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Research Article

Optimization of Parameters of Asymptotically Stable Systems

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This work deals with numerical methods of parameter optimization for asymptotically stable systems. We formulate a special mathematical programming problem that allows us to determine optimal parameters of a stabilizer. This problem involves solutions to a differential equation. We show how to chose the mesh in order to obtain discrete problem guaranteeing the necessary accuracy. The developed methodology is illustrated by an example concerning optimization of parameters for a satellite stabilization system.

1. Introduction

Consider differential equation

$$\dot{x} = f(x, u), \quad x \in R^n, \ t \geq 0, \tag{1.1}$$

that describes a system equipped with a stabilizer. Here, $u \in U \subset R^k$ is a parameter. It is assumed that $0 = f(0, u)$ for all $u \in U$ and the zero equilibrium position of system (1.1) is asymptotically stable whenever $u \in U$. The parameter u should be chosen to optimize, in some sense, the behaviour of the trajectories. The choice of this parameter can be based on various criteria; obviously, it is impossible to construct a stabilizer optimal in all aspects. For example, for a linear controllable system, the pole assignment theorem guarantees the existence of a linear feedback yielding a linear differential equation with any given set of eigenvalues. One can choose a stabilizer with a very high damping speed. However, such a stabilizer is practically useless because of the so called pick-effect (see [1, 2]). Namely, there exists a large deviation of the solutions from the equilibrium position at the beginning of the stabilization process, whenever the module of the eigenvalues is big.

The aim of this paper is to develop an effective numerical tool oriented to optimization of stabilizer parameters according to different criteria that appear in the engineering practice.

Throughout this paper, we denote the set of real numbers by R and the usual n-dimensional space of vectors with components in R by R^n. The space of absolutely continuous functions defined in $[0,T]$ with values in R^n is denoted by $\mathrm{AC}([0,T],R^n)$. We denote by $\langle a,b\rangle$ the usual scalar product in R^n and by $|\cdot|$ the Euclidean norm. By B, we denote the closed unit ball, that is, the set of vectors $x\in R^n$ satisfying $|x|\le 1$. The transpose of a matrix A is denoted by A^*. We use the matrix norm $|A|=\max_{|x|=1}|Ax|$. If P and Q are two subsets in R^n and $\lambda\in R$, we use the following notations: $\lambda P=\{\lambda p \mid p\in P\}$, $P+Q=\{p+q \mid p\in P,\, q\in Q\}$.

2. Statement of the Problem

Denote by $x(t,x_0,u)$ the solution to the Cauchy problem

$$\begin{gathered}\dot{x}=f(x,u),\quad x\in R^n,\ t\in[0,T],\\ x(0)=x_0,\end{gathered}\tag{2.1}$$

where u is a parameter from a compact set $U\subset R^k$. Let $f(0,u)=0$ for all $u\in U$. Consider the functions

$$\varphi_i(u)=\max_{t\in\Delta_i}\max_{x_0\in B_i}|x(t,x_0,u)|_i,\quad i=\overline{0,m}.\tag{2.2}$$

Here, $\Delta_i\subseteq[0,T]$ are compact sets, and $|\cdot|_i$ are norms in R^n, and $B_i=\{x\in R^n \mid |x|_i\le b_i\}$. Consider the following mathematical programming problem:

$$\begin{gathered}\varphi_0(u)\longrightarrow\min,\\ \varphi_i(u)\le\overline{\varphi}_i,\quad i=\overline{1,m},\\ u\in U.\end{gathered}\tag{2.3}$$

Many problems of stabilization systems' parameters optimization can be written in this form.

Minimization of the Final Deviation

The problem is to determine the optimal values of the system parameters that guarantee minimal deviation of the system state from the zero equilibrium position at the final moment of time. This problem can be formalized as follows:

$$\max_{x_0 \in B} |x(T, x_0, u)| \longrightarrow \min, \quad u \in U. \tag{2.4}$$

For linear systems $\dot{x} = A(u)x$ with $T \gg 1$, this problem is an approximation for the maximization of the degree of stability [3].

Minimization of the Maximal Deviation

This problem consists of determination of parameters that correspond to minimization of the maximum deviation of trajectories and satisfy certain restrictions at the final moment of time. This problem can be formalized as follows:

$$\max_{t\in\,[0,T]} \max_{|x_0|=1} |x(t, x_0, u)| \longrightarrow \min, \quad \max_{|x_0|=1} |x(T, x_0, u)| \le \delta, \quad u \in U. \tag{2.5}$$

The above problems are of interest for stabilization theory; they both have form (2.3). Problem (2.3) has some special features, and its solution can be useful for parameter optimization of stabilization systems; however, its study can hardly be performed analytically for more or less complex systems. For this reason, we focus on the numerical aspects of this problem.

3. Numerical Methods

Let $\varepsilon > 0$ be small enough. We approximate problem (2.3) by the following problem

$$\overline{\varphi}_0 \longrightarrow \min, \quad \left|\tilde{x}\left(t^i_k, x^i_j, u\right)\right|_i \le \overline{\varphi}_i + \varepsilon, \quad i = \overline{0, m}, \quad u \in U, \tag{3.1}$$

where $t^i_0 = 0$, $t^i_k \in \Delta_i$, $x^i_j \in B_i$, $j = \overline{1, J}$, and

$$\tilde{x}\left(t^i_{k+1}, x^i_j, u\right) = \tilde{x}\left(t^i_k, x^i_j, u\right) + \tau f\left(\tilde{x}\left(t^i_k, x^i_j, u\right), u\right), \quad \tau = t^i_{k+1} - t^i_k,\ k = \overline{0, N} \tag{3.2}$$

is the Euler approximation for the solution $x(\cdot, x_j^i, u)$. Problem (2.3) can be approximated by problems (3.1) with any given accuracy.

Assume that

$$f(x,u) = A(u)x + g(x,u), \tag{3.3}$$

where matrix $A(u) = \nabla_x f(0,u)$ has eigenvalues with negative real part and the function $g(\cdot,u)$ satisfies $g(0,u) = 0$ and the Lipschitz condition

$$\left|g(x_1,u) - g(x_2,u)\right| \leq L_g^u \max\{|x_1|, |x_2|\}|x_1 - x_2|, \tag{3.4}$$

with $L_g^u > 0$ for all x_1 and x_2 in a neighbourhood of the zero equilibrium position. Consider functions $\varphi_i(\cdot)$ defined by (2.2), assuming that the balls B_i are contained in a sufficiently small neighbourhood of the origin. Consider $\delta > 0$. Let $K_i(\delta)$ and $J_i(\delta)$ be sets of indices such that the points $t_k^i \in \Delta_i$, $k \in K_i(\delta)$, and $x_j^i \in B_i$, $j \in J_i(\delta)$ form a δ-net in Δ_i and B_i, $i = \overline{1,m}$, respectively. Define the functions

$$\varphi_i^\delta(u) = \max_{k \in K_i(\delta)} \max_{j \in J_i(\delta)} \left|\tilde{x}\left(t_k^i, x_j^i, u\right)\right|, \quad i = \overline{0,m}. \tag{3.5}$$

Problem (3.1) can be written as

$$\begin{gathered} \varphi_0^\delta(u) \longrightarrow \min, \\ \varphi_i^\delta(u) \leq \overline{\varphi}_i + \varepsilon, \quad i = \overline{1,m}, \ u \in U. \end{gathered} \tag{3.6}$$

Denote by $\hat{u}$ and u^δ the optimal parameters for problems (2.3) and (3.6), respectively.

Theorem 3.1. *For any $\epsilon > 0$, there exists $\delta > 0$ such that u^δ is an admissible solution to the following problem:*

$$\begin{gathered} \varphi_0(u) \longrightarrow \min, \\ \varphi_i(u) \leq \overline{\varphi}_i + 2\varepsilon, \quad i = \overline{1,m}, \ u \in U, \\ \varphi_0\left(u^\delta\right) \leq \varphi_0(\hat{u}) + 2\varepsilon. \end{gathered} \tag{3.7}$$

This theorem allows one to choose the parameters of discretization in order to obtain optimal stabilizer parameters with a necessary precision. A rigorous formulation of this claim is the following. Denote by $V(\sigma)$ the value of the problem

$$\begin{gathered} \varphi_0(u) \longrightarrow \min, \\ \varphi_i(u) \leq \overline{\varphi}_i + \sigma, \quad i = \overline{1,m}, \ u \in U. \end{gathered} \tag{3.8}$$

Assume that problem (2.3) is *calm* in Clarke's sense (see [4]). Then, there exists a constant $K > 0$ satisfying the inequality

$$\frac{V(2\varepsilon) - V(0)}{2\varepsilon} > -K, \tag{3.9}$$

for all $\varepsilon > 0$ sufficiently small. It follows from Theorem 3.1 that

$$\left|\widetilde{V} - V(0)\right| \leq M\varepsilon, \tag{3.10}$$

where $\widetilde{V} = \varphi_0(\widetilde{u})$, $\widetilde{u}$ is the solution of problem (3.1), and $M = 2\max\{1, K\}$.

The exact formulas for $\delta = \delta(\epsilon)$ leading to the proof of Theorem 3.1 are presented in the Appendix. The main tool used to obtain them is the following version of Filippov-Gronwall inequality [5].

Theorem 3.2. *Let $P = \{p \in R^n \mid \langle p, Vp\rangle \leq 1\}$, where V is a symmetric positive definite matrix. Consider the functions $y(\cdot) \in \mathrm{AC}([0,T], R^n)$ and $\xi(\cdot) \in \mathrm{AC}([0,T], R)$, $\xi(t) \geq 0$ satisfying the following condition*

$$\max_{\langle p, Vp\rangle = 1} \left(\langle \dot{y}(t), Vp\rangle - \langle f\left(y(t) - \xi(t)p\right), Vp\rangle\right) \leq \dot{\xi}(t), \tag{3.11}$$

for almost all $t \in [0,T]$. Then, $x(t) \in y(t) + \xi(t)P$ for all $t \in [0,T]$, whenever $x_0 \in y(0) + \xi(0)P$, where $x(t)$ is the solution to the Cauchy problem $\dot{x} = f(x)$, $x(0) = x_0$.

Note that the use of this theorem allows us to obtain more precise estimates for the number of points in the meshes needed to achieve a given discretization accuracy. The estimates based on the usual Gronwall inequality can be significantly improved for asymptotically stable systems if we take into account the behaviour of the trajectories for large values of time. Theorem 3.2 is a natural tool for this analysis. For example, according to the classical estimates, the number of points in the mesh in t, needed to ensure a given precision, grows exponentially with the length of the time interval. Meanwhile, the estimates obtained from Theorem 3.2 for asymptotically stable systems (see Theorems A.2 and A.6) give a linear growth of the number of points in the mesh. This result is of practical importance. Optimization problem (3.6) is a hard nonsmooth problem. Our computational experience shows that the usage of the Nelder-Mead method is the most adequate approach to solve it. The numerical solution of this problem significantly depends on the structure of the involved functions. The problem of optimal choice of parameters is solved only once, at the stage of the control system's development, so one could afford to dedicate more resources to its solution. However, if the mesh is constructed using the classical precision estimates, the required computational effort can be extremely high, making it impossible to solve the problem in a reasonable time. Our estimates for the number of points of discretization allow us to construct an adequate mesh and to significantly reduce the CPU time.

4. Example: Optimal Parameters for Satellite-Stabilizer System

Consider motion of a connected two-body system in a circular orbit around the Earth. Body 1 is a satellite with the center of mass O_1, and body 2 is a stabilizer with the center of mass O_2 . These two bodies are linked to each other at the point P through a dissipative hinge mechanism (Figure 1). Let O be the center of mass of the system.

We use three reference frames: $OXYZ$ is the orbital coordinate frame, its axis OZ is directed along the radius vector of the point O with respect to the center of the Earth, OX is directed along the velocity of the point O, and OY is normal to the orbit plane. The axes of referential frames $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$ are the central principal axes of inertia for bodies 1 and 2, respectively. Consider motion of the system in the orbit plane supposing that the bodies are connected in their centres of mass; that is, the points O_1, O_2, O, and P coincide. Let α_1 and α_2 be the angles between the axis OX and the axes O_1x_1 and O_2x_2, respectively. Denote by α_i', $i = 1, 2$ the derivative of α_i with respect to time. The equations of motion for this system can be written as [6]

$$\begin{aligned} &B_1\alpha_1'' + 3\omega_0^2(A_1 - C_1)\sin\alpha_1\cos\alpha_1 + \overline{k}_1(\alpha_1' - \alpha_2') = 0, \\ &B_2\alpha_2'' + 3\omega_0^2(A_2 - C_2)\sin\alpha_2\cos\alpha_2 - \overline{k}_1(\alpha_1' - \alpha_2') = 0. \end{aligned} \tag{4.1}$$

Here, A_1, B_1, C_1 and A_2, B_2, C_2 are the principal moments of inertia of the bodies, $\overline{k}_1$ is the damping coefficient of the system, and ω_0 is the constant angular velocity of the orbital motion of the system's center of mass. Introducing a new independent variable $\tau = \omega_0 t$ and the dimensionless parameters

$$p_1 = \frac{A_1 - C_1}{B_1}, \qquad p_2 = \frac{A_2 - C_2}{B_2}, \qquad \mu = \frac{B_2}{B_1}, \qquad k_1 = \frac{\overline{k}_1}{\omega_0 B_1}, \tag{4.2}$$

the equations of motion can be written as

$$\begin{aligned} &\ddot{\alpha}_1 + 3p_1\sin\alpha_1\cos\alpha_1 + k_1(\dot{\alpha}_1 - \dot{\alpha}_2) = 0, \\ &\ddot{\alpha}_2 + 3p_2\sin\alpha_2\cos\alpha_2 - \frac{k_1}{\mu}(\dot{\alpha}_1 - \dot{\alpha}_2) = 0. \end{aligned} \tag{4.3}$$

Here, the dot denotes the derivative with respect to τ. The parameters (p_1, p_2, k_1, μ) satisfy the following conditions:

$$-1 \le p_1 \le 1, \qquad -1 \le p_2 \le 1, \qquad \mu > 0, \qquad k_1 > 0. \tag{4.4}$$

We study small oscillations of system (4.3) in the vicinity of the equilibrium position

$$\alpha_{10} = 0, \qquad \alpha_{20} = 0. \tag{4.5}$$

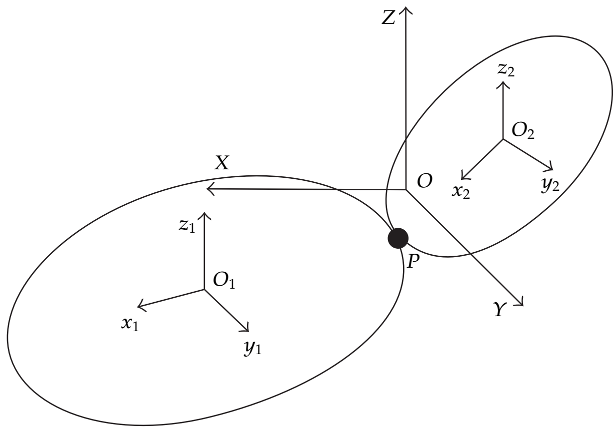

Figure 1: The satellite stabilizer system.

The equations of motion, linearized in the vicinity of the above stationary solution, take the form

$$\ddot{\alpha}_1 + 3p_1\alpha_1 + k_1(\dot{\alpha}_1 - \dot{\alpha}_2) = 0,$$
$$\ddot{\alpha}_2 + 3p_2\alpha_2 - \frac{k_1}{\mu}(\dot{\alpha}_1 - \dot{\alpha}_2) = 0. \tag{4.6}$$

The characteristic equation for system (4.6) is

$$\mu\lambda^4 + k_1(1+\mu)\lambda^3 + 3\mu(p_1+p_2)\lambda^2 + 3k_1(p_1+\mu p_2)\lambda + 9\mu p_1 p_2 = 0. \tag{4.7}$$

Analysis of (4.7) allows one to obtain the necessary and sufficient conditions of asymptotic stability. The region of asymptotic stability is given by

$$\{(k_1,\ p_1,\ p_2) : k_1 > 0,\ p_1 > 0,\ p_2 > 0,\ p_1 \neq p_2\}. \tag{4.8}$$

Taking into account the feasibility conditions for the system parameters, we arrive at the following set of admissible parameters for our optimization problem:

$$U = \{(p_1, p_2, k_1, \mu) : k_1 > 0,\ \mu > 0,\ 0 < p_1 \leq 1,\ 0 < p_2 \leq 1,\ p_1 \neq p_2\}. \tag{4.9}$$

4.1. The Maximal Degree of Stability

Consider the set U described by (4.9). Denote by $u = (p_1, p_2, k_1, \mu)$ a parameter that belongs to the set U. Let $\{\lambda_1(u), \ldots, \lambda_4(u)\}$ be the roots of (4.7). The inclusion $u \in U$ implies that $\operatorname{Re} \lambda_i(u) < 0$, $i = \overline{1,4}$. The degree of stability is defined by

$$\delta(u) = -\max_{i=\overline{1,4}} \operatorname{Re} \lambda_i(u). \tag{4.10}$$

Consider the following problem

$$\begin{gathered}\delta(u) \longrightarrow \max, \\ u \in U.\end{gathered} \tag{4.11}$$

In [7, 8], it is proved that the maximal degree of stability is achieved when all the roots of the characteristic equations are real and equal. This situation becomes possible only when the conditions

$$\begin{gathered}k_1 = 4\delta(u)\frac{\mu}{1+\mu}, \\ p_1 + p_2 = 2\delta^2(u), \\ 3(p_1 + \mu p_2) = (1+\mu)\delta^2(u), \\ 9p_1p_2 = \delta^4(u)\end{gathered} \tag{4.12}$$

are satisfied. The above system has two sets of solutions

$$\begin{gathered}\hat{p}_1 = \left(3 - 2\sqrt{2}\right)^2 \simeq 0.0294, \\ \hat{p}_2 = 1, \\ \hat{k}_1 = \sqrt{6}\left(3 - 2\sqrt{2}\right) \simeq 0.4203, \\ \hat{\mu} = 3 - 2\sqrt{2} \simeq 0.1716,\end{gathered} \tag{4.13}$$

$$\begin{gathered}\hat{p}_1 = 1, \\ \hat{p}_2 = \left(3 - 2\sqrt{2}\right)^2 \simeq 0.0294, \\ \hat{k}_1 = \sqrt{6} \simeq 2.4495, \\ \hat{\mu} = 3 + 2\sqrt{2} \simeq 5.8284.\end{gathered} \tag{4.14}$$

4.2. Numerical Optimization

Denote by $x(\cdot, x_0, p_1, p_2, k_1, \mu)$ the solution of linear system (4.6) with $x(0) = x_0$, defined in the interval $[0, T]$. The parameters (p_1, p_2, k_1, μ)belong to asymptotic stability region defined in (4.9). Consider the following problem:

$$\max_{|x_0|=1} \left| x(T, x_0, p_1, p_2, k_1, \mu) \right| \longrightarrow \min, \quad k_1 > 0,\ \mu > 0,\ 0 < p_1 \leq 1,\ 0 < p_2 \leq 1. \tag{4.15}$$

Problem (4.15) can be reduced to an optimization problem without constrains using quadratic penalty functions; see [9]. If T is big enough, this problem approximates problem (4.11), where the concept of degree of stability is used. The parameters given by (4.13) and (4.14) are close to optimal solutions of problem (4.15) only when $T \gg 1$. Put $T = 10\pi$. The results of simulations show that the value of problem (4.15) is about 10^{-6}–10^{-7}, independently on the values of admissible parameters (p_1, p_2, k_1, μ). For example, the following values

$$\widehat{\widehat{p}}_1 = 0.0779, \qquad \widehat{\widehat{p}}_2 = 0.8574, \qquad \widehat{\widehat{k}}_1 = 0.5540, \qquad \widehat{\widehat{\mu}}_1 = 0.3337 \tag{4.16}$$

are optimal parameters for problem (4.15). The corresponding minimal value is 7.2×10^{-7}. Parameters (4.13) and (4.14) give the values 1.0×10^{-6} and 5.9×10^{-7}, respectively.

In practice, it is important to consider smaller time intervals $[0, T]$. Solve problem (4.15) with $T = 3\pi$. In this case, we see that the value of problem really depends on the choice of parameters. Let us estimate the global minimum in this problem. To this end, we solve problem (4.15) using all combinations of the following values:

$$\begin{aligned} p_1 &= 0.25,\ 0.5,\ 0.75, \\ p_2 &= 0.25,\ 0.5,\ 0.75, \\ k_1 &= 1,\ 2,\ 3, \\ \mu &= 2,\ 4,\ 6, \end{aligned} \tag{4.17}$$

as initial guesses for numerical optimization. We obtain the following two sets of parameters with the best value of the problem:

$$\check{p}_1 = 0.06928, \qquad \check{p}_2 = 1.00757, \qquad \check{k}_1 = 0.59209, \qquad \check{\mu} = 0.33161, \tag{4.18}$$

$$\check{p}_1 = 1.00521, \qquad \check{p}_2 = 0.06920, \qquad \check{k}_1 = 1.78178, \qquad \check{\mu} = 3.01152. \tag{4.19}$$

The estimate for the global minimum m is

$$m = 0.00378. \tag{4.20}$$

Meanwhile, the value corresponding to parameters (4.13) and (4.14) is 0.4660. Thus, we see that the methodology based on resolution of problem (2.3) can be more adequate in the practice than that one using the concept of degree of stability.

Since we are studying the behaviour of a nonlinear system in a vicinity of its equilibrium position, it is also important to estimate the deviation of the linearized system trajectories from zero. The stabilizer constructed for the linearized system makes sense only if its trajectories belong to a small vicinity of the equilibrium position; otherwise, the influence of the nonlinearity can destabilize the system even in a very small neighbourhood of the equilibrium.

Consider the following problem:

$$\begin{gathered}
\max_{t\in[0,3\pi]}\max_{|x_0|=1}\left|x(t,x_0,p_1,p_2,k_1,\mu)\right| \longrightarrow \min, \\
\max_{|x_0|=1}\left|x(3\pi,x_0,p_1,p_2,k_1,\mu)\right| \leq 0.005, \\
k_1>0, \\
\mu>0, \\
0<p_1\leq 1, \\
0<p_2\leq 1.
\end{gathered} \tag{4.21}$$

In this problem, the solutions of system (4.6) at the moment $T = 3\pi$ are constrained to be in a neighbourhood of the equilibrium position with the radius 0.005. The obtained optimal solutions $(\widetilde{p}_1,\widetilde{p}_2,\widetilde{k}_1,\widetilde{\mu})$ minimize the maximum norm of the damping process of linear system (4.6) in the interval $[0,3\pi]$. After numerical optimization, we get the following optimal parameters:

$$\widetilde{p}_1 = 0.07140, \qquad \widetilde{p}_2 = 1.01643, \qquad \widetilde{k}_1 = 0.60004, \qquad \widetilde{\mu} = 0.33887, \tag{4.22}$$

$$\widetilde{p}_1 = 1.01642, \qquad \widetilde{p}_2 = 0.07140, \qquad \widetilde{k}_1 = 1.77067, \qquad \widetilde{\mu} = 2.95097. \tag{4.23}$$

The corresponding value of problem (4.21) is $P = 1.58685$. We can see that the couple of parameters in (4.22) and (4.23) are slightly different from (4.18) and (4.19). Moreover,

$$\max_{t\in[0,3\pi]}\max_{|x_0|=1}\left|x\left(t,x_0,\check{p}_1,\check{p}_2,\check{k}_1,\check{\mu}\right)\right| \simeq 1.5974 \simeq P. \tag{4.24}$$

Taking the optimal parameters $(\widehat{p}_1,\widehat{p}_2,\widehat{k}_1,\widehat{\mu})$ of problem (4.11), we get

$$\max_{t\in[0,3\pi]}\max_{|x_0|=1}\left|x\left(t,x_0,\widehat{p}_1,\widehat{p}_2,\widehat{k}_1,\widehat{\mu}\right)\right| \simeq 1.9169. \tag{4.25}$$

Thus, the stabilizer with the parameters corresponding to the maximal degree of stability yields more significant deviation of the trajectories from the equilibrium position than the stabilizer with the parameters obtained solving problem (4.21).

Our aim is to find optimal parameters for system (4.3). To this end, we solve problem (4.15), with $T = 3\pi$, but now, $x(\cdot,x_0,p_1,p_2,k_1,\mu)$ stands for the solution of system (4.3) with $x(0) = x_0$. We get the following two sets of optimal parameters:

$$\overline{p}_1 = 0.23350, \qquad \overline{p}_2 = 1.08235, \qquad \overline{k}_1 = 0.62791, \qquad \overline{\mu} = 0.62137, \tag{4.26}$$

$$\overline{p}_1 = 1.07743, \qquad \overline{p}_2 = 0.23171, \qquad \overline{k}_1 = 1.02413, \qquad \overline{\mu} = 1.63140. \tag{4.27}$$

Table 1: Values of function $M(\delta)$ corresponding to parameters (4.13), (4.18), (4.22), and (4.26).

δ	set (4.13)	set (4.18)	set (4.22)	set (4.26)
1	2.8463	2.5556	2.6528	0.0671
0.9	2.5605	1.2783	1.6612	0.0572
0.8	2.0915	0.4046	0.6545	0.0529
0.7	1.4216	0.1279	0.2609	0.0483
0.6	0.8687	0.0495	0.1076	0.0446
0.5	0.5296	0.0183	0.0469	0.0390
0.4	0.3183	0.0093	0.0198	0.0324
0.3	0.1896	0.0077	0.0073	0.0249
0.2	0.1054	0.0059	0.0021	0.0166
0.1	0.0481	0.0032	0.0004	0.0084

Consider optimal parameters (4.26) and (4.27), and denote by $x(\cdot, x_0, \overline{p}_1, \overline{p}_2, \overline{k}_1, \overline{\mu})$ the solution of system (4.3) corresponding to these parameters and satisfying $x(0) = x_0$. The optimal value is

$$P_1 = \max_{t\in[0,3\pi]} \max_{|x_0|=1} \left| x\left(t, x_0, \overline{p}_1, \overline{p}_2, \overline{k}_1, \overline{\mu}\right) \right| \simeq 1.3634. \tag{4.28}$$

The above value of P_1 is even smaller than P, obtained for the linearized system.

To compare the solutions of all optimization problem, we consider the following function

$$M(\delta) = \max_{|x_0|\le \delta} \left| x(3\pi, x_0, p_1, p_2, k_1, \mu) \right|, \quad \delta > 0. \tag{4.29}$$

Here, $x(3\pi, x_0, p_1, p_2, k_1, \mu)$ stands for the solution to system (4.3). Table 1 gives the values of the function $M(\delta)$ for parameters (4.13), (4.18), (4.22), and (4.26).

Observe that the values in the last column of Table 1, computed for parameters (4.26), always satisfy the condition $M(\delta) \le 0.1\delta$. On the other hand, this condition is not satisfied for parameters (4.13), obtained maximizing the degree of stability. For the parameters obtained for linearization, the condition is satisfied only if δ is sufficiently small. This illustrates the advantages of applying the introduced methodology, based on numerical solution of optimization problem (2.3), directly to nonlinear systems.

5. Conclusions

The methods usually applied to optimize the parameters of a stabilization system are based on the idea of the maximum stability degree, that is, the minimization of the system's transition time. These methods, however, face the problem of the so-called peak effect when the deviation of the system trajectory from the equilibrium increases with the decrease of the time of response. The approach suggested in this paper consists of a numerical analysis of a stabilization system based on a more complete and flexible description of the system behaviour capable to take into account not only the stability degree but also the maximum deviation of the trajectory on a given time interval or at a given moment. For this optimization problem, we develop a numerical method and prove that it can guarantee a given accuracy for the problem solution. We obtained more precise estimates for the number of points in the meshes needed to achieve a given discretization accuracy than the estimates based on the usual Gronwall inequality. This method is applied to optimization of a stabilization system for a satellite with a gravitational stabilizer. The obtained results show that the above approach can offer solutions more adequate for practical implementation than those given by optimization of the stability degree.

Appendix

A. The Mathematical Basis

In this Appendix, we present a series of theorems, with schematic proofs, containing explicit estimates for the fineness of discretization needed to obtain the necessary precision of approximations and to prove Theorem 3.1.

A.1. Linear Systems

Consider a linear system

$$\dot{x} = Ax, \quad x \in R^n, \quad t \geq 0, \tag{A.1}$$

where A is a matrix. Assume that all its eigenvalues have negative real part. Let V be a symmetric positive definite matrix satisfying the Lyapunov equation [10]

$$A^* V + VA = -I. \tag{A.2}$$

Set

$$P = \{p \in R^n \mid \langle p, Vp \rangle \leq 1\}. \tag{A.3}$$

The quadratic form $V(p) = \langle p, Vp \rangle$ is the Lyapunov function for system (A.1). Denote by η_1 and η_2 the minimal and maximal eigenvalues of V, respectively. Let τ be a positive constant. Consider the Euler approximation for system (A.1),

$$y_{k+1} = y_k + \tau A y_k, \qquad k = 0, 1, \ldots. \tag{A.4}$$

The following theorem provides an explicit estimate for the constant τ guaranteeing the equality $\lim_{k \to \infty} y_k = 0$.

Theorem A.1. *Let $b > 0$. Consider $y_0 \in bP$. If*

$$0 < \tau < \frac{\eta_1}{\eta_2^2 |A|^2}, \tag{A.5}$$

then the following inequalities hold:

$$\langle y_k, V y_k \rangle \le \beta^k \langle y_0, V y_0 \rangle, \quad k = 1, 2, \ldots, \tag{A.6}$$

where

$$\beta = 1 - \frac{\tau}{\eta_2} + \tau^2 \frac{\eta_2}{\eta_1} |A|^2. \tag{A.7}$$

It is easy to see that $0 < \beta < 1$. The proof of this theorem uses the induction and the inequality

$$\eta_1 |p|^2 \le \langle p, V p \rangle \le \eta_2 |\, p|^2. \tag{A.8}$$

Assume that constant τ satisfies condition (A.5). Consider the polygonal Euler approximation to solution of system (A.1)

$$\begin{gathered} y(t) = y_k + (t - t_k) A y_k, \quad t \in [t_k, t_{k+1}], \\ t_k = k\tau, \quad k = 0, 1, \ldots. \end{gathered} \tag{A.9}$$

Let $b > 0$. Set

$$\gamma = - \min_{\substack{\langle p, V p \rangle = 1 \\ \langle y, V y \rangle \le b^2}} \langle A^2 y, V p \rangle, \tag{A.10}$$

$$\xi_\mu(t) = \xi_0 e^{-\mu t}, \quad \xi_0 > 0, \tag{A.11}$$

$$x(t) = e^{At} x(0). \tag{A.12}$$

Theorem A.2. *Let $y_0 \in bP$ be given. Assume that condition $0 \le \mu < 1/2\eta_2$ is satisfied. If*

$$\tau \gamma \le \left(\frac{1}{2\eta_2} - \mu \right) \xi_\mu(t), \quad t \ge 0, \tag{A.13}$$

then the inequality $|y(t) - x(t)| \le \xi_\mu(t)/\sqrt{\eta_1}$ holds for all $t \in [0, T]$, whenever $|y_0 - x(0)| \le \xi_0/\sqrt{\eta_2}$.

This theorem is a consequence of the inequality

$$\max_{\langle p, V p \rangle = 1} \langle A p, V p \rangle \le -\frac{1}{2\eta_2}, \tag{A.14}$$

of the inclusion

$$\frac{1}{\sqrt{\eta_2}} B \subset P \subset \frac{1}{\sqrt{\eta_1}} B, \tag{A.15}$$

and of Theorem 3.2 with function $y(\cdot)$ defined by (A.9) and function $\xi(\cdot)$ defined by (A.11).

The following theorem is also a consequence of Theorem 3.2.

Theorem A.3. *Let* $x(t) = e^{At}x_0$ *and* $z(t) = e^{At}z_0$ *be solutions to system* (A.1). *Assume that* $x_0, z_0 \in P$. *If*

$$0 \leq \mu \leq \frac{1}{2\eta_2}, \tag{A.16}$$

then the inequality $|z(t) - x(t)| \leq \xi_\mu(t)/\sqrt{\eta_1}$, $t \in [0,T]$ *holds whenever* $|z_0 - x_0| \leq \xi_0/\sqrt{\eta_2}$.

Consider now $t \in \Delta \subset [0,T]$, where Δ is a closed interval. Let $x(t, x_0)$ be the solution of system (A.1), with

$$x_0 \in S = \{x \in R^n : |x| = 1\}. \tag{A.17}$$

Let $\delta > 0$. Assume that parameters of function $\xi_\mu(\cdot)$ defined by (A.11) satisfy the following conditions,

$$\xi_0 = \sqrt{\eta_2}\delta, \qquad 0 \leq \mu < \frac{1}{2\eta_2}. \tag{A.18}$$

Assume that $\{x_j\}$, $j = \overline{1, J}$ is a finite set of points uniformly distributed on S. If

$$J \geq 2\left(\left[\frac{1}{\delta}\right] + 1\right)^{n-1}, \tag{A.19}$$

then we have

$$\bigcup_{j=1}^{J}(x_j + \delta B) \supset S. \tag{A.20}$$

Let $|\Delta|$ be the length of the interval Δ. Consider a finite set $\{t_k\} \subset \Delta$, $k = \overline{0, N}$, such that the difference

$$t_{k+1} - t_k = \tau = \frac{\delta}{2\gamma\sqrt{\eta_2}} \tag{A.21}$$

is a constant. If

$$N = \left[\frac{2\gamma\sqrt{\eta_2}}{\delta}|\Delta|\right] + 1, \tag{A.22}$$

then the set

$$\bigcup_{k=0}^{N}\left[t_k - \frac{\delta}{4\gamma\sqrt{\eta_2}}, t_k + \frac{\delta}{4\gamma\sqrt{\eta_2}}\right] \tag{A.23}$$

contains Δ. Consider the Euler approximation of solution $x(t, x_0)$

$$\tilde{x}(t_{k+1}, x_j) = \tilde{x}(t_k, x_j) + \tau A \tilde{x}(t_k, x_j), \quad k = \overline{0, N}. \tag{A.24}$$

Theorem A.4. *Let $\varepsilon > 0$. If*

$$\delta = \frac{4\gamma\sqrt{\eta_1}}{8\gamma\sqrt{\eta_2} + |A|}\varepsilon, \tag{A.25}$$

then the following inequality holds:

$$\left| \max_{k=\overline{0,N}} \max_{j=\overline{1,J}} \left| \tilde{x}(t_k, x_j) \right| - \max_{t\in\Delta} \max_{x_0\in S} |x(t, x_0)| \right| \leq \varepsilon. \tag{A.26}$$

The proof of this theorem uses the results of Theorems A.1, A.2, and A.3.

A.2. Nonlinear Systems

Assume that $g : R^n \to R^n$ is a twice continuously differentiable function satisfying $g(0) = 0$. Consider the function $f(x) = Ax + g(x)$, where A is a matrix. Assume that the eigenvalues of the matrix A have negative real parts. Consider the system

$$\dot{x} = f(x), \quad t \geq 0. \tag{A.27}$$

Since g is twice continuously differentiable, there exists a constant $L_g > 0$ such that function $g(\cdot)$ satisfies the following Lipschitz condition:

$$\left| g(x_1) - g(x_2) \right| \leq L_g \max\{|x_1|, |x_2|\} |x_1 - x_2|, \tag{A.28}$$

for all x_1 and x_2 in a small neighbourhood of the equilibrium position $x = 0$. Consider the set P defined by (A.3) and constants η_1, η_2 as before. Define the Euler approximation for system (A.27),

$$y_{k+1} = y_k + \tau f(y_k), \quad k = 0, 1, \ldots, \tag{A.29}$$

where τ is a positive constant.

Theorem A.5. *Assume that b is a constant satisfying*

$$0 < b < \frac{\sqrt{\eta_1}}{4L_g\eta_2}. \tag{A.30}$$

Let $y_0 \in bP$. If

$$0 < \tau < \frac{8\eta_1^2}{\eta_2 (4|A|\eta_2 + 1)^2}, \tag{A.31}$$

then the following inequalities hold:

$$\langle y_k, V y_k \rangle \leq \beta^k \langle y_0, V y_0 \rangle, \qquad k = 1, 2 \ldots, \tag{A.32}$$

where

$$\beta = 1 - \frac{\tau}{2\eta_2} + \tau^2 \frac{(4|A|\eta_2 + 1)^2}{16\eta_1^2}. \tag{A.33}$$

The proof of this theorem uses the Lipschitz condition (A.28) in the form

$$|g(y_k)| \leq L_g |y_k|^2, \tag{A.34}$$

and the mathematical induction method. It is easy to see that $0 < \beta < 1$.

Assume that the constant τ satisfies condition (A.31) and consider the function

$$y(t) = y_k + (t - t_k) f(y_k), \qquad t \in [t_k, t_{k+1}], \ \ t_k = k\tau, \ \ k = 0, 1, \ldots. \tag{A.35}$$

Put

$$\gamma_1 = \gamma + \frac{8\eta_2 |A| + 1}{64\eta_2^2 \sqrt{\eta_1} L_g}, \tag{A.36}$$

where γ is defined by (A.10). Denote by $x(t)$ the solution to system (A.27) with the initial condition $x(0) = x_0$.

Theorem A.6. *Assume that b is a constant satisfying*

$$\xi_0 \leq b \leq \frac{\sqrt{\eta_1}}{4 L_g \eta_2}, \tag{A.37}$$

and let $y_0 \in bP$. If $0 \leq \mu < 1/4\eta_2$ and

$$\tau\gamma_1 \leq \left(\frac{1}{4\eta_2} - \mu \right) \xi_\mu(t), \qquad t \geq 0, \tag{A.38}$$

then the inequality $|y(t) - x(t)| \leq \xi_\mu(t)/\sqrt{\eta_1}$ holds for all $t \in [0, T]$, whenever $|y_0 - x(0)| \leq \xi_0/\sqrt{\eta_2}$.

This theorem is a consequence of Theorem 3.2 with function $y(\cdot)$ defined by (A.35) and function $\xi(\cdot)$ defined by (A.11).

Consider now the Cauchy problems

$$\begin{aligned} \dot{x} &= f(x), \\ x(0) &= x_0, \\ \dot{z} &= f(z), \\ z(0) &= z_0. \end{aligned} \tag{A.39}$$

Denote by $x(\cdot)$ and $z(\cdot)$ the respective solutions. The following theorem is also a consequence of Theorem 3.2.

Theorem A.7. *Let b be a constant satisfying*

$$\xi_0 \le b \le \frac{\sqrt{\eta_1}}{4L_g\eta_2}. \tag{A.40}$$

Assume that the trajectories $x(\cdot)$ and $z(\cdot)$ belong to the set bP. If

$$0 \le \mu \le \frac{1}{4\eta_2}, \tag{A.41}$$

then we have $|z(t) - x(t)| \le \xi_\mu(t)/\sqrt{\eta_1}$ for all $t \in [0, T]$, whenever $|z_0 - x_0| \le \xi_0/\sqrt{\eta_2}$.

Let $\delta > 0$. Assume that the parameters of function $\xi_\mu(\cdot)$ defined by (A.11) satisfy the following conditions:

$$\xi_0 = \sqrt{\eta_2}\delta, \quad 0 \le \mu \le \frac{1}{4\eta_2}. \tag{A.42}$$

Consider the balls

$$B_{b_i} = \left\{ x \in R^n : |x| \le \frac{b_i}{\sqrt{\eta_2}} \right\} \subset b_i P, \quad i = \overline{0, m}, \tag{A.43}$$

where the constants b_i satisfy the following conditions:

$$\xi_0 \le b_i \le \frac{\sqrt{\eta_1}}{4L_g\eta_2}, \quad i = \overline{0, m}. \tag{A.44}$$

For each index i, take a finite set $\{x_j^i\}$ of points, $j = \overline{1, J_i}$, uniformly distributed in the ball B_{b_i}. If

$$J_i \geq \left(\left[\frac{b_i}{\sqrt{\eta_2}\delta}\right] + 1\right)^n, \tag{A.45}$$

then we have

$$\bigcup_{j=1}^{J_i}\left(x_j^i + \delta B\right) \supset B_{bi}. \tag{A.46}$$

Let $\Delta_i \subset [0, T]$, $i = \overline{0, m}$, be closed intervals with length $|\Delta_i|$. Consider a finite set of points $\{t_k^i\}$, $k = \overline{0, N_i}$, in each interval Δ_i. It is assumed that the difference $\tau = t_{k+1}^i - t_k^i$ is a constant. Let

$$\tau = \frac{\delta}{4\gamma_1\sqrt{\eta_2}}. \tag{A.47}$$

Define the sets

$$\Delta_{N_i} = \bigcup_{k=0}^{N_i}\left[t_k^i - \frac{\delta}{8\gamma_1\sqrt{\eta_2}}, t_k^i + \frac{\delta}{8\gamma_1\sqrt{\eta_2}}\right], \quad i = \overline{0, m}. \tag{A.48}$$

If

$$N_i = \left[\frac{4\gamma_1\sqrt{\eta_2}}{\delta}|\Delta_i|\right] + 1, \tag{A.49}$$

then we have $\Delta_{N_i} \supset \Delta_i$. Let $x_0 \in B_{b_i}$. Denote by $x(t, x_0)$ the solution to the Cauchy problem

$$\begin{gathered}\dot{x} = Ax + g(x), \quad x \in R^n,\ t \in \Delta_i, \\ x(0) = x_0.\end{gathered} \tag{A.50}$$

Consider the Euler approximation of the solution $x(t, x_0)$

$$\tilde{x}\left(t_{k+1}^i, x_j^i\right) = \tilde{x}\left(t_k^i, x_j^i\right) + \tau\left[A\tilde{x}\left(t_k^i, x_j^i\right) + g\left(\tilde{x}\left(t_k^i, x_j^i\right)\right)\right], \quad k = \overline{0, N_i}, \tag{A.51}$$

with τ satisfying condition (A.47).

Theorem A.8. *Let $\varepsilon > 0$ be given. If*

$$\delta = \frac{2^7 L_g\gamma_1\eta_2^2\sqrt{\eta_1\eta_2}}{2^8 L_g\gamma_1\eta_2^3 + 4\eta_2\sqrt{\eta_1}|A| + \sqrt{\eta_1}}\varepsilon, \tag{A.52}$$

then the following inequalities:

$$\left| \max_{k=\overline{0,N_i}} \max_{j=\overline{1,J_i}} \left| \tilde{x}\left(t_k^i, x_j^i\right) \right| - \max_{t\in\Delta_i} \max_{x_0\in B_{b_i}} |x(t,x_0)| \right| \leq \varepsilon, \quad i=\overline{0,m}, \tag{A.53}$$

hold.

The proof of this theorem follows from Theorems A.5, A.6, and A.7.

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