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Research Article

Numerical Solution of Nonlinear Fredholm Integrodifferential Equations of Fractional Order by Using Hybrid of Block-Pulse Functions and Chebyshev Polynomials

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A numerical method for solving nonlinear Fredholm integral equations of second kind is proposed. The Fredholm-type equations, which have many applications in mathematical physics, are then considered. The method is based upon hybrid function approximate. The properties of hybrid of block-pulse functions and Chebyshev series are presented and are utilized to reduce the computation of nonlinear Fredholm integral equations to a system of nonlinear. Some numerical examples are selected to illustrate the effectiveness and simplicity of the method.

1. Introduction

Over the last years, the fractional calculus has been used increasingly in different areas of applied science. This tendency could be explained by the deduction of knowledge models which describe real physical phenomena. In fact, the fractional derivative has been proved reliable to emphasize the long memory character in some physical domains especially with the diffusion principle. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives, and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow [1]. In the fields of physics and chemistry, fractional derivatives and integrals are presently associated with the application of fractals in the modeling of electrochemical reactions, irreversibility, and electromagnetism [2], heat conduction in materials with memory, and radiation problems. Many mathematical formulations of mentioned phenomena contain nonlinear integrodifferential equations with fractional order. Nonlinear phenomena are also of fundamental importance in various fields of science and engineering. The nonlinear models of real-life problems are still difficult to be solved either numerically or theoretically. There has recently been much attention devoted to the search for better and more efficient solution methods for determining a solution, approximate or exact, analytical or numerical, to nonlinear models [3–5].

In this paper, we study the numerical solution of a nonlinear fractional integrodifferential equation of the second:

$$D^{\alpha} f(x) - \lambda \int_0^1 k(x,t) [f(t)]^m dt = g(x), \quad m > 1, \tag{1.1}$$

with the initial condition

$$f^{(i)}(0) = \delta_i, \quad i = 0,1,\ldots,r-1, \; r-1 < \alpha \leq r, \; r \in N \tag{1.2}$$

by hybrid of block-pulse functions and Chebyshev polynomials. Here, $g \in L^2([0,1))$, $k \in L^2([0,1)^2)$ are known functions; $f(x)$ is unknown function. D^{α} is the Caputo fractional differentiation operator and mis a positive integer.

During the last decades, several methods have been used to solve fractional differential equations, fractional partial differential equations, fractional integrodifferential equations, and dynamic systems containing fractional derivatives, such as Adomian's decomposition method [6–11], He's variational iteration method [12–14], homotopy perturbation method [15, 16], homotopy analysis method [3], collocation method [17], Galerkin method [18], and other methods [19–21]. But few papers reported application of hybrid function to solve the nonlinear fractional integro-differential equations.

The paper is organized as follows: in Section 2, we introduce the basic definitions and properties of the fractional calculus theory. In Section 3, we describe the basic formulation of hybrid block-pulse function and Chebyshev polynomials required for our subsequent. Section 4 is devoted to the solution of (1.1) by using hybrid functions. In Section 5, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples.

2. Basic Definitions

We give some basic definitions and properties of the fractional calculus theory, which are used further in this paper.

Definition 2.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ is defined as [22]

$$\begin{aligned} J^{\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt, \quad \alpha > 0, \; x > 0, \\ J^0 f(x) &= f(x). \end{aligned} \tag{2.1}$$

It has the following properties:

$$J^{\alpha} x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad \gamma > -1. \tag{2.2}$$

Definition 2.2. The Caputo definition of fractal derivative operator is given by

$$D^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t) dt, \tag{2.3}$$

where $m-1 \leq \alpha \leq m,\ m \in N,\ x > 0$. It has the following two basic properties:

$$\begin{aligned} D^{\alpha} J^{\alpha} f(x) &= f(x), \\ J^{\alpha} D^{\alpha} f(x) &= f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^{+}) \frac{x^{k}}{k!}, \quad x > 0. \end{aligned} \tag{2.4}$$

3. Properties of Hybrid Functions

3.1. *Hybrid Functions of Block-Pulse and Chebyshev Polynomials*

Hybrid functions $h_{nm}(x),\ n = 1, 2, \ldots, N,\ m = 0, 1, 2, \ldots, M-1$, are defined on the interval $[0, 1)$ as

$$h_{nm}(x) = \begin{cases} T_m(2Nx - 2n + 1), & x \in \left[\left(\frac{n-1}{N}\right), \frac{n}{N}\right) \\ 0, & \text{otherwise} \end{cases} \tag{3.1}$$

and $\omega_n(t) = \omega(2Nt - 2n + 1)$, where n and m are the orders of block-pulse functions and Chebyshev polynomials.

3.2. *Function Approximation*

A function $y(x)$ defined over the interval 0 to 1 may be expanded as

$$y(x) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} h_{nm}(x), \tag{3.2}$$

where

$$c_{nm} = \left(y(x), h_{nm}(x)\right), \tag{3.3}$$

in which $(\cdot, \cdot)$ denotes the inner product.

If $y(x)$ in (3.2) is truncated, then (3.2) can be written as

$$y(x) = \sum_{n=1}^{N} \sum_{m=0}^{M-1} c_{nm} h_{nm}(x) = C^{T} H(x) = H^{T}(x) C, \tag{3.4}$$

where C and $H(x)$, given by

$$C = [c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots c_{N0}, \ldots, c_{NM-1}]^T, \tag{3.5}$$

$$H(x) = [h_{10}(x), h_{11}(x), \ldots, h_{1M-1}(x), h_{20}(x), \ldots, h_{2M-1}(x), \ldots h_{N0}(x), \ldots, h_{NM-1}(x)]^T. \tag{3.6}$$

In (3.4) and (3.5), c_{nm}, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M-1$, are the coefficients expansions of the function $y(x)$ and $h_{nm}(x)$, $n = 1, 2, \ldots, N$, $m = 0, 1, \ldots, M-1$, are defined in (3.1).

3.3. Operational Matrix of the Fractional Integration

The integration of the vector $H(x)$ defined in (3.6) can be obtained as

$$\int_0^x H(t)dt \approx PH(x), \tag{3.7}$$

see [23], where P is the $MN \times MN$operational matrix for integration.

Our purpose is to derive the hybrid functions operational matrix of the fractional integration. For this purpose, we consider an m-set of block pulse function as

$$b_n(x) = \begin{cases} 1, & \dfrac{i}{m} \leq t \leq \dfrac{i+1}{m}, \\ 0, & \text{otherwise}, \end{cases} \quad i = 0, 1, 2, \ldots, m-1. \tag{3.8}$$

The functions $b_i(x)$ are disjoint and orthogonal. That is,

$$b_i(x)b_j(x) = \begin{cases} 0, & i \neq j, \\ b_j(x), & i = j. \end{cases} \tag{3.9}$$

From the orthogonality of property, it is possible to expand functions into their block pulse series.

Similarly, hybrid function may be expanded into an NM-set of block pulse function as

$$H(x) = \Phi B(x), \tag{3.10}$$

where $B(x) = [b_1(t), b_2(t), \ldots, b_{NM}(t)]$ and Φ is an $MN \times MN$ product operational matrix.

In [24], Kilicman and Al Zhour have given the block pulse operational matrix of the fractional integration F^α as follows:

$$J^\alpha B(x) \approx F^\alpha B(x), \tag{3.11}$$

where

$$F^{\alpha} = \frac{1}{l^{\alpha}} \frac{1}{\Gamma(\alpha+2)} \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \cdots & \xi_{l-1} \\ 0 & 1 & \xi_1 & \xi_2 & \cdots & \xi_{l-2} \\ 0 & 0 & 1 & \xi_1 & \cdots & \xi_{l-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \xi_1 \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{3.12}$$

with $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$.

Next, we derive the hybrid function operational matrix of the fractional integration. Let

$$J^{\alpha} H(x) \approx P^{\alpha} H(x), \tag{3.13}$$

where matrix P^{α} is called the hybrid function operational matrix of fractional integration.

Using (3.10) and (3.11), we have

$$J^{\alpha} H(x) \approx J^{\alpha} \Phi B(x) = \Phi J^{\alpha} B(x) \approx \Phi F^{\alpha} B(x). \tag{3.14}$$

From (3.10) and (3.13), we get

$$P^{\alpha} H(x) = P^{\alpha} \Phi B(x) = \Phi F^{\alpha} B(x). \tag{3.15}$$

Then, the hybrid function operational matrix of fractional integration P^{α} is given by

$$P^{\alpha} = \Phi F^{\alpha} \Phi^{-1}. \tag{3.16}$$

Therefore, we have found the operational matrix of fractional integration for hybrid function.

3.4. The Product Operational of the Hybrid of Block-Pulse and Chebyshev Polynomials

The following property of the product of two hybrid function vectors will also be used.

Let

$$H(x) H^{T}(x) C \cong \tilde{C} H(x), \tag{3.17}$$

where

$$\tilde{C} = \begin{pmatrix} \tilde{C}_1 & 0 & \cdots & 0 \\ 0 & \tilde{C}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{C}_N \end{pmatrix} \tag{3.18}$$

is an $MN \times MN$ product operational matrix. And, $\tilde{C}_i$ $i = 1, 2, 3, \ldots N$ are $M \times M$ matrices given by

$$\tilde{C}_i = \frac{1}{2}\begin{pmatrix} 2c_{i0} & 2c_{i1} & 2c_{i2} & 2c_{i3} & \cdots & 2c_{i,M-2} & 2c_{i,M-1} \\ c_{i1} & 2c_{i0} + c_{i2} & c_{i1} + c_{i3} & c_{i2} + c_{i4} & \cdots & c_{i,M-3} + c_{i,M-1} & c_{i,M-2} \\ c_{i2} & c_{i1} + c_{i3} & 2c_{i0} + c_{i4} & c_{i1} + c_{i5} & \cdots & c_{i,M-4} & c_{i,M-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cdots & \cdots & \cdots & 2c_{i0} + c_{iu} & c_{i1} + c_{i,u+1} & \cdots & c_{iv} \\ \cdots & \cdots & \cdots & c_{i1} + c_{iu} & 2c_{i0} & \vdots & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & 2c_{i0} & c_{i1} \\ c_{i,M-1} & c_{i,M-2} & c_{i,M-3} & c_{i,M-4} & \cdots & c_{i1} & 2c_{i0} \end{pmatrix}. \tag{3.19}$$

We also define the matrix D as follows:

$$D = \int_0^1 H(x)H^T(x)dx. \tag{3.20}$$

For the hybrid functions of block-pulse and Chebyshev polynomials, D has the following form:

$$D = \begin{pmatrix} L & 0 & \cdots & 0 \\ 0 & L & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L \end{pmatrix}, \tag{3.21}$$

where L is $M \times M$ nonsingular symmetric matrix given in [23].

4. Nonlinear Fredholm Integral Equations

Consider (1.1); we approximate $g(x), k(x,t)$ by the way mentioned in Section 3 as

$$g(x) = H^T(x)G,$$
$$k(x,t) = H^T(x)KH(t). \tag{4.1}$$

(see [25]), Now, let

$$D^\alpha f(x) \approx A^T H(x). \tag{4.2}$$

For simplicity, we can assume that $\delta_i = 0$ (in the initial condition). Hence by using (2.4) and (3.13), we have

$$f(x) \approx A^T P^\alpha H(x). \tag{4.3}$$

Define

$$C = [c_0, c_1, c_2, \dots, c_{l-1}]^T = A^T P^\alpha,$$
$$[f(t)]^m = \left[H^T(t)C\right]^m = \left[C^T H(t)\right]^m = C^T H(t) \cdot H^T(t)C \left[H^T(t)C\right]^{m-2}. \tag{4.4}$$

Applying (3.17) and (4.4),

$$[f(t)]^m = A^T \tilde{C} H(t)\left[H^T(t)C\right]^{m-2} = A^T \tilde{C} H(t) \cdot H^T(t)C \left[B^T(t)C\right]^{m-3},$$
$$[f(t)]^m = C^T\left[\tilde{C}\right]^{m-1} H(t) = C^* H(t). \tag{4.5}$$

With substituting in (1.1), we have

$$H^T(x)A - \lambda \int_0^1 H^T(x)KH(t)H^T(t)C^{*T} dt = H^T(x)G,$$
$$H^T(x)A - \lambda H^T(x)K \int_0^1 H(t)H^T(t)dt \cdot C^{*T} = H^T(x)G \tag{4.6}$$

Applying (3.20), we get

$$A - \lambda KD \cdot C^{*T} = G, \tag{4.7}$$

which is a nonlinear system of equations. By solving this equation, we can find the vector C.

We can easily verify the accuracy of the method. Given that the truncated hybrid function in (3.4) is an approximate solution of (1.1), it must have approximately satisfied these equations. Thus, for each $x_i \in [0,1]$,

$$E(x_i) = A^T H(x_i) - \lambda \int_0^1 k(x_i, t) C^* H(t) dt - g(x_i) \approx 0. \tag{4.8}$$

If max $E(x_i) = 10^{-k}$ (k is any positive integer) is prescribed, then the truncation limit N is increased until the difference $E(x_i)$ at each of the points x_i becomes smaller than the prescribed 10^{-k}.

5. Numerical Examples

In this section, we applied the method presented in this paper for solving integral equation of the form (1.1) and solved some examples.

Example 5.1. Let us first consider fractional nonlinear integro-differential equation:

$$D^{\alpha} f(x) - \int_0^1 xt \left[f(t)\right]^2 dt = 1 - \frac{x}{4}, \quad 0 \le x < 1,\ 0 < \alpha \le 1, \tag{5.1}$$

(see [26]), with the initial condition $f(0) = 0$.

The numerical results for $M = 1$, $N = 2$, and $\alpha = 1/4, 1/2, 3/4$, and 1 are plotted in Figure 1. For $\alpha = 1$, we can get the exact solution $f(x) = x$. From Figure 1, we can see the numerical solution is in very good agreement with the exact solution when $\alpha = 1$.

Example 5.2. As the second example considers the following fractional nonlinear integro-differential equation:

$$D^{1/2} f(x) - \int_0^1 xt \left[f(t)\right]^4 dt = g(x), \quad 0 \le x < 1, \tag{5.2}$$

with the initial condition $f(0) = 0$ and $g(x) = (1/\Gamma(1/2))((8/3)\sqrt{x^3} - 2\sqrt{x}) - (x/1260)$, the exact solution is $f(x) = x^2 - x$. Table 1 shows the numerical results for Example 5.2.

Example 5.3.

$$D^{5/3} f(x) - \int_0^1 (x+t)^2 \left[f(t)\right]^3 dt = g(x), \quad 0 \le x < 1, \tag{5.3}$$

(see [12]), where

$$g(x) = \frac{6}{\Gamma(1/3)} \sqrt[3]{x} - \frac{x^2}{7} - \frac{x}{4} - \frac{1}{9}, \tag{5.4}$$

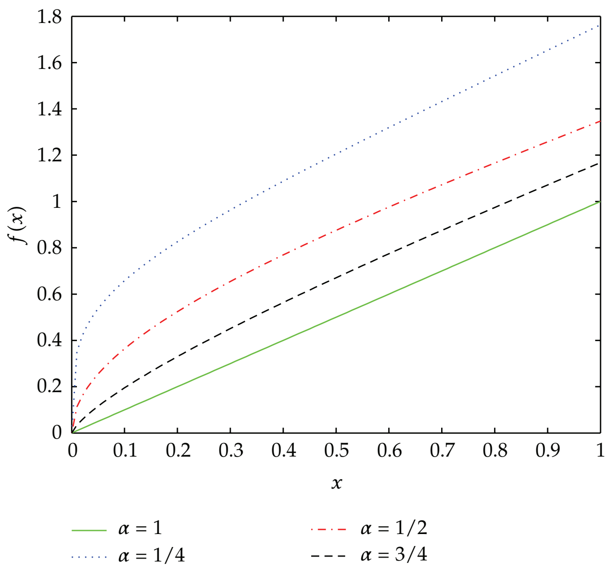

Figure 1: The approximate solution of Example 5.1 for $N = 1$, $M = 2$.

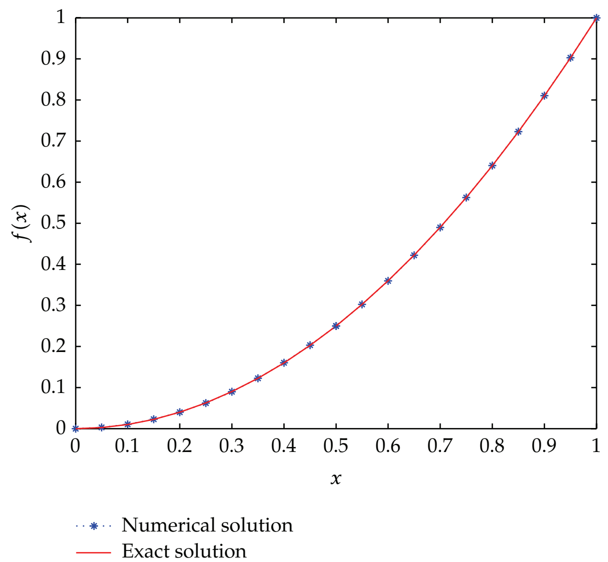

Figure 2: Exact and numerical solutions of Example 5.3 for $N = 2$, $M = 3$.

and with these supplementary conditions $f(0) = f'(0) = 0$. The exact solution is $f(x) = x^2$. Figures 2 and 3 illustrates the numerical results of Example 5.3 with $N = 2, M = 3$.

6. Conclusion

We have solved the nonlinear Fredholm integro-differential equations of fractional order by using hybrid of block-pulse functions and Chebyshev polynomials. The properties of hybrid

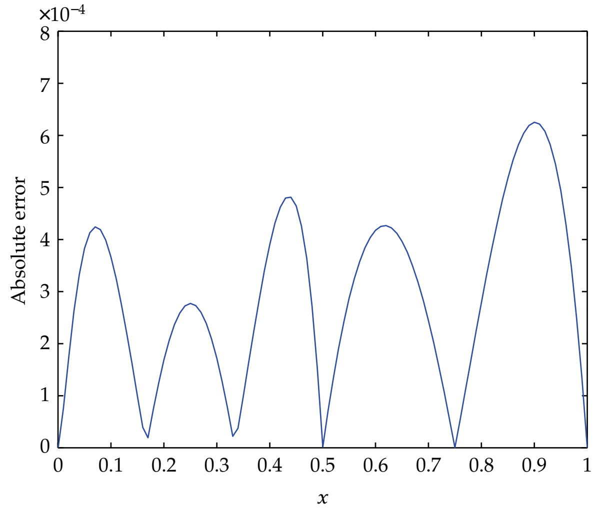

Figure 3: Absolute error of Example 5.3 for $N = 2$, $M = 3$.

Table 1: Absolute error for $\alpha = 1/2$ and different values of M, N for Example 5.2.

x	$N = 2$, $M = 3$	$N = 3$, $M = 3$	$N = 4$, $M = 3$
0.1	5.1985e-003	1.5106e-004	2.8496e-006
0.2	1.1372e-003	2.4887e-004	3.9120e-006
0.3	7.4698e-004	3.2711e-004	4.6808e-006
0.4	1.2729e-003	7.0337e-005	3.1231e-006
0.5	4.6736e-003	4.3451e-004	3.2653e-006
0.6	1.2160e-003	2.5000e-006	2.6369e-006
0.7	6.0767e-004	6.2935e-005	4.7123e-007
0.8	6.0442e-004	3.2421e-004	4.8631e-006
0.9	1.2039e-003	6.2276e-005	2.0707e-006

of block-pulse functions and Chebyshev polynomials are used to reduce the equation to the solution of nonlinear algebraic equations. Illustrative examples are given to demonstrate the validity and applicability of the proposed method. The advantages of hybrid functions are that the values of N and M are adjustable as well as being able to yield more accurate numerical solutions. Also hybrid functions have good advantage in dealing with piecewise continuous functions.

The method can be extended and applied to the system of nonlinear integral equations, linear and nonlinear integro-differential equations, but some modifications are required.

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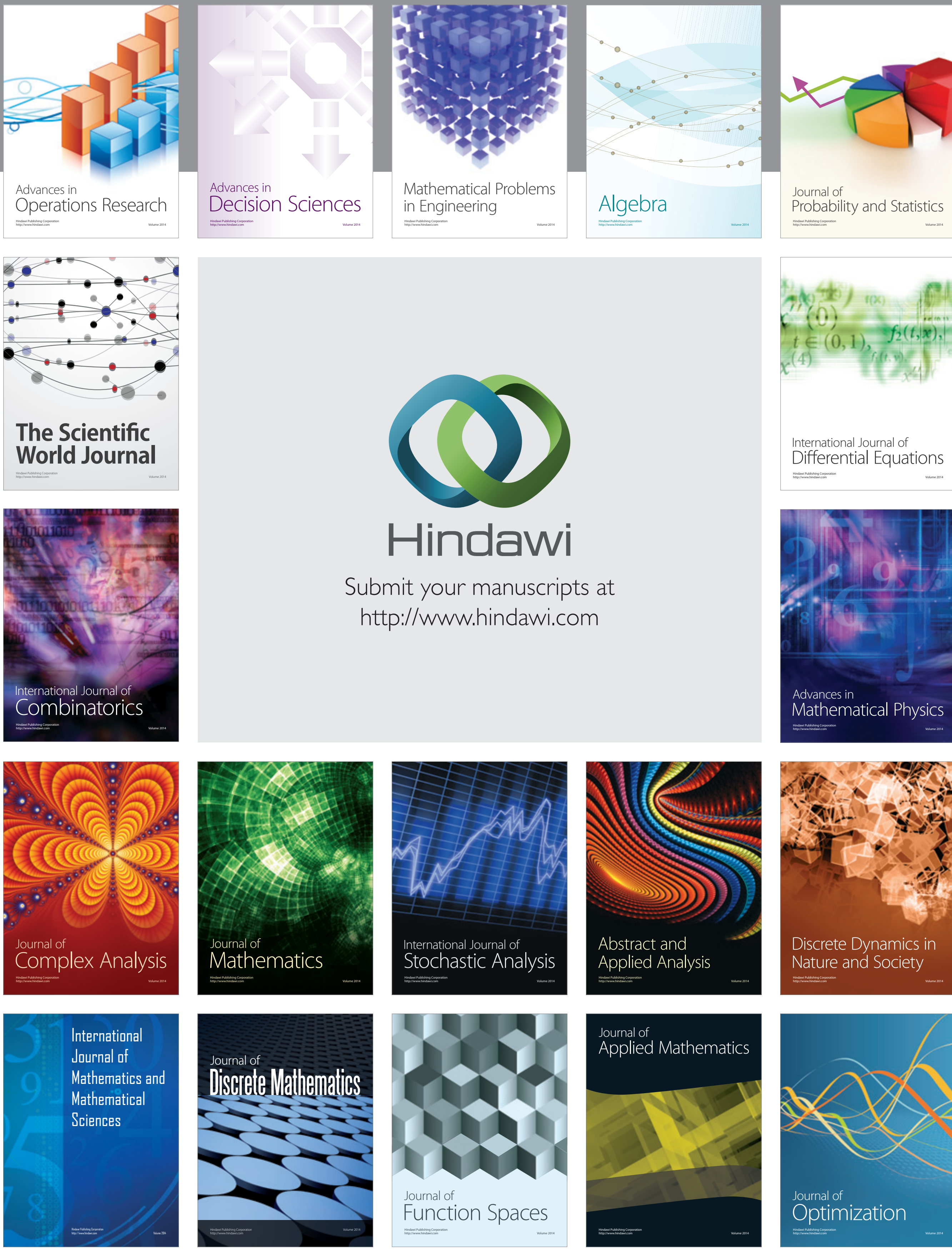
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