Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2011, Article ID 351269, 26 pages
doi:10.1155/2011/351269

Research Article

Vector Rotators of Rigid Body Dynamics with Coupled Rotations around Axes without Intersection

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Received 27 January 2011; Revised 30 May 2011; Accepted 1 June 2011

Academic Editor: Massimo Scalia

Vector method based on mass moment vectors and vector rotators coupled for pole and oriented axes is used for obtaining vector expressions for kinetic pressures on the shaft bearings of a rigid body dynamics with coupled rotations around axes without intersection. Mass inertia moment vectors and corresponding deviational vector components for pole and oriented axis are defined by K. Hedrih in 1991. These kinematical vectors rotators are defined for a system with two degrees of freedom as well as for rheonomic system with two degrees of mobility and one degree of freedom and coupled rotations around two coupled axes without intersection as well as their angular velocities and intensity. As an example of defined dynamics, we take into consideration a heavy gyrorotor disk with one degree of freedom and coupled rotations when one component of rotation is programmed by constant angular velocity. For this system with nonlinear dynamics, a series of tree parametric transformations of system nonlinear dynamics are presented. Some graphical visualization of vector rotators properties are presented too.

1. Introduction

Well-known toy top or a tern is just a simple toy for many that has the unusual property that when it rotates sufficiently by high angular velocity about its axis of symmetry and it keeps in the state of stationary rotation around this axis. This feature has attracted scientists around the world and as a result of years of research created many devices and instruments, from simple to very complex structures, which operate on the principle of a spinning top that plays an important role in stabilizing the movement. Ability gyroscope that keeps the line was used in many fields of mechanical engineering, mining, aviation, navigation, military industry, and in celestial mechanics.

Gyroscopes' name comes from the Greek words $\gamma\upsilon\rho o$ (turn) and $\sigma\kappa o\pi\varepsilon\omega$ (observed) and is related to the experiments that the 1852nd were painted by Jean Bernard Leon Foucault. The principle of gyroscope based on the principle of precession pseudoregular.

Gyroscopes are very responsible parts of instruments for aircraft, rockets, missiles, transport vehicles, and many weapons. This gives them a very important role, and they need to be under the strict control of the design and inner workings because in case of damage they could lead to catastrophic consequences. Gyroscope (gyro, top) is a homogeneous, axis-symmetric rotating body that rotates by large angular velocity about its axis of symmetry and is now one of the most inertial sensors that measure angular velocity and small angular disturbances angular displacement around the reference axis.

Properties of gyroscopes possess heavenly bodies in motion, artillery projectiles in motion, rotors of turbines, different mobile installations on ships, aircraft propeller rotating, and so forth. The modern technique of gyroscope is an essential element of powerful gyroscopic devices and accessories, which are used to automatically control movement of aircraft, missiles, ships, torpedoes, and so on. They are used in navigation to stabilize the movement of ships in a seaway, to change direction, and direction of angular and translator velocity projectiles, and in many other special purposes.

There are many devices that are applied to the military, and their design is based on the principles of gyroscopes. Technical applications gyros today are so manifold and diverse that there is a need to get out of the general theory of gyroscopes allocates a separate discipline, called "applied theory of gyroscopes."

An overview of optical gyroscopes theory with practical aspects, applications, and future trends is presented in [1] written by Adi in 2006.

The original research results of dynamics and stability of gyrostats were given in 1979 by Ančev and Rumjancev [2].

Three of papers [3–5] written by Rumjancev related to stability of rotation of a heavy rigid body with one fixed point in S. V. Kovalevskaya's case, on the stability of motion of gyrostats and Stability of rotation of a heavy gyrostat on a horizontal plane pointed out important research results in this area.

Subjects of series of published papers (see [3–15]) are construction models, dynamics, and applications of gyroscopes as well as special phenomena of nonlinear vibration properties of the gyroscope, analysis of gyroscope dynamics for a satellites, analytical research results on a synchronous gyroscopic vibration absorber, inertial rotation sensing in three dimensions using coupled electromagnetic ring-gyroscopes, gyroscopes for orientation and inertial navigation, and others.

By Cavalca et al. [10] published in 2005 an investigation result on the influence of the supporting structure on the dynamics of the rotor system is presented.

Each mechanical gyroscope is based on coupled rotations around more axes with one point intersection. Most of the old equipment was based on rotation of complex and coupled component rotations which resulting in rotation about fixed point gyroscopes.

The classical book [16] by Andonov et al. contains a classical and very important elementary dynamical model of heavy mass particle relative motion along rotate circle around vertical axis through its centre, whose nonlinear dynamics and singularities are primitive model of the simple case of the gyrorotor, and present an analogous and useful dynamical and mathematical model of nonlinear dynamics.

No precisions and errors appear in the functions of gyroscopes caused by eccentricity and unbalanced gyrorotor body as well by distance between axis of rotations are reason to investigate determined task as in the title of our paper.

This vector approach proposed by us is very suitable to obtain new view to the properties of dynamics of pure classical task, investigated by numerous generations of the researchers and serious scientists around the world.

Using Hedrih's (see [17–22]) mass moment vectors and vector rotators, some characteristics members of the vector expressions of derivatives of linear momentum and angular momentum for the gyrorotor coupled rotations around two axes without intersection obtain physical and dynamical visible properties of the complex system dynamics.

Between them there are vector terms that present deviational couple effect containing vector rotators whose directions are the same as kinetic pressure components on corresponding gyrorotor shaft bearings.

Also, we can conclude that the impact of different possibilities to establish the phenomenological analogy of different physical vector models (see [17, 20]) expressed by vectors connected to the pole and the axis and the influence of such possibilities to applications allows researchers and scientists to obtain larger views within their specialization fields. This is the reason for introducing mass moment vectors to the rotor dynamics, as well as vector rotators.

The primary-main vector is $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ vector of the body mass inertia moment at the point $A = O$ for the axis oriented by the unit vector $\vec{n}$ and there is a corresponding $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ vector of the rigid body mass deviational moment for the axis through the point A (see [17, 20]).

Also, there are a number of the *vector rotators*, pure kinematics vectors depending on angular velocity and angular acceleration of the body rotation as well as of the mass center position or deviational plane of the body in relation to the axis.

For the case of a rigid body simple rotation about one axis there are two orthogonal vector rotators with same intensity depending on angular acceleration and angular velocity. Directions of these vector rotators are the same as components of kinetic pressures to shaft bearings. The vector rotators correspond to the rotation axis and one in the deviational plane through the axis and second orthogonal to the deviational plane and both with intensity $\mathfrak{R} = \sqrt{\dot{\omega}^2 + \omega^4}$. In the listed papers [17–22] as well as in others, written by first author of this paper, no listed heir, many applications of the discovered vector method by using mass moment vectors are presented for to express kinetic parameters of heavy rotors dynamics as well as of coupled multistep rotors dynamics and for gyrorotors dynamics.

Organizations of this paper based on the vector method applications with use of the mass moment vectors and vector rotators for obtaining vector expressions for linear momentum and angular momentum and their derivatives of the rigid body coupled rotations around two axes without intersections. These obtained expressions are analyzed and series of conclusions are pointed out, all useful for analysis of the rigid body coupled rotations around two axes without intersections when system dynamics is with two degrees of mobility as well as with two degrees of freedom, or for constrained by programmed rheonomic constraint and with one degree of freedom.

By using two vector equations of dynamic equilibrium of rigid body dynamics with coupled rotations around two axes without intersection for two degrees of freedom it is possible to obtain two nonlinear differential equations in scalar form for rotations about each axes and also corresponding kinetic pressures in vector form bearing of both shafts.

2. Mass Moment Vectors for the Axis to the Pole

The monograph [20], IUTAM extended abstract [17], and monograph paper [21] contain definitions of three mass moment vectors coupled to an axis passing through a certain point as a reference pole. Now, we start with necessary definitions of mass momentum vectors.

Definitions of selected mass moment vectors for the axis and the pole, which are used in this paper are as follows.

(1) Vector $\vec{\mathfrak{S}}_{\vec{n}}^{(O)}$ of the body mass linear moment for the axis, oriented by the unit vector $\vec{n}$, through the point—pole O, in the following form (see Figure 1):

$$\vec{\mathfrak{S}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint_V [\vec{n}, \vec{\rho}]\, dm = [\vec{n}, \vec{\rho}_C] M, \quad dm = \sigma dV, \tag{2.1}$$

where $\vec{\rho}$ is the position vector of the elementary body mass particle dm in point N, between pole O and mass particle position N.

(2) Vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment for the axis, oriented by the unit vector $\vec{n}$, through the point—pole O, in the following form

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} \stackrel{\text{def}}{=} \iiint_V [\vec{\rho}, [\vec{n}, \vec{\rho}]]\, dm. \tag{2.2}$$

For special cases, the details can be seen in [17–22]. In the previously cited references, the spherical and deviational parts of the mass inertia moment vector and the inertia tensor are analysed. In monograph [20] knowledge about the change (rate) in time and the derivatives of the mass moment vectors of the body mass linear moment, the body mass inertia moment for the pole, and a corresponding axis for different properties of the body, is shown, on the basis of results from the first author's reference [22].

This expression

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(O_1)} + \left[\vec{\rho}_O, \vec{\mathfrak{S}}_{\vec{n}}^{(O_1)}\right] + \left[\vec{\mathrm{M}}_C^{(O_1)}, [\vec{n}, \vec{\rho}_O]\right] + [\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]] \mathrm{M} \tag{2.3}$$

is *the vector form of the theorem for the relation of material body mass inertia moment vectors,* $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ and $\vec{\mathfrak{J}}_{\vec{n}}^{(O_1)}$, *for two parallel axes through two corresponding points, pole O and pole* O_1. We can see that all the terms in the last expression have the same structure. These structures are $[\vec{\rho}_O, [\vec{n}, \vec{r}_O]]\mathrm{M}$, $[\vec{r}_C, [\vec{n}, \vec{\rho}_O]]\mathrm{M}$, and $[\vec{\rho}_O, [\vec{n}, \vec{\rho}_O]]\mathrm{M}$.

In the case when the pole O_1 is the centre C of the body mass, the vector $\vec{r}_C$ (the position vector of the mass centre with respect to the pole O_1) is equal to zero whereas the vector $\vec{\rho}_O$ turns into $\vec{\rho}_C$ so that the last expression (2.3) can be written in the following form:

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{\mathfrak{J}}_{\vec{n}}^{(C)} + [\vec{\rho}_C, [\vec{n}, \vec{\rho}_C]] \mathrm{M}. \tag{2.4}$$

This expression (2.4) *represents the vector form of the theorem of the rate change of the mass inertia moment vector for the axis and the pole, when the axis is translated from the pole at the mass centre C to the arbitrary point, pole O.*

The *Huygens-Steiner theorems* (see [20, 21]) for the body mass axial inertia moments, as well as for the mass deviational moments, emerged *from this theorem* (2.4) on the change of the vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ of the body mass inertia moment at point O for the axis oriented by the unit vector $\vec{n}$ passing through the mass center C, and when the axis is moved by translate to the other point O.

Mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ for the axis to the pole is possible to decompose in two parts: first $\vec{n}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)})$ collinear with axis and second $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$ normal to the axis. So we can write

$$\vec{\mathfrak{J}}_{\vec{n}}^{(O)} = \vec{n}\left(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}\right) + \vec{\mathfrak{D}}_{\vec{n}}^{(O)} = J_{\vec{n}}^{(O)}\vec{n} + \vec{\mathfrak{D}}_{\vec{n}}^{(O)}. \tag{2.5}$$

Collinear component $\vec{n}(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)})$ to the axis corresponds to the axial mass inertia moment $J_{\vec{n}}^{(O)}$ of the body. Second component, $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$, orthogonal to the axis, we denote by the $\vec{\mathfrak{D}}_{O}^{(\vec{n})}$, and it is possible to obtain by both side double vector products by unit vector $\vec{n}$ with mass moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ in the following form:

$$\vec{\mathfrak{D}}_{\vec{n}}^{(O)} = \left[\vec{n}, \left[\vec{\mathfrak{J}}_{\vec{n}}^{(O)}, n\right]\right] = \vec{\mathfrak{J}}_{\vec{n}}^{(O)}(\vec{n}, \vec{n}) - \vec{n}\left(\vec{n}, \vec{\mathfrak{J}}_{\vec{n}}^{(O)}\right) = \vec{\mathfrak{J}}_{\vec{n}}^{(O)} - J_{O}^{\vec{n}}\vec{n}. \tag{2.6}$$

In case when rigid body is balanced with respect to the axis the mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}}^{(O)}$ is collinear to the axis and there is no deviational part. In this case axis of rotation is main axis of body inertia. When axis of rotation is not main axis then mass inertial moment vector for the axis contains deviation part $\vec{\mathfrak{D}}_{\vec{n}}^{(O)}$. That is case of rotation unbalanced rotor according to axis and bodies skew positioned to the axis of rotation.

3. Linear Momentum and Angular Momentum Vector Expressions for Rigid Body Dynamic with Coupled Rotation around Axes without Intersection

3.1. Model of a Rigid Body Rotation around Two Axes without Intersection

Let us to consider rigid body rotation around two axes first oriented by unit vector $\vec{n}_1$ with fixed position and second oriented by unit vector $\vec{n}_2$ which is rotating around fixed axis with angular velocity $\vec{\omega}_1 = \omega_1\vec{n}_1$. Axes of rotation are without intersection. Rigid body is positioned on the moving rotating axis oriented by unit vector $\vec{n}_2$ and rotate around self-rotating axis with angular velocity $\vec{\omega}_2 = \omega_2\vec{n}_2$ and around fixed axis oriented by unit vector $\vec{n}_1$ with angular velocity $\vec{\omega}_1 = \omega_1\vec{n}_1$. Then, axes of rigid body coupled rotations are without intersection. The shortest orthogonal distance between axes is defined by length $\overline{O_1O_2}$ and it are perpendicular to both axes that is to the direction of angular velocities $\vec{\omega}_1 = \omega_1\vec{n}_1$ and $\vec{\omega}_2 = \omega_2\vec{n}_2$. This vector is $\vec{r}_0 = \overrightarrow{O_1O_2}$ (see Figure 1):

$$\vec{r}_0 = r_0\frac{[\vec{n}_1, \vec{n}_2]}{\left|[\vec{n}_1, \vec{n}_2]\right|} = r_0\vec{u}_{01}, \tag{3.1}$$

and it can be seen on Figure 1.

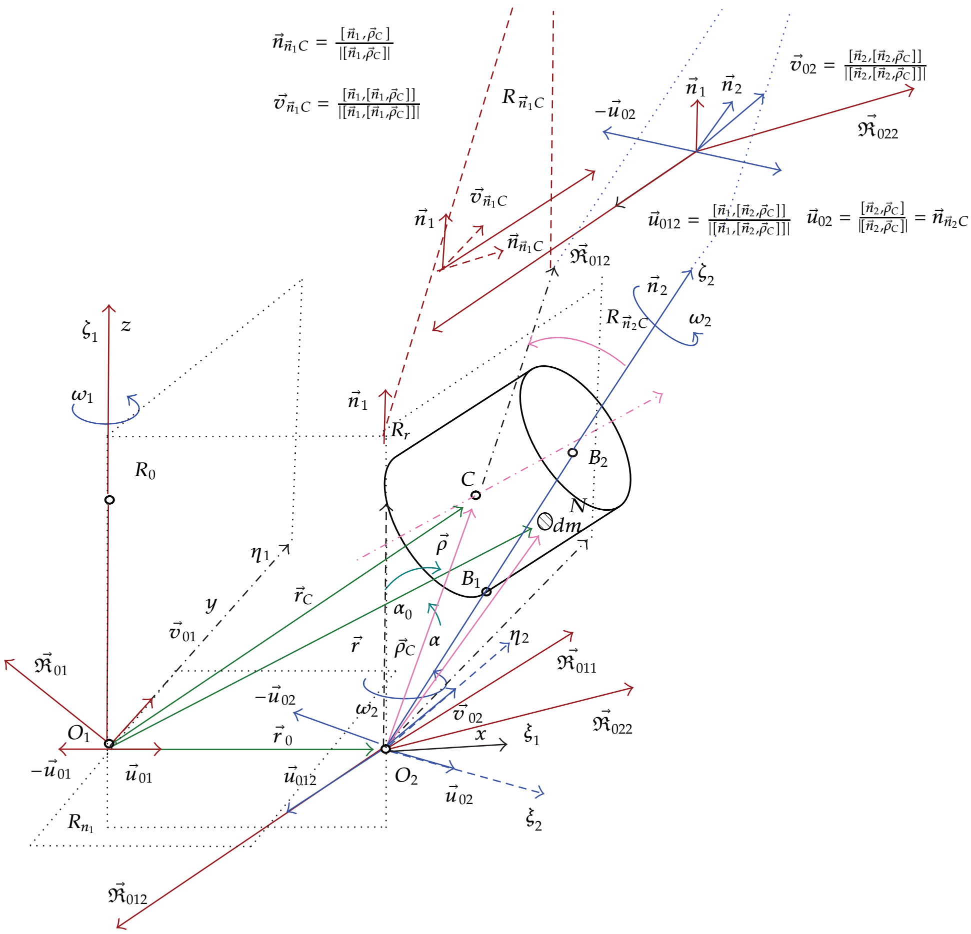


Figure 1: Arbitrary position of rigid body coupled rotations around two axes without intersection. System is with two degrees of mobility (two freedom or one degree of freedom and one rheonomic constraint) where φ_1 and φ_2 are generalized coordinates fixed coordinate system and two moveable coordinate systems $O_1\xi_1\eta_1\zeta_1 = O_1\xi_1\eta_1 z$ and $O_2\xi_2\eta_2\zeta_2 = O_2\xi_2\eta_2 z_2$ that are rotating with component angular velocities of rigid body coupled rotations: independent generalized (/or rheonomic) coordinates are φ_1 coordinate of precession rotation and φ_2 coordinate around self rotation axis. Vector rotators $\vec{\mathfrak{R}}_{01}$, $\vec{\mathfrak{R}}_{011}$, and $\vec{\mathfrak{R}}_{022}$ are presented.

When any of three main central axes of rigid body mass inertia moment is not in direction of self rotation axis, then we can see that rigid body is skew positioned. The angles β_i, $i = 1, 2$ are angles of skew position of rigid body to the self rotation axis. When center C of the mass of rigid body is not on self rotation axis of rigid body rotation, we can say that rigid body is skew. Eccentricity of position is normal distance between mass center C and axis of self rotation and it is defined by $\overrightarrow{e} = [\overrightarrow{n}_2, [\overrightarrow{\rho}_C, \overrightarrow{n}_2]]$. Here $\overrightarrow{\rho}_C$ is vector position of mass center C with origin in point O_2, and position vector of mass center with fixed origin in point O_1 is $\overrightarrow{r}_C = \overrightarrow{r}_O + \overrightarrow{\rho}_C$.

A plane in which lies the shortest distance, lenght $\overline{O_1O_2}$, that is perpendicular to fixed axis of precession rotation by angular velocity $\overrightarrow{\omega}_1 = \omega_1\overrightarrow{n}_1$ is denoted as R_{n_1}. A plane that is formed by the shortest distance and fixed axis of component (transmission) rotation gyrorotor system is denoted as R_0 and in referent position with O_1x we denote axis of fixed coordinate system, with O_1z we denote axis in line with axis of component rotation by angular velocity $\vec{\omega}_1 = \omega_1\vec{n}_1$ while third axis O_1y is perpendicular to it. Lets choose a moveable axis $O_1\xi_1$ in line to vector $\vec{r}_0 = \overrightarrow{O_1O_2}$, axis $O_1\zeta_1 = O_1z$ that rotates by angular velocity $\vec{\omega}_1 = \omega_1\vec{n}_1$ around the moveable coordinate system is rotating $O_1\xi_1\eta_1\zeta_1 = O_1\xi_1\eta_1 z$ as it can be seen on Figure 2.

In the rigid body, an elementary mass around point N we denote dm with position vector $\vec{\rho}$, and with origin in the point O_2 on the movable self rotation axis and with $\vec{r}$ vector positions of the same body elementary mass with origin in the point O_1 where point O_1 is fixed on the axis oriented by unit $\vec{n}_1$ and O_2 is on self-axis rotation oriented by unit $\vec{n}_2$ and both points are on the end of shortest orthogonal distance betwen axis of body coupled rotations. Position vector of elementary mass with origin in pole O_1 is $\vec{r} = \vec{r}_0 + \vec{\rho}$, and velocity of mass particle dm is: $\vec{v} = [\vec{\omega}_1, \vec{r}_0] + [\vec{\omega}_1 + \vec{\omega}_2, \vec{\rho}]$.

3.2. Linear Momentum and Angular Momentum of a Rigid Body Coupled Rotations around Two Axes without Intersection

By using basic definition of linear momentum and angular momentum as well as expresson for velocity of rotation elementary body mass $\vec{v} = [\vec{\omega}_1, \vec{r}_0] + [\vec{\omega}_1 + \vec{\omega}_2, \vec{\rho}]$, we can write the following vector expressions:

(a) for linear momentum in the following vector form (see [20, 23]):

$$\vec{\mathfrak{K}} = [\vec{\omega}_1, \vec{r}_0]M + \omega_1\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)} + \omega_2\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}, \tag{3.2}$$

where $\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)} = \iiint_V [\vec{n}_1, \vec{\rho}]dm$ and $\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)} = \iiint_V [\vec{n}_2, \vec{\rho}]dm$ are correspond body mass linear moment of the rigid body for the axes oriented by direction of component angular velocities of coupled rotations through the movable pole O_2 on self-rotating axis;

(b) for angular momentum in the following vector form (see [20, 23]):

$$\vec{\mathfrak{L}}_{O_1} = \omega_1\vec{n}_1 r_0^2 M + \omega_1[\vec{\rho}_C, [\vec{n}_1, \vec{r}_0]]M + \omega_1\left[\vec{r}_0, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right] + \omega_2\left[\vec{r}_0, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right] + \omega_1\vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)} + \omega_2\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}, \tag{3.3}$$

where $\vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)} \stackrel{\text{def}}{=} \iiint_V [\vec{\rho}, [\vec{n}_1, \vec{\rho}]]dm$ and $\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)} \stackrel{\text{def}}{=} \iiint_V [\vec{\rho}, [\vec{n}_2, \vec{\rho}]]dm$ are corresponding rigid body mass inertia moment vectors for the axes oriented by directions of component rotations through the pole O_2 on self-rotating axes.

First term in expression (2.6) presents transmission part of linear momentum as if all rigid body mass is concentrate in pole O_2 on self-rotating axis and rotate around fixed axes with angular velocity ω_1. This part is equal to zero in case when axes are with intersection. Second and third terms in expression for linear momentum present linear momentum of pure rotation, as relative motion around two axes with intersection in the pole O_2 on self rotation axes. This two parts are different from zero in all case.

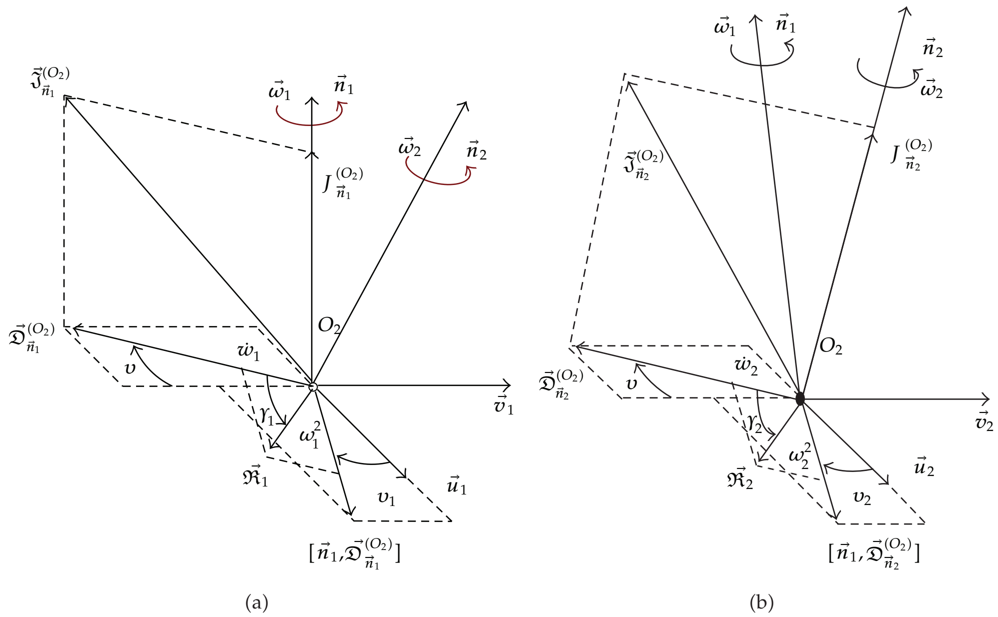


Figure 2: Vector rotators $\overrightarrow{\mathfrak{R}}_1$ (a) and $\overrightarrow{\mathfrak{R}}_2$ (b) in relations to corresponding mass moment vectors $\overrightarrow{\mathfrak{J}}^{(O_2)}_{\vec{n}_1}$ and $\overrightarrow{\mathfrak{J}}^{(O_2)}_{\vec{n}_2}$, and their corresponding deviational components $\overrightarrow{\mathfrak{D}}^{(O_2)}_{\vec{n}_1}$ and $\overrightarrow{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}$ as well as to corresponding deviational planes.

Term $\overrightarrow{\mathfrak{S}}^{O}_{\vec{n}_1} = [\overrightarrow{n}_1, \overrightarrow{r}_O]M$ is corresponding linear mass moment vector as if all rigid body mass M is concentrate in pole O_2 on the self rotation axis for the axis oriented by direction of precision rotation, threw the pole O_1.

First term in expression (3.3) presents transmission part of angular momentum as if all rigid body is concentrate in pole O_2 on self-rotating axes and rotate arround fixed axis with angular velocity ω_1. This part is equal to zero in case when axes are with intersection. First, second, third, and fourth members present transmission parts and fifth and sixth parts present relative angular momentum with respect to pole O_2 of pure rotation by two axes as they are with intersection in pole O_2 on self axis rotation. In case when axes are with intersection first four members in expression for angular moment are equal to zero.

3.3. Derivatives of Linear Momentum and Angular Momentum of Rigid Body Coupled Rotations around Two Axes without Intersection

By using expressions for linear momentum (3.2) after taking in account derivatives of parts, the derivative of linear momentum of rigid body coupled rotations around two axes without intersection, we can write the following vector expression:

$$\begin{aligned}\frac{d\overrightarrow{\mathfrak{K}}}{dt} &= \dot{\omega}_1[\overrightarrow{n}_1, \overrightarrow{r}_0]M + \omega_1^2[\overrightarrow{n}_1, [\overrightarrow{n}_1, \overrightarrow{r}_0]]M + \dot{\omega}_1\overrightarrow{\mathfrak{S}}^{(O_2)}_{\vec{n}_1} + \omega_1^2\left[\overrightarrow{n}_1, \overrightarrow{\mathfrak{S}}^{(O_2)}_{\vec{n}_1}\right] \\ &\quad + \dot{\omega}_2\overrightarrow{\mathfrak{S}}^{(O_2)}_{\vec{n}_2} + \omega_2^2\left[\overrightarrow{n}_2, \overrightarrow{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}\right] + 2\omega_1\omega_2\left[\overrightarrow{n}_1, \overrightarrow{\mathfrak{S}}^{(O_2)}_{\vec{n}_2}\right].\end{aligned} \tag{3.4}$$

After analysis structure of linear momentum derivative terms, we can see that it is possible to introduce pure kinematic vectors depending on component angular velocitie and component angular accelerations of component coupled rotations that are useful to express derivatives of linear moment in following form

$$\frac{d\vec{\mathfrak{K}}}{dt} = \vec{\mathfrak{R}}_{01}\left|\left[\vec{n}_1, \vec{r}_0\right]\right| M + \vec{\mathfrak{R}}_{011}\left|\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right| + \vec{\mathfrak{R}}_{022}\left|\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right| + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]. \tag{3.5}$$

By using vector expressions for angular momentum (4.1) after taking in account derivatives of parts, the derivative of angular momentum of rigid body coupled rotations around two axes without intersection, we can write the folowing expression:

$$\begin{aligned}
\frac{d\vec{\mathfrak{L}}_{O_1}}{dt} &= \dot{\omega}_1\left[\vec{r}_0, \left[\vec{n}_1, \vec{r}_0\right]\right] M + \omega_1\omega_2\left[\vec{r}_0, \left[\left[\vec{n}_1, \vec{n}_2\right], \vec{\rho}_C\right]\right] M + \omega_1\omega_2\left[\left[\vec{n}_2, \vec{\rho}_C\right], \left[\vec{n}_1, \vec{r}_0\right]\right] M \\
&\quad + \dot{\omega}_1\left[\vec{\rho}_C, \left[\vec{n}_1, \vec{r}_0\right]\right] M - \omega_1^2\left[\vec{\rho}_C, \vec{r}_0\right] M + \omega_1^2\left[\left[\vec{n}_1, \vec{\rho}_C\right], \left[\vec{n}_1, \vec{r}_0\right]\right] M \\
&\quad + \dot{\omega}_1\left[\vec{r}_0, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right] + \omega_1^2\left[\left[\vec{n}_1, \vec{r}_0\right], \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right] + \omega_1^2\left[\vec{r}_0, \left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right]\right] \\
&\quad + \dot{\omega}_2\left[\vec{r}_0, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right] + \omega_2^2\left[\vec{r}_0, \left[\vec{n}_2, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]\right] + \omega_1\omega_2 M\left\{\left[\vec{r}_0, \vec{n}_1\right]\left(\vec{\rho}_C, \vec{n}_2\right) - \left[\vec{r}_0, \vec{\rho}_C\right]\left(\vec{n}_1, \vec{n}_2\right)\right\} \\
&\quad + \omega_1\omega_2 M\left\{\vec{n}_2\left(\vec{\rho}_C, \left[\vec{n}_1, \vec{r}_0\right]\right) - \vec{\rho}_C\left(\vec{n}_2, \left[\vec{n}_1, \vec{r}_0\right]\right) + \left[\vec{r}_0, \vec{n}_2\left(\vec{n}_1, \vec{\rho}_C\right)\right] - \left[\vec{r}_0, \vec{\rho}_C\left(\vec{n}_1, \vec{n}_2\right)\right]\right\} \\
&\quad + \dot{\omega}_1\vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)} + \omega_1^2\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)}\right] + \dot{\omega}_2\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)} + \omega_2^2\left[\vec{n}_2, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right] + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right].
\end{aligned} \tag{3.6}$$

After analysis structure of angular momentum terms, we can see, as in previous chapter for the derivatives of linear momentum, that it is possible to introduce pure kinematic vectors rotators depending on angular velocities and angular accelerations of component coupled rotations and that is used to express derivatives of angular momentum in the following shorter form:

$$\begin{aligned}
\frac{d\vec{\mathfrak{L}}_{O_1}}{dt} &= \vec{\chi}_{12}\left(\vec{r}_0, \vec{\rho}_C, M, \dot{\omega}_1, \dot{\omega}_2, \omega_1, \omega_2, \vec{n}_1, \vec{n}_2\right) + \dot{\omega}_1\vec{n}_1 r_0^2 M + 2\omega_1\omega_2\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right] \\
&\quad + \dot{\omega}_1\left(\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)}\right)\vec{n}_1 + \dot{\omega}_2\left(\vec{n}_2, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right)\vec{n}_2 + \vec{\mathfrak{R}}_1\left|\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}\right| + \vec{\mathfrak{R}}_2\left|\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|,
\end{aligned} \tag{3.7}$$

where the following denotation is used:

$$
\begin{aligned}
&\vec{\chi}_{12}(\vec{r}_0,\vec{\rho}_C,M,\dot{\omega}_1,\dot{\omega}_2,\omega_1,\omega_2,\vec{n}_1,\vec{n}_2)\\
&\quad= \dot{\omega}_1[\vec{\rho}_C,[\vec{n}_1,\vec{r}_0]]M+\omega_1^2[\vec{n}_1,[\vec{\rho}_C,[\vec{n}_1,\vec{r}_0]]]M\\
&\qquad+\dot{\omega}_1\left[\vec{r}_0,\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right]+\dot{\omega}_2\left[\vec{r}_0,\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]+\omega_1^2\left[\vec{n}_1,\left[\vec{r}_0,\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right]\right]+\omega_2^2\left[\vec{n}_2,\left[\vec{r}_0,\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]\right]\\
&\qquad+\omega_1^2\vec{n}_1\left(\vec{r}_0,\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right)M+\omega_2^2\left[\vec{r}_0,\left[\vec{n}_2,\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]\right]-\omega_1\omega_2\left[[\vec{n}_1,\vec{r}_0],\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]M\\
&\qquad+\omega_1^2M\vec{n}_1(\vec{\rho}_C,[\vec{n}_1,\vec{r}_0])+\omega_1\omega_2\left[[\vec{n}_1,\vec{r}_0],\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]M\\
&\qquad+\omega_2\omega_1\left[\vec{r}_0,\left[\vec{n}_1,\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]\right]+\omega_1\omega_2\left[\vec{r}_0,\left[\vec{n}_2,\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right]\right]\\
&\qquad+\omega_1\omega_2(\vec{\rho}_C,\vec{n}_1)[\vec{r}_0,\vec{n}_2]M-\omega_1\omega_2(\vec{\rho}_C,\vec{n}_2)[\vec{r}_0,\vec{n}_1]M.
\end{aligned}
\tag{3.8}
$$

4. Vector Rotators of Rigid Body Coupled Rotations around Two Axes without Intersection

We can see that in previous expression (3.5) for derivative of linear momentum the following three vectors are introduced:

$$
\begin{aligned}
&\vec{\mathfrak{R}}_{01}=\dot{\omega}_1\vec{u}_{01}+\omega_1^2\vec{v}_{01},\\
&\vec{\mathfrak{R}}_{01}=\dot{\omega}_1\left[\vec{n}_1,\frac{\vec{r}_0}{r_0}\right]+\omega_1^2\left[\vec{n}_1,\left[\vec{n}_1,\frac{\vec{r}_0}{r_0}\right]\right],\\
&\vec{\mathfrak{R}}_{011}=\dot{\omega}_1\vec{u}_{011}+\omega_1^2\vec{v}_{011},\\
&\vec{\mathfrak{R}}_{011}=\dot{\omega}_1\frac{\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right|}+\omega_1^2\left[\vec{n}_1,\frac{\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_1}^{(O_2)}\right|}\right]=\dot{\omega}_1\frac{[\vec{n}_1,\vec{\rho}_C]}{|[\vec{n}_1,\vec{\rho}_C]|}+\omega_1^2\frac{[\vec{n}_1,[\vec{n}_1,\vec{\rho}_C]]}{|[\vec{n}_1,\vec{\rho}_C]|},\\
&\vec{\mathfrak{R}}_{022}=\dot{\omega}_1\vec{u}_{022}+\omega_1^2\vec{v}_{022},\\
&\vec{\mathfrak{R}}_{022}=\dot{\omega}_2\frac{\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right|}+\omega_2^2\left[\vec{n}_2,\frac{\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right|}\right]=\dot{\omega}_2\frac{[\vec{n}_2,\vec{\rho}_C]}{|[\vec{n}_2,\vec{\rho}_C]|}+\omega_1^2\frac{[\vec{n}_2,[\vec{n}_2,\vec{\rho}_C]]}{|[\vec{n}_2,\vec{\rho}_C]|}.
\end{aligned}
\tag{4.1}
$$

The first two vector rotators $\vec{\mathfrak{R}}_{01}$ and $\vec{\mathfrak{R}}_{011}$ are orthogonal to the direction of the first fixed axis and third vector rotator $\vec{\mathfrak{R}}_{022}$ is orthogonal to the self rotation axis. But, first vector rotator $\vec{\mathfrak{R}}_{01}$ is coupled for pole O_1 on the fixed axis and second and third vector rotators, $\vec{\mathfrak{R}}_{011}$ and $\vec{\mathfrak{R}}_{022}$, are coupled for the pole O_2 at self rotation axis and for corresponding direction oriented by directions of component angular velocities of coupled rotations. Intensity of two first rotators is equal and is expressed by angular velocity and angular acceleration of the first component rotation, and intensity of third vector rotators is expressed by angular velocity and angular acceleration of the second component rotation, and are in the following forms:

$$\mathfrak{R}_{01} = \mathfrak{R}_{011} = \sqrt{\dot{\omega}_1^2 + \omega_1^4}, \qquad \mathfrak{R}_{022} = \sqrt{\dot{\omega}_2^2 + \omega_2^4}. \tag{4.2}$$

Lets introduce notation γ_{01}, γ_{011}, and γ_{022} denote difference between corresponding component angles of rotation φ_1 and φ_2 of the rigid body component rotations and corresponding absolute angles of pure kinematics vector rotators about axes oriented by unit vectors $\vec{n}_1$ and $\vec{n}_2$. These angles are determined by the following relations:

$$\gamma_{01} = \gamma_{011} = \arctan \frac{\dot{\varphi}_1^2}{\ddot{\varphi}_1}, \qquad \gamma_{02} = \arctan \frac{\dot{\varphi}_2^2}{\ddot{\varphi}_2}. \tag{4.3}$$

Angular velocity of relative kinematics vectors rotators $\vec{\mathfrak{R}}_{01}$, $\vec{\mathfrak{R}}_{011}$, and $\vec{\mathfrak{R}}_{022}$ which rotate about corresponding axes in relation to the component angular velocities of the rigid body component rotations are

$$\dot{\gamma}_{01} = \dot{\gamma}_{011} = \frac{\dot{\varphi}_1 (2\ddot{\varphi}_1 - \dot{\varphi}_1 \dddot{\varphi}_1)}{\ddot{\varphi}_1^2 + \dot{\varphi}_1^4}, \qquad \dot{\gamma}_{02} = \frac{\dot{\varphi}_2 (2\ddot{\varphi}_2 - \dot{\varphi}_2 \dddot{\varphi}_2)}{\ddot{\varphi}_2^2 + \dot{\varphi}_2^4}. \tag{4.4}$$

In Figure 1. Vector rotators $\vec{\mathfrak{R}}_{01}$, $\vec{\mathfrak{R}}_{011}$, and $\vec{\mathfrak{R}}_{022}$ are presented.

Fourth vector rotator $\vec{\mathfrak{R}}_{012}$ is in the following vector form and with intensity $\mathfrak{R}_{012}$:

$$\vec{\mathfrak{R}}_{012} = 2\omega_1\omega_2 \frac{\left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]}{\left|\left[\vec{n}_1, \vec{\mathfrak{S}}_{\vec{n}_2}^{(O_2)}\right]\right|} = 2\omega_1\omega_2 \frac{[\vec{n}_1, [\vec{n}_2, \vec{\rho}_C]]}{|[\vec{n}_1, [\vec{n}_2, \vec{\rho}_C]]|}, \qquad \left|\vec{\mathfrak{R}}_{012}\right| = \mathfrak{R}_{012} = 2\omega_1\omega_2. \tag{4.5}$$

This vector rotator $\vec{\mathfrak{R}}_{012}$ depends on both components of coupled rotations.

We can see that in previous vector expression (3.6) or (3.7) for derivative of angular momentum are introduced following two vectors rotators: $\vec{\mathfrak{R}}_1 = \dot{\omega}_1 \vec{u}_1 + \omega_1^2 \vec{v}_1$ and $\vec{\mathfrak{R}}_2 = \dot{\omega}_1 \vec{u}_2 + \omega_1^2 \vec{v}_2$ in the following vector form:

$$\begin{aligned}
\vec{\mathfrak{R}}_1 &= \dot{\omega}_1 \frac{\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}\right|} + \omega_1^2 \left[\vec{n}_1, \frac{\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}\right|}\right] = \dot{\omega}_1 \vec{u}_1 + \omega_1^2 \vec{v}_1, \\
\vec{\mathfrak{R}}_2 &= \dot{\omega}_2 \frac{\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|} + \omega_2^2 \left[\vec{n}_2, \frac{\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|}\right] = \dot{\omega}_2 \vec{u}_2 + \omega_2^2 \vec{v}_2.
\end{aligned} \tag{4.6}$$

The first $\vec{\mathfrak{R}}_1$ is orthogonal to the fixed axis oriented by unit vector $\vec{n}_1$ and second $\vec{\mathfrak{R}}_2$ is orthogonal to the self rotation axis oriented by unit vector $\vec{n}_2$. Intensity of first rotator $\vec{\mathfrak{R}}_1$ is equal to intensity of previous defined rotator $\mathfrak{R}_{01}$ and intensity of second rotator $\vec{\mathfrak{R}}_2$ is equal to intensity of previous defined rotator $\mathfrak{R}_{022}$ defined by expressions (3.7). Their intensities are

$$\mathfrak{R}_1 = \sqrt{\dot{\omega}_1^2 + \omega_1^4}, \qquad \mathfrak{R}_2 = \sqrt{\dot{\omega}_2^2 + \omega_2^4}. \tag{4.7}$$

In Figure 2 vector rotators $\vec{\mathfrak{R}}_1$ (in Figure 2(a)) and $\vec{\mathfrak{R}}_2$ (in Figure 2(b)) in relations to corresponding mass moment vectors $\vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)}$ and $\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}$, and their corresponding deviational components $\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}$ and $\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}$ as well as to corresponding deviational planes are presented.

Vector rotators $\vec{\mathfrak{R}}_1$ and $\vec{\mathfrak{R}}_2$ are pure kinematical vectors first presented in [20, 21] as a function on angular velocity and angular acceleration in a form $\vec{\mathfrak{R}} = \ddot{\varphi}\vec{u} + \dot{\varphi}^2\vec{w} = \mathfrak{R}\vec{\mathfrak{R}}_0$. Also from Section 3.3 expressions (3.5) and (3.6) or (3.7) for derivatives for linear and angular momentum contain members with in tree types of different pure kinematical vectors rotators which rotate around first and second axis in corresponding directions of coupled rotation components, but with pole in O_1 or in O_2. These vector rotators are possible to separate by following criteria: (1) intensity of vector rotator is expressed by angular velocity ω_1 and angular acceleration $\dot{\omega}_1$ in the form $\mathfrak{R}_1 = \sqrt{\dot{\omega}_1^2 + \omega_1^4}$ or angular velocity ω_2 and angular acceleration $\dot{\omega}_2$ in the form and $\mathfrak{R}_2 = \sqrt{\dot{\omega}_2^2 + \omega_2^4}$; (2) intensity of the vector rotators is expressed by both angular velocity components ω_1 and ω_2, and no contain angular accelerations $\dot{\omega}_1$ and $\dot{\omega}_2$; (3) vector rotators are coupled by pole in O_1 or in O_2; (4) type of angular velocities components of vector rotators.

Rotators from first set are rotated around through pole O_2 axis in direction of first component rotation angular velocity and depend of angular velocity ω_1 and angular acceleration $\dot{\omega}_1$. There are two vectors of such type and all trees have equal intensity. Rotators from second set are rotated around axis in direction of second component rotation and depend of angular velocity ω_2 and angular acceleration $\dot{\omega}_2$. There are two vectors of such type and they have equal intensity.

Let us introduce notation, γ_1 and γ_2 denote difference between corresponding component angles of rotation φ_1 and φ_2 of the rigid body component rotations and corresponding absolute angles of pure kinematics vector rotators about axes oriented by unit vectors $\vec{n}_1$ and $\vec{n}_2$ through pole O_2. These angles are determined by following relations:

$$\gamma_1 = \arctan\frac{\dot{\varphi}_1^2}{\ddot{\varphi}_1}, \qquad \gamma_2 = \arctan\frac{\dot{\varphi}_2^2}{\ddot{\varphi}_2}. \tag{4.8}$$

Angular velocity of relative kinematics vectors rotators $\vec{\mathfrak{R}}_1$ and $\vec{\mathfrak{R}}_2$ which rotate about axes in corresponding directions in relation to the component angular velocities of the rigid body component rotations through pole O_2 are

$$\dot{\gamma}_1 = \frac{\dot{\varphi}_1\left(2\ddot{\varphi}_1^2 - \dot{\varphi}_1\dddot{\varphi}_1\right)}{\ddot{\varphi}_1^2 + \dot{\varphi}_1^4}, \qquad \dot{\gamma}_2 = \frac{\dot{\varphi}_2\left(2\ddot{\varphi}_2^2 - \dot{\varphi}_2\dddot{\varphi}_2\right)}{\ddot{\varphi}_2^2 + \dot{\varphi}_2^4}. \tag{4.9}$$

Also, it is possible to separate a few numbers of rotators and between the following:

$$\vec{\mathfrak{R}}_{12} = 2\omega_1\omega_2 \frac{\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right]}{\left|\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right]\right|} = 2\omega_1\omega_2 \vec{u}_{12}, \tag{4.10}$$

where $\vec{u}_{12} = [\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}]/|[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}]|$ unit vector orthogonal to the axis oriented by unit vector $\vec{n}_1$ and mass moment vector $\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}$ for the axis oriented by unit vector $\vec{n}_2$ through pole O_2, and intensity equal $\mathfrak{R}_{12} = 2\omega_1\omega_2$ twice multiplication of product of intensities of component angular velocities ω_1 and ω_2 of rigid body coupled rotations around exes without intersection.

5. Vector Rotators of Rigid Body-Disk Dynamics with Coupled Rotations around Two Orthogonal Axes without Intersection

Let us consider vector rotators for the special case when rigid body-disk rotate around two orthogonal axes without intersection.

Vector of relative mass center position $\vec{\rho}_C$ in relation to the pole O_2 and self rotation axis oriented by unit vector $\vec{n}_2$, we can express in the movable coordinate systems with axes oriented by basic unit vectors: $\vec{n}_2$, $\vec{u}_{02}$ and $\vec{v}_{02}$ which rotate around self rotation axis with angular velocity ω_2 in the form $\vec{\rho}_C = \rho_C(\cos\beta\vec{n}_2 + \sin\beta\vec{u}_{02})$, as well as by basic unit vectors $\vec{u}_{01}$, $\vec{v}_{01}$ and $\vec{n}_1$ which rotate around fixed axis oriented by unit vector $\vec{n}_1$ with angular velocity ω_1 in the following form: $\vec{\rho}_C = \rho_C\langle\cos\beta\vec{u}_{01} - \sin\beta\cos\varphi_2\vec{v}_{01} + \sin\beta\sin\varphi_2\vec{n}_1\rangle$. β is angle between mass center vector position $\vec{\rho}_C$ and self rotation axis oriented by unit vector $\vec{n}_2$. Vector of the orthogonal distance between orthogonal axes without intersection is $\vec{r}_0 = -r_0\vec{v}_{01}$.

For this case unit vectors $\vec{n}_1$ and $\vec{n}_2$ are orthogonal, and after taking into account this orthogonality and corresponding formulas (4.1), (4.5), (4.6), and (4.10) for vector rotators we obtain the following vector expressions:

$$\vec{\mathfrak{R}}_{011} = \frac{\langle\vec{v}_{01}(\dot{\omega}_1\cos\beta - \omega_1^2\sin\beta\sin\varphi_2) - \vec{u}_{01}(\dot{\omega}_1\sin\beta\sin\varphi_2 + \omega_1^2\cos\beta)\rangle}{\sqrt{\cos^2\beta + \sin^2\beta\sin^2\varphi_2}}, \qquad \left|\vec{\mathfrak{R}}_{011}\right| = \sqrt{\dot{\omega}_1^2 + \omega_1^4}$$

$$\vec{\mathfrak{R}}_{022} = \dot{\omega}_2\vec{v}_{02} - \omega_1^2\vec{u}_{02}, \qquad \left|\vec{\mathfrak{R}}_{022}\right| = \sqrt{\dot{\omega}_2^2 + \omega_2^4}$$

$$\vec{\mathfrak{R}}_{012} = -2\omega_1\omega_2\vec{u}_{01}, \qquad \left|\vec{\mathfrak{R}}_{012}\right| = \mathfrak{R}_{012} = 2\omega_1\omega_2$$

$$\vec{\mathfrak{R}}_1 = -\frac{\vec{u}_{01}\langle\dot{\omega}_1\cos\beta + \omega_1^2\sin\beta\cos\varphi_2\rangle + \vec{v}_{01}\langle-\dot{\omega}_1\sin\beta\cos\varphi_2 + \omega_1^2\cos\beta\rangle}{\sqrt{\cos^2\beta + \sin^2\beta\cos^2\varphi_2}}, \qquad \left|\vec{\mathfrak{R}}_1\right| = \sqrt{\dot{\omega}_1^2 + \omega_1^4}$$

$$\vec{\mathfrak{R}}_2 = \dot{\omega}_2 \vec{v}_{02} - \omega_1^2 \vec{u}_{02}, \qquad \left|\vec{\mathfrak{R}}_2\right| = \sqrt{\dot{\omega}_2^2 + \omega_2^4}$$

$$\vec{\mathfrak{R}}_{12} = 2\omega_1\omega_2 \frac{\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right]}{\left|\left[\vec{n}_1, \vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}\right]\right|} = 2\omega_1\omega_2 \vec{u}_{12}, \qquad \left|\vec{\mathfrak{R}}_{12}\right| = \mathfrak{R}_{12} = 2\omega_1\omega_2. \tag{5.1}$$

Previous expressions for vectors rotators are derived with supposition that rigid body is disk and that unit vectors in different deviation planes are:

$$\frac{\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}\right|} = -\frac{\langle \cos\beta \vec{u}_{01} - \sin\beta\cos\varphi_2 \vec{v}_{01_1} \rangle}{\sqrt{\cos^2\beta + \sin^2\beta\cos^2\varphi_2}}, \qquad \left[\vec{n}_1, \frac{\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}\right|}\right] = -\frac{\cos\beta \vec{v}_{01} + \sin\beta\cos\varphi_2 \vec{u}_{01}}{\sqrt{\cos^2\beta + \sin^2\beta\cos^2\varphi_2}}$$

$$\vec{v}_{02} = \vec{u}_2 = \frac{\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|}, \qquad \vec{u}_{02} = \vec{v}_2 = \left[\vec{n}_2, \frac{\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}}{\left|\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|}\right]. \tag{5.2}$$

In Figure 3. four schematic presentations of deviational planes and component directions of the vector rotators of rigid body-disk dynamics with coupled rotation around two orthogonal axes without intersection are presented. In Figure 3(a) deviation plane containing body mass center C, vector of relative mass center position $\vec{\rho}_C$ in relation to the pole O_2 and self rotation axis oriented by unit vector $\vec{n}_2$ is visible. In Figure 3(b) deviation plane containing self rotation axis oriented by unit vector $\vec{n}_2$ and body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}$ and its deviational component vector of mass deviational moment $\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}$ for self rotation axis and pole O_2 is visible. In Figure 3(c) two deviational planes through pole O_2: deviation plane containing self rotation axis oriented by unit vector $\vec{n}_2$ and body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}$ and its deviational component vector of mass deviational moment $\vec{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}$ for self rotation axis and pole O_2 and deviation plane containing axis parallel to fixed axis oriented by unit vector $\vec{n}_1$ and body mass inertia moment vector $\vec{\mathfrak{J}}_{\vec{n}_1}^{(O_2)}$ and its deviational component vector of mass deviational moment $\vec{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}$ for axis oriented by unit vector $\vec{n}_1$ and through pole O_2 are visible. In Figure 3(d) schematic presentation of the rigid body-disk skew and eccentrically positioned on the self rotation axis with corresponding mass moment vectors and deviation plane as a detail of the rigid body-disk coupled rotation around two orthogonal axes without intersection is visible. In all form of the parts in Figure 3. the component directions of the vector rotators components are visible.

By use derived vector expressions of the vector rotators we can obtain some angles between corresponding vector rotator and basic vectors of corresponding movable coordinate

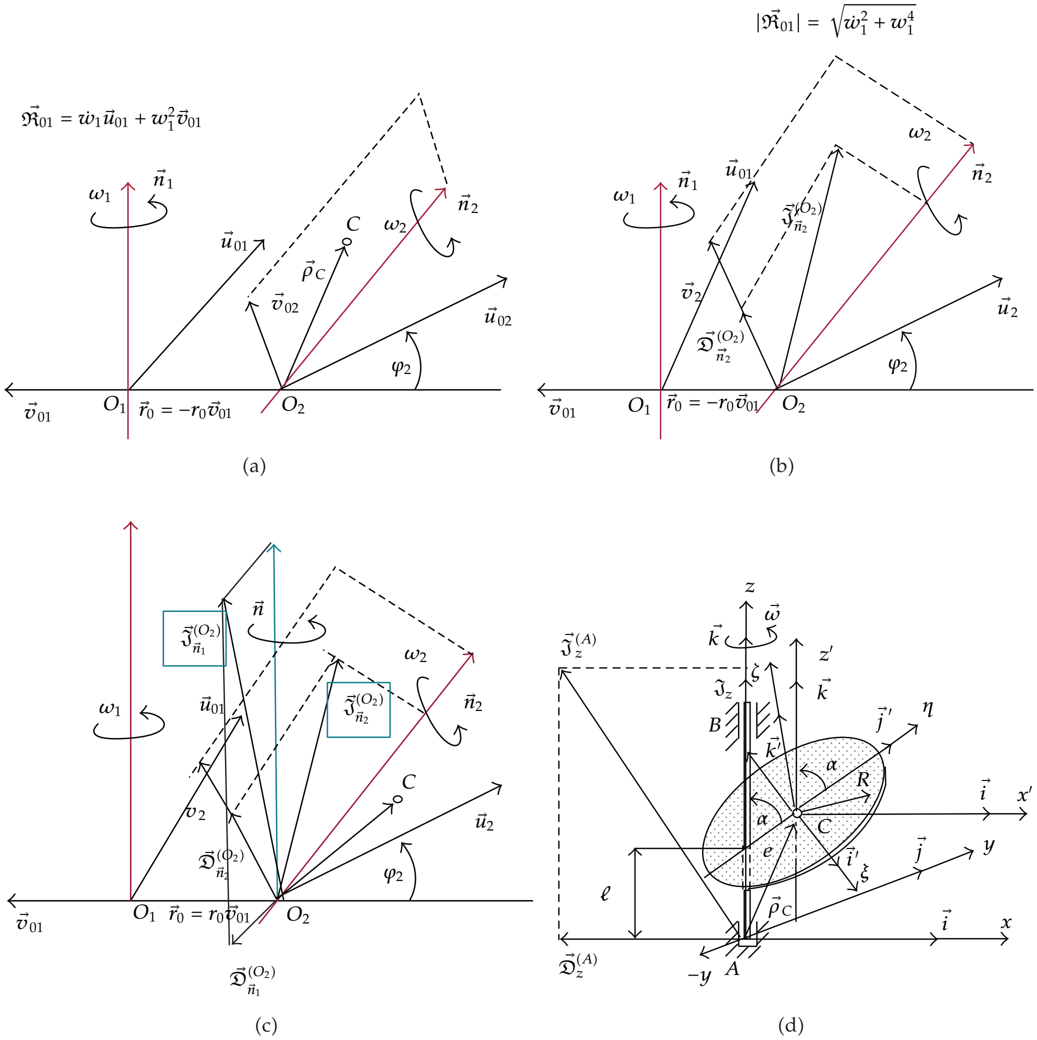


Figure 3: Schematic presentation of deviational planes and component directions of the vector rotators of rigid body-disk dynamics with coupled rotation around two orthogonal axes without intersection. (a) Deviation plane containing body mass center C, vector of relative mass center position $\overrightarrow{\rho}_C$ in relation to the pole O_2, and self rotation axis oriented by unit vector $\overrightarrow{n}_2$. (b) Deviation plane containing self rotation axis oriented by unit vector $\overrightarrow{n}_2$ and body mass inertia moment vector $\overrightarrow{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}$ and its deviational component vector of mass deviational moment $\overrightarrow{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}$ for self rotation axis and pole O_2. (c) Two deviational planes through pole O_2: deviation plane containing self rotation axis oriented by unit vector $\overrightarrow{n}_2$ and body mass inertia moment vector $\overrightarrow{\mathfrak{J}}_{\vec{n}_2}^{(O_2)}$ and its deviational component vector of mass deviational moment $\overrightarrow{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}$ for self rotation axis and pole O_2 and deviation plane containing axis parallel to fixed axis oriented by unit vector $\overrightarrow{n}_1$ and body mass inertia moment vector $\overrightarrow{\mathfrak{J}}_{\vec{n}_1}^{(O_2)}$ and its deviational component vector of mass deviational moment $\overrightarrow{\mathfrak{D}}_{\vec{n}_1}^{(O_2)}$ for axis oriented by unit vector $\overrightarrow{n}_1$ and through pole O_2. (d) Schematic presentation of the rigid body-disk skew and eccentrically positioned on the self rotation axis with corresponding mass moment vectors and deviation plane as a detail of the rigid body-disk coupled rotation around two orthogonal axes without intersection.

systems coupled with corresponding compounding axis of component coupled rotations in the following form:

$$tg\gamma_1 = \frac{\dot{\omega}_1}{\omega_1^2}, \qquad tg\gamma_{011} = \frac{\dot{\omega}_1}{\omega_1^2} = tg\gamma_1$$

$$tg\tilde{\gamma}_1 = \frac{1 - (\dot{\omega}_1/\omega_1^2)tg\beta\cos\varphi_2}{(\dot{\omega}_1/\omega_1^2) + tg\beta\cos\varphi_2} \quad \text{or in the form} \quad tg\tilde{\gamma}_1 = \frac{1 - tg\gamma_1 tg\beta\cos\varphi_2}{tg\gamma_1 + tg\beta\cos\varphi_2}$$
$$tg\tilde{\gamma}_{011} = \frac{(\dot{\omega}_1/\omega_1^2) - tg\beta\sin\varphi_2}{1 + (\dot{\omega}_1/\omega_1^2)tg\beta\sin\varphi_2} \quad \text{or in the form} \quad tg\tilde{\gamma}_{011} = \frac{tg\gamma_{011} - tg\beta\sin\varphi_2}{1 + tg\gamma_{011} tg\beta\sin\varphi_2}. \tag{5.3}$$

For the case that $\dot{\omega}_1 = 0$, $\omega_1 = \text{constant}$

$$tg\tilde{\gamma}_{011} = \frac{(\dot{\omega}_1/\omega_1^2) - tg\beta\sin\varphi_2}{1 + (\dot{\omega}_1/\omega_1^2)tg\beta\sin\varphi_2} = tg\beta\sin\varphi_2,$$
$$tg\tilde{\gamma}_1 = \frac{1}{tg\beta\cos\varphi_2} = ctg\beta\frac{1}{os\varphi_2}, \tag{5.4}$$

where γ_1 is relative angle of rotation in comparison with angle of rotation φ_1, when $\tilde{\gamma}_1$ is absolute angle of rotor rotation about axis oriented by unit vector $\vec{n}_1$, taking into account its rotation about axis oriented by unit vector $\vec{n}_2$.

6. Dynamic of Rigid Body Coupled Rotation around Two Orthogonal Axes without Intersection and with One Degree of Freedom

6.1. Model Description of a Gyrorotor Coupled Rotations around Two Orthogonal Axes without Intersection and with One Degree of Freedom

We are going to take into consideration special case of the considered heavy rigid body with coupled rotations about two axes without intersection with one degree of freedom, and in the gravitation field. For this case generalized coordinate φ_2 is independent, and coordinate φ_1 is programmed. In that case, we say that coordinate φ_1 is rheonomic coordinate and system is with kinematical excitation, programmed by forced support rotation by constant angular velocity. When the angular velocity of shaft support axis is constant, $\dot{\varphi}_1 = \omega_1 = \text{constant}$, we have that rheonomic coordinate is linear function of time, $\varphi_1 = \omega_1 t + \varphi_{10}$, and angular acceleration around fixed axis is equal to zero $\dot{\omega}_1 = 0$.

Special case is when the support shaft axis is vertical and the gyrorotor shaft axis is horizontal, and all time in horizontal plane, and when axes are without intersection at normal distance a. So we are going to consider that example presented in Figure 5. The normal distance between axes is a. The angle of self rotation around moveable self rotation axis oriented by the unit vector $\vec{n}_2$ is φ_2 and the angular velocity is $\omega_2 = \dot{\varphi}_2$. The angle of rotation around the shaft support axis oriented by the unit vector $\vec{n}_1$ is φ_1 and the angular velocity is $\omega_1 = \text{constant}$. The angular velocity of rotor is $\vec{\omega} = \omega_1\vec{n}_1 + \omega_2\vec{n}_2 = \dot{\varphi}_1\vec{n}_1 + \dot{\varphi}_2\vec{n}_2$. The angle φ_2 is generalized coordinates in case when we investigate system with one degree of freedom, but system has two degrees of mobility. Also, without loss of generality we take that

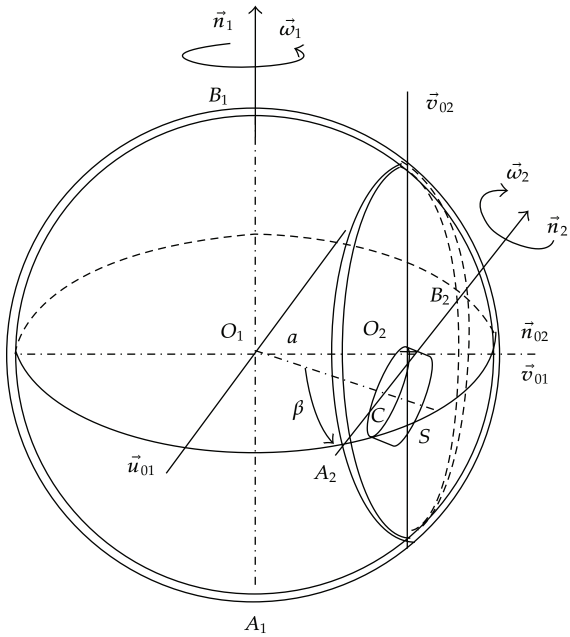


Figure 4: Model of heavy gyrorotor with two component coupled rotations around orthogonal axes without intersections.

rigid body is a disk, eccentrically positioned on the self rotation shaft axis with eccentricity e, and that angle of skew inclined position between one of main axes of disk and self rotation axis is β, as it is visible in Figure 4.

For that example, differential equation of the heavy gyrorotor-disk self rotation of reviewed model in Figure 4, for the case coupled rotations about two orthogonal axes, we can obtain in the following form:

$$\ddot{\varphi}_2 + \Omega^2(\lambda - \cos\varphi_2)\sin\varphi_2 + \Omega^2\psi\cos\varphi_2 = 0, \tag{6.1}$$

where

$$\Omega^2 = \omega_1^2 \frac{J^{(C)}_{\vec{u}_2} - J^{(C)}_{\vec{v}_2}}{J^{(C)}_{\vec{n}_2}}, \quad \lambda = \frac{mge\sin\beta}{\omega_1^2\left(J^{(C)}_{\vec{u}_2} - J^{(C)}_{\vec{v}_2}\right)}, \quad \psi = \frac{2mea\sin\beta}{J^{(C)}_{\vec{u}_2} - J^{(C)}_{\vec{v}_2}}, \quad \varepsilon = 1 + 4\left(\frac{e}{r}\right)^2. \tag{6.2}$$

Here it is considered an eccentric disc (eccentricity is e), with mass m and radius r, which is inclined to the axis of its own self rotation by the angle β (see Figure 5.), so that previous constants (6.11) in differential equation (6.10) become the following forms:

$$\Omega^2 = \omega_1^2 \frac{\left(\varepsilon\sin^2\beta - 1\right)}{\left(\varepsilon\sin^2\beta + 1\right)}, \quad \varepsilon = 1 + 4\left(\frac{e}{r}\right)^2, \quad \lambda = \frac{g(\varepsilon - 1)\sin\beta}{e\omega_1^2\left(\varepsilon\sin^2\beta - 1\right)}, \quad \psi = \frac{2ea\sin\beta}{er\left(\varepsilon\sin^2\beta - 1\right)}. \tag{6.3}$$

6.2. Phase Portrait of the Heavy Gyrorotor Disk Coupled Rotations About Two Axes without Intersection and Their Three Parameter Transformations

Relative nonlinear dynamics of the heavy gyrorotor-disk around self rotation shaft axis is possible to present by means of phase portrait method. Forms of phase trajectories and their transformations by changes of initial conditions, and for different cases of disk eccentricity and angle of its skew, as well as for different values of orthogonal distance between axes of component rotations may present character of nonlinear oscillations.

For that reason it is necessary to find first integral of the differential (6.10). After integration of the differential (6.3), the nonlinear equation of the phase trajectories of the heavy gyrorotor disk dynamics with the initial conditions $t_0 = 0$, $\varphi_1(t_0) = \varphi_{10}$, $\dot{\varphi}_1(t_0) = \dot{\varphi}_{10}$, we obtain in the following for

$$\dot{\varphi}_2^2 = \dot{\varphi}_{02}^2 + 2\Omega^2\left(\lambda \cos\varphi_2 - \frac{1}{2}\cos^2\varphi_2 + \psi \sin\varphi_2\right) - 2\Omega^2\left(\lambda \cos\varphi_{02} - \frac{1}{2}\cos^2\varphi_{02} + \psi \sin\varphi_{02}\right). \tag{6.4}$$

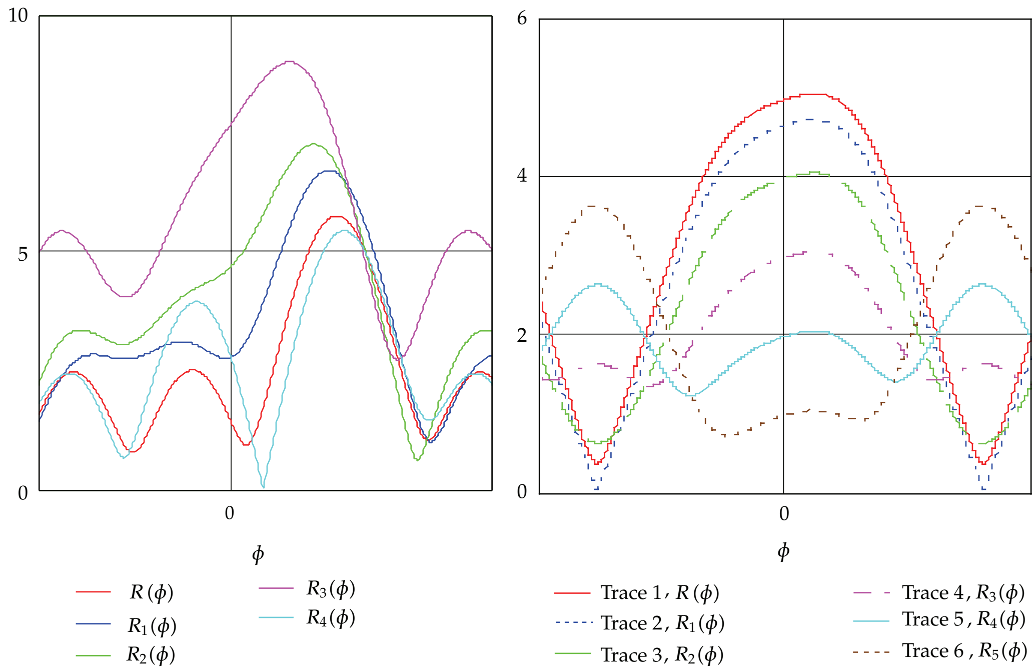
As the analyzed system is conservative it is the energy integral.

6.3. Kinematical Vector Rotators of the Heavy Gyrorotor Disk Coupled Rotations about Two Axes without Intersection and Their Three Parameter Transformations

In the considered case for the heavy gyrorotor-disk nonlinear dynamics in the gravitational field with one degree of freedom and with constant angular velocity about fixed axis, we have three sets of vector rotators.

Three of these vector rotators $\overrightarrow{\mathfrak{R}}_{01}$, $\overrightarrow{\mathfrak{R}}_{011}$, and $\overrightarrow{\mathfrak{R}}_1$, from first set, are with same constant intensity $|\overrightarrow{\mathfrak{R}}_{01}| = |\overrightarrow{\mathfrak{R}}_{011}| = |\overrightarrow{\mathfrak{R}}_1| = \omega_1^2 =$ constant and rotate with constant angular velocity ω_1 and equal to the angular velocity of rigid body precession rotation about fixed axis, but two of these three vector rotators, $\overrightarrow{\mathfrak{R}}_{011}$ and $\overrightarrow{\mathfrak{R}}_1$ are connected to the pole O_2 on the self rotation axis, and are orthogonal to the axis parallel direction as direction of the fixed axis. All these three vector rotators $\overrightarrow{\mathfrak{R}}_{01}$, $\overrightarrow{\mathfrak{R}}_{011}$, and $\overrightarrow{\mathfrak{R}}_1$ are in different directions (see Figures 3(a), 3(b), 3(c), and 4). Two of these vector rotators, $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$, from second set, are with same intensity equal to $\mathfrak{R}_{022} = \sqrt{\dot{\omega}_2^2 + \omega_2^4}$, and connecter to the pole O_2 and orthogonal to the self rotation axis oriented by unit vector $\vec{n}_2$ and rotate about this axis with relative angular velocity $\dot{\gamma}_2$ defined by second expression (4.9), $\dot{\gamma}_2 = (\dot{\varphi}_2(2\ddot{\varphi}_2^2 - \dot{\varphi}_2\dddot{\varphi}_2))/(\ddot{\varphi}_2^2 + \dot{\varphi}_2^4)$, in respect to the self rotation angular velocity ω_2. These two of these vector rotators, $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$ are oriented in the following directions:

$$\overrightarrow{\mathfrak{R}}_{022} = \dot{\omega}_2 \frac{[\vec{n}_2, \vec{\rho}_C]}{|[\vec{n}_2, \vec{\rho}_C]|} + \omega_2^2 \frac{[\vec{n}_2, [\vec{n}_2, \vec{\rho}_C]]}{|[\vec{n}_2, \vec{\rho}_C]|}, \qquad \overrightarrow{\mathfrak{R}}_2 = \dot{\omega}_2 \frac{\overrightarrow{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}}{\left|\overrightarrow{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|} + \omega_2^2 \left[\vec{n}_2, \frac{\overrightarrow{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}}{\left|\overrightarrow{\mathfrak{D}}_{\vec{n}_2}^{(O_2)}\right|}\right]. \tag{6.5}$$



(a) (b)

Figure 5: Intensity of vector rotators $\vec{\mathfrak{R}}_{022}$ and $\vec{\mathfrak{R}}_2$, for different disk eccentricity (a) and for different initial conditions (b).

By use expressions (5.1) we can list following series of vector rotators of the gyrorotor-disk with coupled rotation around orthogonal axes without intersection and with ω_1 = constant:

$$\begin{gathered}
\vec{\mathfrak{R}}_{01} = \omega_1^2 \vec{v}_{01}, \qquad \left|\vec{\mathfrak{R}}_{01}\right| = \omega_1^2, \\
\vec{\mathfrak{R}}_{011} = -\omega_1^2 \frac{\langle \sin\beta \sin\varphi_2 \vec{v}_{01} + \cos\beta \vec{u}_{01}\rangle}{\sqrt{\cos^2\beta + \sin^2\beta \sin^2\varphi_2}}, \qquad \left|\vec{\mathfrak{R}}_{011}\right| = \omega_1^2, \\
\vec{\mathfrak{R}}_{022} = \dot{\omega}_2 \vec{v}_{02} - \omega_1^2 \vec{u}_{02}, \qquad \left|\vec{\mathfrak{R}}_{022}\right| = \sqrt{\dot{\omega}_2^2 + \omega_2^4}, \\
\vec{\mathfrak{R}}_{012} = -2\omega_1\omega_2 \vec{u}_{01}, \qquad \left|\vec{\mathfrak{R}}_{012}\right| = \mathfrak{R}_{012} = 2\omega_1\omega_2, \\
\vec{\mathfrak{R}}_1 = -\omega_1^2 \frac{\vec{u}_{01}\langle \sin\beta \cos\varphi_2\rangle + \vec{v}_{01}\langle \cos\beta\rangle}{\sqrt{\cos^2\beta + \sin^2\beta \cos^2\varphi_2}}, \qquad \left|\vec{\mathfrak{R}}_1\right| = \omega_1^2, \\
\vec{\mathfrak{R}}_2 = \dot{\omega}_2 \vec{v}_{02} - \omega_2^2 \vec{u}_{02}, \qquad \left|\vec{\mathfrak{R}}_2\right| = \sqrt{\dot{\omega}_2^2 + \omega_2^4}, \\
\vec{\mathfrak{R}}_{12} = 2\omega_1\omega_2 \frac{\left[\vec{n}_1, \vec{\mathrm{J}}_{\vec{n}_2}^{(O_2)}\right]}{\left|\left[\vec{n}_1, \vec{\mathrm{J}}_{\vec{n}_2}^{(O_2)}\right]\right|} = 2\omega_1\omega_2 \vec{u}_{12}, \qquad \left|\vec{\mathfrak{R}}_{12}\right| = \mathfrak{R}_{12} = 2\omega_1\omega_2.
\end{gathered} \tag{6.6}$$

One of the vectors rotators from the third set is $\overrightarrow{\mathfrak{R}}_{012}$ with intensity $|\overrightarrow{\mathfrak{R}}_{012}| = 2\omega_1\omega_2$ and direction: $\overrightarrow{\mathfrak{R}}_{012} = 2\omega_1\omega_2[\vec{n}_1, [\vec{n}_2, \vec{\rho}_C]]/|[\vec{n}_1, [\vec{n}_2, \vec{\rho}_C]]| = -2\omega_1\omega_2\vec{u}_{01}$. This vector rotator is connecter to the pole O_2 and orthogonal to the axis oriented by unit vector $\vec{n}_1$ and relative rotate about this axis. Intensity of this vector rotator expressed by generalized coordinate φ_2, angle of self rotation of heavy disk, taking into account first integral (6.4) of the differential equation (6.1) obtain the following form:

$$\left|\overrightarrow{\mathfrak{R}}_{012}\right| = 2\omega_1\sqrt{\dot{\varphi}_{02}^2 + 2\Omega^2\left(\lambda\cos\varphi_2 - \frac{1}{2}\cos^2\varphi_2 + \psi\sin\varphi_2\right) - \mathfrak{U}}, \tag{6.7}$$

where $\mathfrak{U}$ denotes $2\Omega^2(\lambda\cos\varphi_{02} - 1/2\cos^2\varphi_{02} + \psi\sin\varphi_{02})$.

Intensity $\mathfrak{R}_{022} = \sqrt{\dot{\omega}_2^2 + \omega_2^4}$ of two of these vector rotators, $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$, from second set, depends on angular velocity ω_2 and angular acceleration $\dot{\omega}_2$. For the considered system of the heavy gyrorotor-disk dynamics, for obtaining expression of intensity of vector rotators, $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$, from second set, in the function of the generalized coordinate φ_2, angle of self rotation of heavy disk self rotation, we take into account a first integral (6.4) of nonlinear differential equation (6.1), and by using these result and previous expressions (6.6) of vector rotator we can write the following.

(i) Intensity of the vectors rotators, $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$, connected for the pole O_2 and rotate around self rotation axis, in the following form:

$$\begin{aligned}
&\left|\overrightarrow{\mathfrak{R}}_{022}\right| \\
&= \left|\overrightarrow{\mathfrak{R}}_{022}(\varphi_2)\right| \\
&= \Omega^2\sqrt{\left[-(\lambda - \cos\varphi_2)\sin\varphi_2 + \psi\cos\varphi_2\right]^2 + \left[\dot{\varphi}_{02}^2 + 2\Omega^2\left(\lambda\cos\varphi_2 - \frac{1}{2}\cos^2\varphi_2 + \psi\sin\varphi_2\right) - \mathfrak{U}\right]^2}.
\end{aligned} \tag{6.8}$$

(ii) Vector rotators orthogonal to the self rotation axes are in the following vector forms:

$$\begin{aligned}
\overrightarrow{\mathfrak{R}}_{022}(\varphi_2) = {} & \Omega^2\left[-(\lambda - \cos\varphi_2)\sin\varphi_2 + \psi\cos\varphi_2\right]\frac{[\vec{n}_2, \vec{\rho}_C]}{|[\vec{n}_2, \vec{\rho}_C]|} \\
& + \Omega^2\left[\dot{\varphi}_{02}^2 + 2\Omega^2\left(\lambda\cos\varphi_2 - \frac{1}{2}\cos^2\varphi_2 + \psi\sin\varphi_2\right)\right. \\
& \left. - 2\Omega^2\left(\lambda\cos\varphi_{02} - \frac{1}{2}\cos^2\varphi_{02} + \psi\sin\varphi_{02}\right)\right]\frac{[\vec{n}_2, [\vec{n}_2, \vec{\rho}_C]]}{|[\vec{n}_2, \vec{\rho}_C]|},
\end{aligned} \tag{6.9}$$

$$\overrightarrow{\mathfrak{R}}_2(\varphi_2) = \Omega^2[-(\lambda-\cos\varphi_2)\sin\varphi_2+\psi\cos\varphi_2]\frac{\overrightarrow{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}}{\left|\overrightarrow{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}\right|}$$
$$+\Omega^2\left[\dot{\varphi}^2_{02}+2\Omega^2\left(\lambda\cos\varphi_2-\frac{1}{2}\cos^2\varphi_2+\psi\sin\varphi_2\right)\right.$$
$$\left.-2\Omega^2\left(\lambda\cos\varphi_{02}-\frac{1}{2}\cos^2\varphi_{02}+\psi\sin\varphi_{02}\right)\right]\left[\vec{n}_2,\frac{\overrightarrow{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}}{\left|\overrightarrow{\mathfrak{D}}^{(O_2)}_{\vec{n}_2}\right|}\right]. \tag{6.10}$$

Parametric equations of the trajectory of the vector rotators $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$ are in the following same forms:

$$u_{\mathfrak{R}}(\varphi_2) = \Omega^2[-(\lambda-\cos\varphi_2)\sin\varphi_2+\psi\cos\varphi_2],$$
$$v_{\mathfrak{R}}(\varphi_2) = \Omega^2\left[\dot{\varphi}^2_{02}+2\Omega^2\left(\lambda\cos\varphi_2-\frac{1}{2}\cos^2\varphi_2+\psi\sin\varphi_2\right)\right.$$
$$\left.-2\Omega^2\left(\lambda\cos\varphi_{02}-\frac{1}{2}\cos^2\varphi_{02}+\psi\sin\varphi_{02}\right)\right], \tag{6.11}$$

but it is necessary to take into consideration that is not in same directions, but is in the same plane orthogonal to the axis oriented by unit vector $\vec{n}_2$ and through pole O_2.

Relative angular velocity $\dot{\gamma}_2$ of both vector rotators $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$ in plane orthogonal to the axis oriented by unit vector $\vec{n}_2$ and through pole O_2. in relation on angular velocity of self rotation, $\omega_2 = \dot{\varphi}_2$ is possible to express by using second expression (4.9), $\dot{\gamma}_2 = (\dot{\varphi}_2(2\ddot{\varphi}_2^2 - \dot{\varphi}_2\dddot{\varphi}_2))/(\ddot{\varphi}_2^2+\dot{\varphi}_2^4)$, and we can write the following:

$$\dot{\gamma}_2 = \frac{\pm\sqrt{h+\Omega^2(2\lambda\cos\varphi_2-\cos^2\varphi_2)}\left(2\Omega^4(\lambda-\cos\varphi_2)^2\sin^2\varphi_2-\dddot{\varphi}_2\right)}{(-\Omega^2(\lambda-\cos\varphi_2)\sin\varphi_2)^2+\left(h+\Omega^2(2\lambda\cos\varphi_2-\cos^2\varphi_2)\right)^2}. \tag{6.12}$$

By using previous derived expression (6.8) for intensity of the vectors rotators, $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$, connected for the pole O_2 and rotate around self rotation axis, oriented by unit vector $\vec{n}_2$ in the orthogonal plane through pole O_2 and by changing some parameters of heavy gyrorotor structure, as it is eccentricity e, angle of disk inclination β, orthogonal distance between axes a, as well as parameter ψ contained in the coefficients of the nonlinear differential equation (6.1) and presented by expressions (6.5), we obtain series of the graphical presentation, and some of these are presented in Figure 5.

By using parametric equations, in the form (6.11), of the trajectory of the vector rotators $\overrightarrow{\mathfrak{R}}_{022}$ and $\overrightarrow{\mathfrak{R}}_2$ connected for the pole O_2 and rotate around self rotation axis, oriented by unit vector $\vec{n}_2$ in the orthogonal plane through pole O_2 and by changing some parameters of heavy gyrorotor-disk, as it is eccentricity e, angle of disk inclination β, orthogonal distance between axes a, as well as parameter ψ contained in the coefficients of the nonlinear differential equation (6.1) and presented by expressions (6.5), we obtain series of the graphical presentation, and some of these are presented in Figure 6.

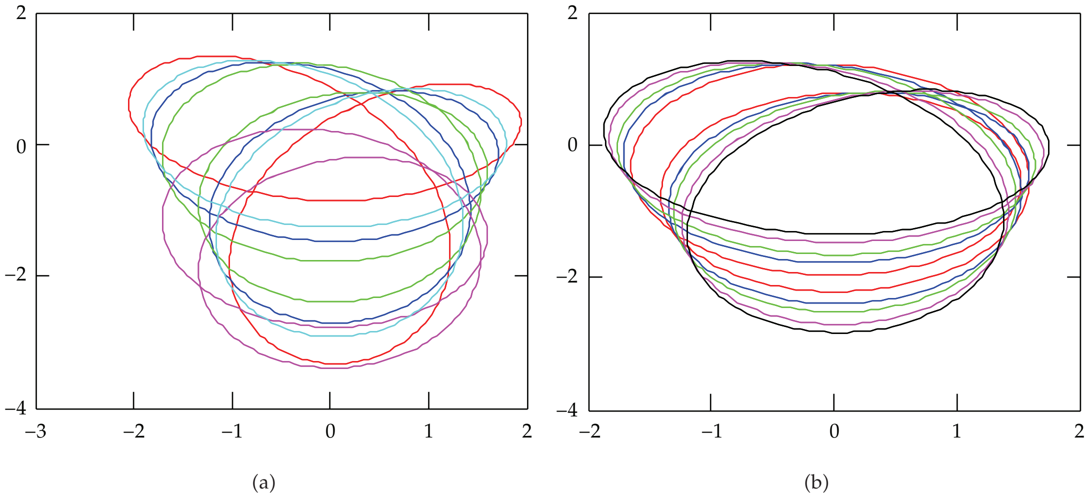


Figure 6: Transformation of the trajectory of the vector rotator $\vec{\mathfrak{R}}_{022}$ (and $\vec{\mathfrak{R}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis for different values of parameter ψ.

In Figure 6. transformation of the trajectory (hodograph) of the vector rotator $\vec{\mathfrak{R}}_{022}$ (and $\vec{\mathfrak{R}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis for different values of parameter ψ is presented.

In Figure 9. transformation of the trajectory (hodograph) of the vector rotator $\vec{\mathfrak{R}}_{022}$ (and $\vec{\mathfrak{R}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis for different values of parameter λ is presented.

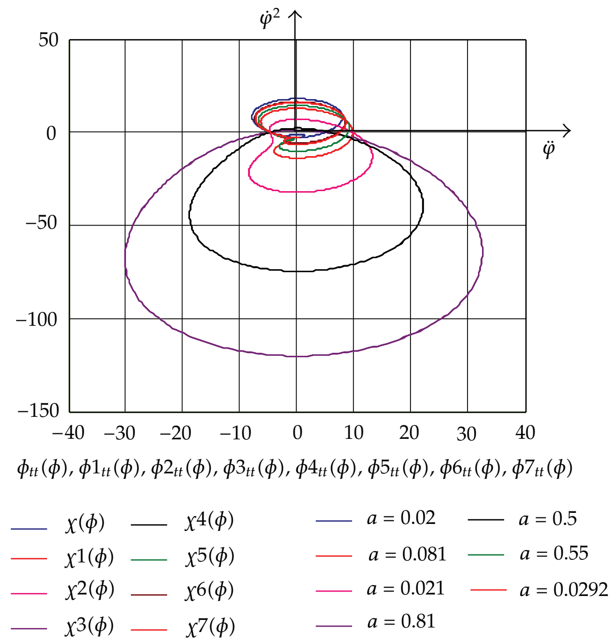
In Figure 7. transformation of the trajectory (hodograph) of the vector rotator $\vec{\mathfrak{R}}_{022}$ (and $\vec{\mathfrak{R}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis, for different values of parameter a, orthogonal distance between axes of gyrorotor-disk coupled component rotations is presented.

By using expression, in the form (6.12), of relative angular velocity $\dot{\gamma}_2$ of the vector rotator $\vec{\mathfrak{R}}_{022}$ (and $\vec{\mathfrak{R}}_2$) rotation in the plane through pole O_2 and orthogonal to the self rotation axis, oriented by unit vector $\vec{n}_2$ and by changing some parameters of heavy gyrorotor structure, as it is eccentricity e, angle of disk inclination β, orthogonal distance between axes a, as well as parameter ψ contained in the coefficients of the nonlinear differential (6.1) and presented by expressions (6.5), we obtain series of the graphical presentation, and some of these are presented in Figure 8.

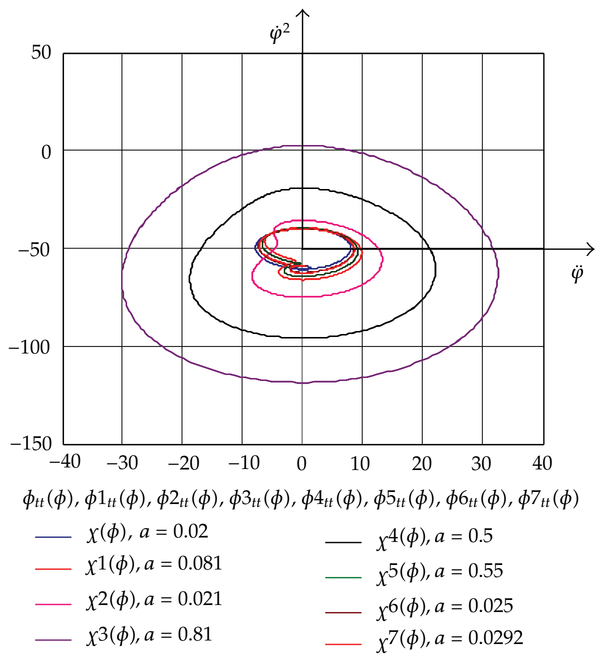
In Figure 8. relative angular velocity $\dot{\gamma}_2$ of the vector rotator $\vec{\mathfrak{R}}_{022}$ (and $\vec{\mathfrak{R}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis, for different values of parameter a, orthogonal distance between axes of gyrorotor-disk coupled component rotations is presented.

7. Concluding Remarks

First main result presented is successful application the vector method by use mass moment vectors for investigation of the rigid body coupled rotation around two axes without

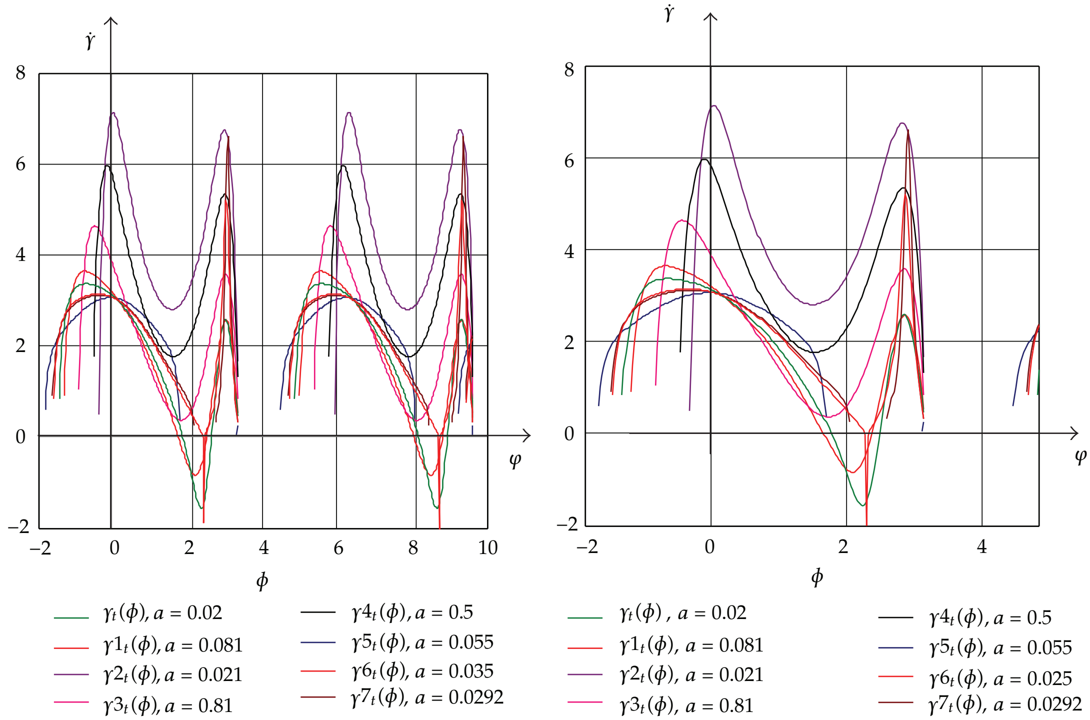


(a)



(b)

Figure 7: Transformation of the trajectory of the vector rotator $\overrightarrow{\mathfrak{R}}_{022}$ (and $\overrightarrow{\mathfrak{R}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis, for different values of parameter a, orthogonal distance between axes of gyrorotor-disk coupled component rotations.



(a) (b)

Figure 8: Relative angular velocity $\dot{\gamma}_2$ of the vector rotator $\vec{\vec{\mathfrak{R}}}_{022}$ (and $\vec{\vec{\mathfrak{R}}}_2$) in the plane through pole O_2 and orthogonal to the self rotation axis, for different values of parameter a, orthogonal distance between axes of gyrorotor-disk coupled component rotations.

cross-sections and vector decomposition of the dynamic structure into series of the vector parameters useful for analysis of the coupled rotation kinetic properties.

By introducing mass moment vectors and vector rotators we expressed linear momentum and angular momentum, as well as their derivatives with respect to time for the case of the rigid body coupled rotations around two axes without intersections. By applications of the new vector approach for the investigations of the kinetic properties of the nonlinear dynamics of the rigid body coupled rotations around two axes without intersections, we show that vector method, as well as applications of the mass moment vectors and vector rotators simples way show characteristic vector structures of coupled rotation kinetic properties.

Appearance, as it is visible, of the vector rotators, their intensity, and their directions as well as their relative angular velocity of rotation around component directions parallel to components of the coupled rotations, is very important for understanding mechanisms of coupled rotations as well as kinetic pressures on shaft bearings of both shafts.

Special attentions are focused to the vector rotators, as well as to the absolute and relative angular velocities of their rotations. These kinematical vector rotators of the heavy gyrorotor disk coupled rotations about two axes without intersection and their three parameter transformations are done as a second main result of this pepar.

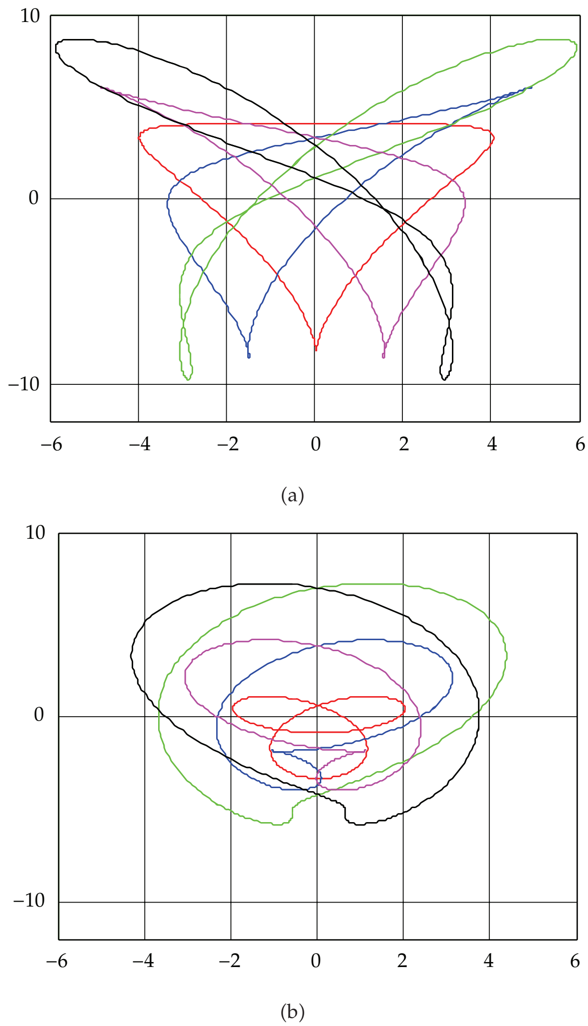


Figure 9: Transformation of the trajectory of the vector rotator $\overrightarrow{\mathfrak{R}}_{022}$ (and $\overrightarrow{\mathfrak{R}}_{2}$) in the plane through pole O_2 and orthogonal to the self rotation axis for different values of parameter λ.

A complete analysis of obtained vector expressions for derivatives of linear momentum and angular momentum give us a series of the kinematical vectors rotators around both directions determined by axes of the rigid body coupled rotations around axes without intersection. These kinematical vectors rotators are defined for a system with two degrees of freedom as well as for rheonomic system with two degrees of mobility and one degree of freedom and coupled rotations around two coupled axes without intersection as well as their angular velocities and intensity.

Acknowledgments

Parts of this research were supported by the Ministry of Sciences and Technology of Republic of Serbia through Mathematical Institute, SANU, Belgrade Grant ON174001 “Dynamics of hybrid systems with complex structures. Mechanics of materials", supported by the Faculty of Mechanical Engineering University of Niš and Faculty of Mechanical Engineering University of Kragujevac.

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