(t)\| = \sqrt{x_1^2 + y_1^2}. \tag{3.7}$$

It follows that

$$s_i' = \sum_j \left[A_{ij} - m(t)\delta_{ij} + \left(f_{ij} - n(t)\delta_{ij}\right)\right]s_j, \tag{3.8}$$

where $m(t) = \sum_{k,l} A_{kl}s_k s_l$, $n(t) = \sum_{k,l} f_{kl}s_k s_l$, $\delta_{ij} = 1(i = j)$, $\delta_{ij} = 0(i \neq j)$, and

$$a' = [m(t) + n(t)]a. \tag{3.9}$$

Thus, the largest Lyapunov exponent can be expressed as

$$\lambda = \lim_{t\to\infty}\frac{1}{t}\log a = \lim_{t\to\infty}\frac{1}{t}\int_0^t [m(\tau) + n(\tau)]d\tau. \tag{3.10}$$

Now the top Lyapunov exponent can be obtained by numerical integration of (3.10).

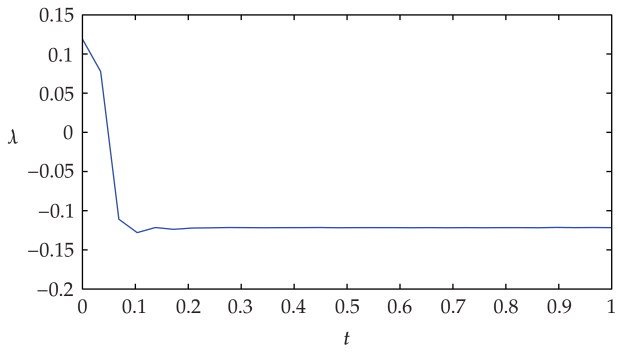

Figure 3: Top Lyapunov exponents versus noise intensity.

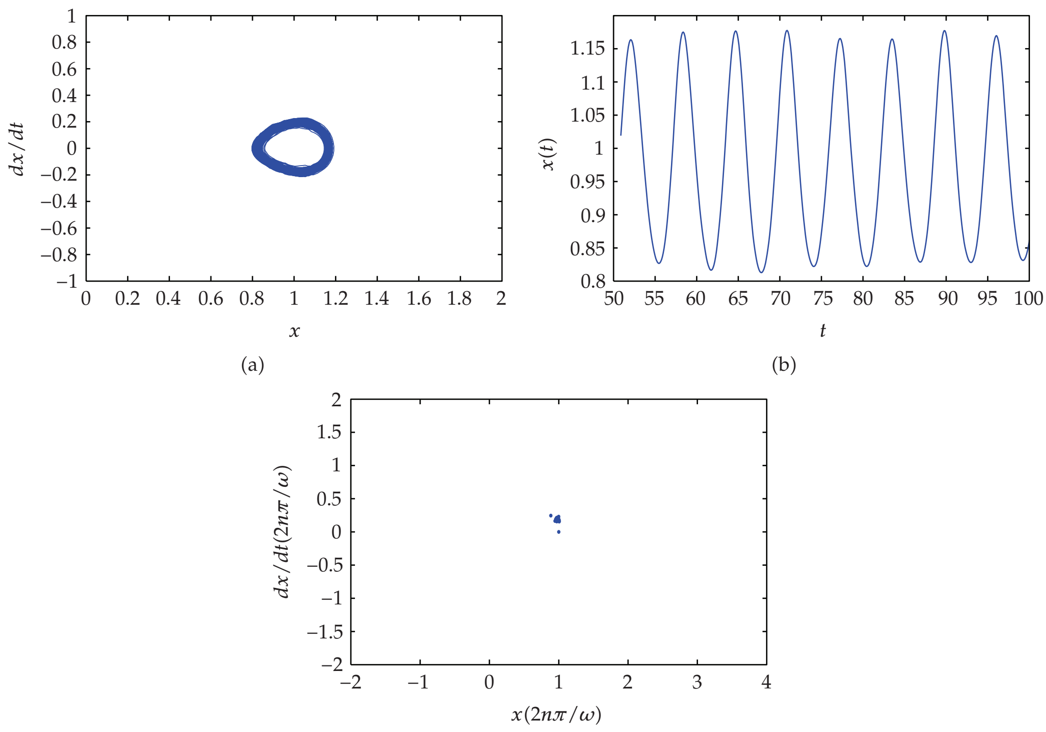

Figure 4: (a) Phase portrait, (b) time history, and (c) Poincaré surface of section.

Take $\delta = 0.25$, $f = 0.27$, $\omega = 1.0$; we solve (3.1) and (3.3) with (3.8)–(3.10) using the Runge-Kutta-Verner method of sixth order. We plot the top Lyaponov exponent depending on the intensity of the noise in Figure 3. From Figure 3, the top Lyaponov exponent keeps positive when σ is smaller than the critical value $\sigma_c = 0.05$. When the intensity is greater than the critical value, the sign of the top Lyaponov exponent suddenly turns from positive to negative, namely, the behavior of this system turns from chaotic to stable abruptly. From then on, the increase of the intensity of stochastic phase would not affect the sign of the top

Lyaponov exponent any longer in the interested parameter range. This suggests the random phase noise effectively stabilizes the system for the parameter range, $\sigma \in (\sigma_c, 1.0]$.

Now we apply Poincaré map of (2.1) to verify the above results. Set Poincaré may as

$$P : \Sigma \longrightarrow \Sigma, \quad \Sigma = \left\{ (x(t), \dot{x}(t)) \mid t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \ldots \right\} \subset R^2. \tag{3.11}$$

For the given initial condition as in Figure 4, the differential (2.1) is solved by the sixth-order Runge-Kutta-Verner method and the solution is plotted for every $T = 2\pi/\omega$, and after deleting the first 500 transient points, the succeeded 200 iteration points are used to plot the Poincaré map for $\sigma = 0.2$ in Figure 4(c). For $\sigma = 0.2$, the phase portrait and time history are plotted in Figures 4(a) and 4(b), too.

From the comparison of Figures 2 and 4, the chaotic phase portrait corresponding to ($\sigma = 0.0$) is changed to a circle. The chaotic state of the time history took place by the periodic state. Poincaré surface of section turns from the chaotic attractor to the stable attractor. It appears that the chaotic state of the original system has been controlled to the stable state by using a random phase.

4. Conclusions

Based on the work of Khasminskii and Wedig, we derive the top Lyaponov exponent for the Duffing-Holmes random system. We have shown that the chaotic dynamical behavior will be suppressed as the noise intensity increases slightly by the criterion of the top Lyaponov exponent. The Poincaré map analysis, the phase portrait, and the time history fully verified the proposed results. We can point out that the random phase is the important tool for the suppressing chaos as a nonfeedback control method.

Acknowledgment

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